MAT 247, Winter 2014
Assignment 9
Due March 25

1. The purpose of this exercise is to prove that trace is the unique (up to scalar) similarity-invariant linear functional on the space of square matrices.

   Let $\mathbb{F}$ be any field (but you may assume $1+1 \neq 0$ in $\mathbb{F}$) and let M_n be the set of $n \times n$ matrices over $\mathbb{F}$. Let

   $$W = \{\alpha \in M_n^* : \alpha(A) = \alpha(B) \text{ if } A \text{ is similar to } B\}$$

   (a) Prove that W is a subspace of M_n^*.

   (b) Prove that $tr \in W$.

   (c) Let $X_{i,j}$ denote the matrix which is 1 in the (i,j) slot and 0 elsewhere. Since $\{X_{i,j}\}$ forms a basis for M_n, we have a dual basis $\{X_{i,j}^*\}$ for M_n^*. Let $\alpha \in W$. Then we can write $\alpha = \sum_{i,j} c_{i,j} X_{i,j}^*$ for some scalars $c_{i,j}$. Prove that $c_{i,j} = 0$ if $i \neq j$. [Hint: first show that if $i \neq j$ then $X_{i,j}$ is similar to $aX_{i,j}$ for any non-zero $a \in \mathbb{F}$.]

   (d) Prove that $c_{i,i} = c_{j,j}$ for all i, j.

   (e) Conclude that W is a 1-dimensional vector space with basis tr.

2. Let V be a finite-dimensional vector space over a field $\mathbb{F}$. Define a linear map $\phi : V^* \otimes V \to \mathbb{F}$ by $\phi(\alpha \otimes v) = \alpha(v)$. In Assignment 7, you constructed an isomorphism $\psi : V^* \otimes V \to L(V,V)$. Prove that if $T \in L(V,V)$, then $tr(T) = \phi(\psi^{-1}(T))$.

3. Let $V = \mathbb{F}^3$. Find a specific element of $V^{\otimes 3}$ which does not lie in $Sym^3 V \oplus \Lambda^3 V$.

4. Let V be a finite-dimensional vector space and let $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ be two linear independent collections of elements of V. Recall that we proved in class that $v_1 \wedge \cdots \wedge v_k \neq 0$ and $w_1 \wedge \cdots \wedge w_k \neq 0$.

   (a) Prove that there exists a non-zero scalar $c \in \mathbb{F}$ such that

   $$v_1 \wedge \cdots \wedge v_k = c w_1 \wedge \cdots \wedge w_k$$

   if and only if $span(v_1, \ldots, v_k) = span(w_1, \ldots, w_k)$.

   [Hint: It maybe useful to use the following fact: given two subspaces W_1, W_2 of a vector space V, there exists a basis for V containing bases of W_1 and W_2.]

   (b) Let $Gr(p, V)$ denote the set of p-dimensional subspaces of V. Use (a) to construct an injective map

   $$Gr(k, V) \to Gr(1, \Lambda^k V)$$