MAT 347
Semidirect products
November 6, 2015

Semidirect products

Recall the following definitions.

Definition: Let H, K be two groups. We define the *direct product* of H and K to be $H \times K = \{(h, k) : h \in H, k \in K\}$ with component-wise multiplication.

Now, let G be a group and let H, K be subgroups. Define $HK = \{hk : h \in H, k \in K\} \subset G$.

1. Suppose that H, K are subgroups of G. Suppose that $H \cap K = \{1\}$. Prove that every element of HK can be written uniquely as hk for $h \in H, k \in K$.

2. Suppose that H, K are normal subgroups of G and $H \cap K = \{1\}$. Explain how to multiply h_1k_1 with h_2k_2. Prove that HK is isomorphic to $H \times K$.

3. Prove that $D_{4n} \cong D_{2n} \times Z_2$ if n is odd.

4. Suppose that H is normal in G, but K is not. Explain how to multiply h_1k_1 with h_2k_2 (express you answer as hk for some $h \in H$, $k \in K$).

5. Suppose now that H, K are two abstract groups (i.e. not embedded as subgroups of a third group). Suppose that we are given a homomorphism $\phi : K \to AutH$. In other words, for each element $k \in K$, we are given an automorphism $\phi_k : H \to H$ of H. Explain how we can use this to define a new group structure on the set $H \times K$, motivated by your computation in 4.

The set $H \times K$ with this group structure will be denoted $H \rtimes_\phi K$ and is called the *semidirect product* of H and K with respect to ϕ.

6. Show that H, K are both subgroups of $H \rtimes_\phi K$ and that H is a normal subgroup.

7. Show that D_{2n} is isomorphic to a semidirect product of Z_n and Z_2.

8. Let F be a field. Consider $H = F$, $K = F^\times$. Define a natural map $K \to AutH$ and form the semidirect product $H \rtimes K$. How can you think about this group?

Isometries

Definition An *isometry* of the plane is a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|f(x) - f(y)| = |x - y|$ (where $|\cdot|$ denotes the length of a vector. The set of isometries of the plane forms a group $Isom(\mathbb{R}^2)$.

9. Show that any translation is an isometry. What can you say about the subgroup of translations inside $Isom(\mathbb{R}^2)$?

10. Show that any orthogonal linear operator on $\mathbb{R}^2$ is an isometry.

11. Show that any isometry is the composition of an translation and an orthogonal linear map. [You may use the following fact without proof: if f is an isometry such that $f(0) = 0$, then f is an orthogonal linear map.]

12. Express $Isom(\mathbb{R}^2)$ as a semi-direct product.