MAT 347
The symmetric and the alternating groups
October 16, 2015

Recall some definitions.:

- A *permutation* of n elements is an element of the group S_n.
- A *cycle of length* m is a permutation that can be written as $(a_1 \cdots a_m)$.
- A *transposition* is a permutation of length 2.
- The *type* of a permutation is the set of lengths of the cycles in its decomposition as product of disjoint cycles. For example the type of $(12345)(67)(89)$ in S_{11} is $(5, 2, 2, 1, 1)$.

1. In general, for an arbitrary group G, the *conjugacy class* of $g \in G$ is the orbit of g in the action of G on G by conjugation. Find a description of the conjugacy classes of S_n.

   Hint: Fix your favourite $\sigma \in S_n$. Then for various $\tau \in S_n$ compute $\tau\sigma\tau^{-1}$. Can you find a formula for $\tau\sigma\tau^{-1}$? Can you describe the conjugacy class of σ?

2. List all the conjugacy classes of S_5 and the size of each class.

   Hint: You know the sum of the sizes of all the conjugacy classes should be 120, so you can check your final answer.

3. Which of the following sets are generators of S_n?

   (a) The set of all cycles.

   (b) The set of all transpositions.

   (c) The set $\{(12), (23), (34), \ldots, (n-1, n)\}$.

   (d) The set $\{(12), (13), (14), \ldots, (1n)\}$.

4.

   (a) Write the permutation (123) as product of transpositions. This can be done in more than one way. Try to write (123) as product of N transpositions, for different values of N. Not all values of N are possible. Which ones are?

   (b) Repeat the same question with the permutation (1234).

   Note: At this point, you can probably make a conjecture for which values of N are not possible, but most likely you won't be able to prove it. For that, we need to introduce some sophistication.

Building the alternating group

Let us fix a positive integer n. Let R be the set of polynomials in the n variables $X_1, \ldots X_n$. We can define an action of S_n on R as follows:

$$\sigma \cdot p(x_1, \ldots, x_n) := p(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

Make sure you understand what this notation means before continuing. Convince yourself that it is, indeed, an action. You will work further with this action in HW 5 (both to understand this proof, and because it will be relevant in Galois Theory at the end of the course.) We define the following polynomial:

$$\Delta := \prod_{1 \leq i < j \leq n} (X_i - X_j)$$

For example, if $n = 3$, then $\Delta = (X_1 - X_2)(X_1 - X_3)(X_2 - X_3)$.

5. Prove that for every $\sigma \in S_n$ there exists a number $\varepsilon_\sigma \in \{1, -1\}$ such that $\sigma \cdot \Delta = \varepsilon_\sigma \Delta$
6. Prove that the map $\varepsilon : S_n \to \{1, -1\}$ is a group homomorphism!

We say that a permutation σ is *even* when $\varepsilon_\sigma = 1$ and it is *odd* when $\varepsilon_\sigma = -1$. When we mention the *parity* of a permutation, we are referring to whether it is odd or even. We define A_n to be the set of all even permutations.

7. Complete: "A cycle of length m is an even permutation iff m is ...."
8. Go back to the conjecture you made in Question 4. Now you can prove it!
9. Prove that A_n is a normal subgroup of S_n.
10. What is $|A_n|$?

    Hint: Use the first isomorphism theorem.
11. Review your answer to Question 2. Which of the conjugacy classes are in A_5? Do their sizes add up to the right number?

The platonic solids

12. Each one of the five platonic solids has a group of rotations that is isomorphic to either some S_n or some A_n. Find them all (with proof).