MAT 347
Irreducibility criteria
January 29, 2016

Let R be a UFD and let F be its field of fractions. Let $f(X) = a^n X^n + \ldots + \ldots a_0 \in R[X]$.

Gauss' Lemma

- $f(X)$ is reducible in $R[X] \iff f(X)$ reducible in $F[X]$.

About roots

- $f(X)$ has a degree-1 factor in $F[X]$ iff it has a root in F.
- Assume $\deg f(X) = 2$ or 3. If f has no roots in F, then it is irreducible in $F[X]$.
- Assume that $\frac{r}{s}$ is a root of $f(X)$ written as a fraction in R in lowest terms.

  Then $r|a_0$ and $s|a_n$.

Reduction

- Let $I \trianglelefteq R$ be a proper ideal. Assume $f(X)$ is monic. Let $\overline{f(X)} \in R/I[X]$ be the projected polynomial.

  If $\overline{f(X)}$ is irreducible in $R/I[X]$, then $f(X)$ is irreducible in $R[X]$.

Eisenstein criterion

- Let $P \trianglelefteq R$ be a prime ideal. Assume $f(X)$ is monic; $a_{n-1}, \ldots, a_0 \in P$; and $a_0 \notin P^2$.

  Then $f(X)$ is irreducible in $R[X]$.

Translation

- Let $a \in R$. The map $T_a : f(X) \in R[X] \to f(X+a) \in R[X]$ is an isomorphism.

Exercises

Prove whether each of the following polynomials is irreducible on the given polynomial ring. If they are not, factor them.

1. $f(X) = X^3 + 4X^2 + X - 6$ in $\mathbb{Q}[X]$.
2. $f(X) = X^4 + X^2 + 1$ in $\mathbb{Z}/2\mathbb{Z}[X]$.
3. $f(X) = X^4 + 1$ in $\mathbb{Z}[X]$
4. $f(X) = X^5 + 3X^4 + 30X^2 - 9X + 12$ in $\mathbb{Q}[X]$.
5. $f(X) = X^5 + 4X^3 - X + iX + 3 + 3i$ in $\mathbb{Z}[i][X]$.
6. $f(X) = X^3 + 6$ in $\mathbb{Z}/7\mathbb{Z}[X]$.
7. $f(X, Y) = X^3 + X^2Y + 3XY^2 + 5XY + 2Y$ in $\mathbb{Z}[X, Y]$.
8. $f(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ in $\mathbb{Z}[X]$.

Hints

1. Check all the candidates for roots.
2. Freshman's dream.
3. Apply Eisenstein to $f(X + 1)$.
4. Eisenstein.
5. Eisenstein.
6. Look for roots.
7. Eistenstein.
8. Apply Eisenstein $f(X + 1)$.