MAT 347
Free groups
October 30, 2015

Let S be a set. We want to define $F(S)$, the free group on the set S. In the end, we will define $F(S)$ using "words" in the set S. But before we do that, let us think about the properties that $F(S)$ should have. The first property of $F(S)$ is that it contains the set S. Moreover, if H is another group containing the set S, then we should be able to construct a group homomorphism from $F(S)$ to H.

Universal property of the free group

Recall the following fact about the group $\mathbb{Z}$.

1. Prove that for any group H and any element $h \in H$, there exists a unique group homomorphism $\Phi : \mathbb{Z} \to H$ such that $\Phi(1) = h$.

Definition: Let G be a group and $\iota : S \to G$ a map of sets. The pair (G, ι) is a called **a free group** on the set S, if for any other group H and any map $\phi : S \to H$, ...

2. Fill in the ... in the above definition.
3. Suppose that S is a set with one element. Explain how to choose ι so that $(\mathbb{Z}, \iota)$ is a free group on the set S.
4. Keep S a one element set with the same $\iota : S \to \mathbb{Z}$. Prove that $(\mathbb{Z} \times Z_2, \iota)$ is not a free group.
5. Define $\iota : \{1, 2\} \to \mathbb{Z}^2$ by $\iota(1) = (1, 0)$, $\iota(2) = (0, 1)$. Prove that $(\mathbb{Z}^2, \iota)$ is not a free group on the set S. Define a notion of "free abelian group" and prove that $(\mathbb{Z}^2, \iota)$ is a free abelian group.
6. Prove that if S is a finite set, and (G, ι) is a free group on the set S, then ι is injective. [Hint: first construct any group H admitting an injective map $\phi : S \to H$.]
7. Suppose that $(G, \iota), (G', \iota')$ are two free groups on the set S. Prove that there exists a unique group isomorphism $\psi : G \to G'$ such that $\psi \circ \iota = \iota'$.
8. Is it obvious that for any S, there is a free group on the set S?

Construction of the free group

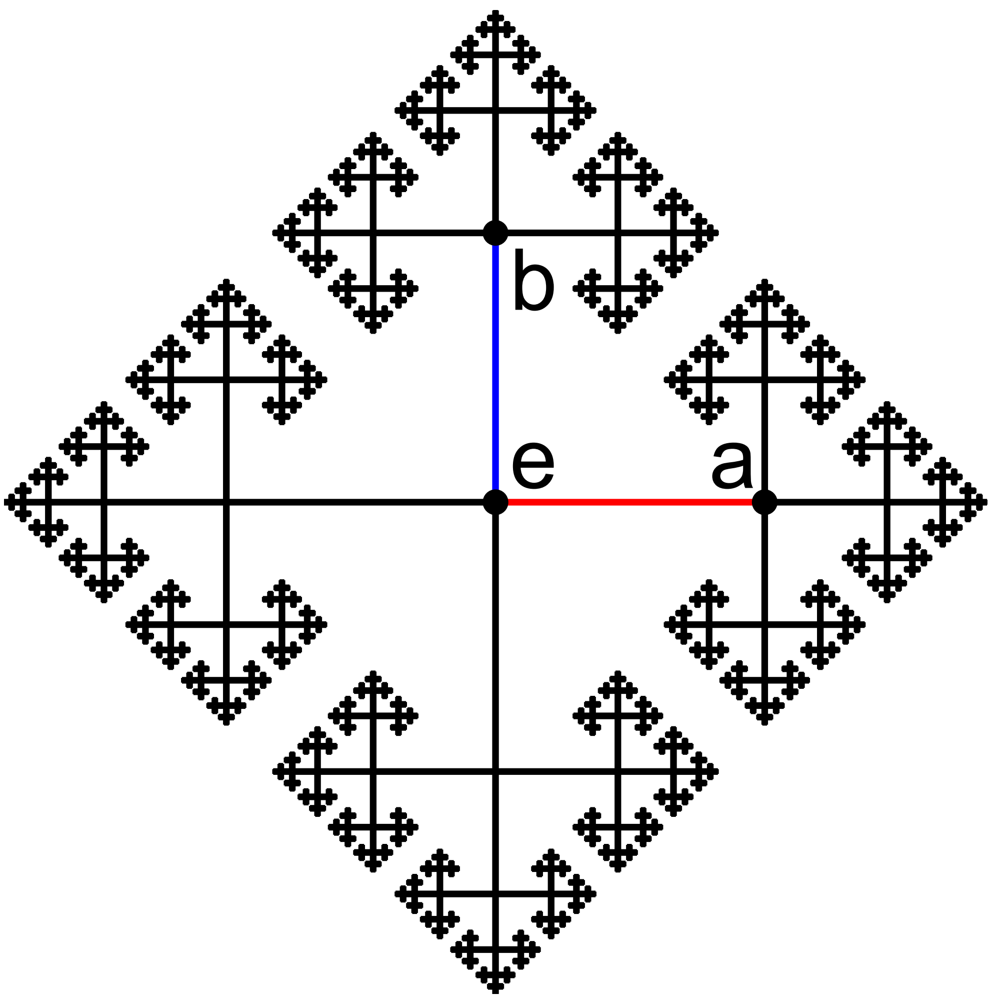
Definition: Let S be a set. Define a new set S^{-1} to be the set of all symbols s^{-1} for $s \in S$. A **word** in the set $S \cup S^{-1}$ is a finite sequence $(x_1, \ldots, x_k)$, where each x_i is either an element of S or S^{-1} (we allow any non-negative integer k). Let $W(S)$ be the set of all words in the set S. $W(S)$ has a binary operation given by concatenation of sequences.

9. Define an equivalence relation on the set $W(S)$ to implement cancellation of neighbouring inverse symbols. Let $F(S)$ be the set of equivalence classes.

10. Prove that $F(S)$ is a group.

11. Prove that $F(S)$ is a free group on the set S.

A picture



12. What does this picture have to do with the free group on two generators?