MAT 347
Counting, group actions, and the Orbit-Stabilizer Lemma
September 25, 2014

Actions

Definition. Let G be a group and let A be a set. An *action* of G on A is a map

$$\begin{aligned} G \times A &\longrightarrow A \\ (g, a) &\mapsto g \cdot a \end{aligned}$$

that satisfies the following two properties:

- $g \cdot (h \cdot a) = (gh) \cdot a$ for all $a \in A$, $g, h \in G$.
- $1 \cdot a = a$ for all $a \in A$,

Definition. An action of G on A is called *faithful* if for all $g \in G$, there exists $a \in A$ such that $g \cdot a \neq a$.

An action of G on A is called *transitive* if for all $a, b \in A$, there exists $g \in G$ such that $g \cdot a = b$.

1. Explain how the following groups act on the following sets. Which of these actions are faithful? Which are transitive?

   (a) A is any set, $G = S_A$.

   (b) $G = D_{2n}$, A is the set of vertices of a regular n-gon.

   (c) $G = D_{2n}$, A is the set of diagonals of a regular n-gon.

   (d) G is any group, $A = G$ as a set. (There are a few possibilities here.)

   (e) A is the set of k-element subsets of $\{1, \ldots, n\}$ and G is the group S_n. (Here k, n are natural numbers with $0 \leq k \leq n$.)

2. Assume we have an action of the group G on the set A. For each $g \in G$, let us define a map $\phi_g : A \to A$ by the equation $\phi_g(a) := g \cdot a$. Show that ϕ_g is a bijection. This defines a map $\phi : G \to S_A$ by the equation $\phi(g) := \phi_g$. Show that ϕ is a group homomorphism.

3. Conversely, show that every group homomorphism $G \to S_A$ comes from an action of G on A. In other words, there is a natural one-to-one correspondence between actions of G on A and group homomorphisms from G to S_A. This is why some authors define an action as a group homomorphism $G \to S_A$ instead.

The Orbit-Stabilizer Lemma

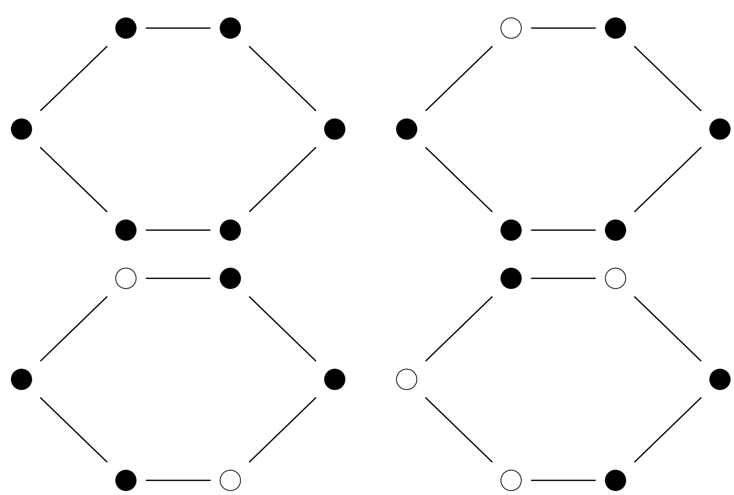
Definitions. Let G be a group acting on a set A.

- Given $g \in G$, we define the *fixed set* of g as the set
$$\mathrm{Fix}(g) := \{a \in A \mid g \cdot a = a\} \subseteq A$$
- Given $a \in A$, we define the *stabilizer* of a as the set
$$\mathrm{Stab}(a) := \{g \in G \mid g \cdot a = a\} \subseteq G$$
- Given $a \in A$ we define the *orbit* of a as the set
$$\Omega_a := \{g \cdot a \mid g \in G\} \subseteq A$$

4. Say we want to count how many *different* necklaces we can build with 6 stones each, if we have stones of two different colours. Define a *diagram* to be any way of colouring each of the six vertices of a hexagon with black or white. Notice that $|A| = 64$. Show that D_{12} acts on A, and that the number of orbits of this action equals the number of different necklaces.

   Note: This shows that the problem of counting the number of orbits of an action is a interesting problem in combinatorics.

5. Regarding the previous question, consider the following diagrams:



   For each one of them, compute the size of its orbit and the size of its stabilizer. Make a conjecture or a formula that relates these two numbers for an arbitrary element in an arbitrary action. Then prove it.