MAT 347
Review problems
April 8, 2016

Groups

1. Let A, B, C be finite abelian groups. Suppose that $A \times B \cong A \times C$. Prove that $B \cong C$. Can you find a counterexample if you remove the conditions on A, B, C?
2. (a) Construct a transitive action of Z_n on $\{1, \ldots, n\}$.
   (b) Are there any other groups of size n which act transitively on $\{1, \ldots, n\}$?
   (c) Find all the orbits for the action of Z_n on the set of 2-element subsets of $\{1, \ldots, n\}$.
   (d) Count the number of orbits for the action of Z_n on the set of 2-element subsets of $\{1, \ldots, n\}$ using Burnside's Lemma.
3. For every integer n, find how many elements of order n there are in S_5.
4. Give an example of a group G and a homomorphism $\phi : G \to G$ such that
   (a) ϕ is injective but not surjective
   (b) ϕ is surjective but not injective
5. Obtain a formula for the number of *different* necklaces that can be made with n stones, if we have stones of k different colours.

   Hint: I suggest you try to solve the cases $n = 5$ and $n = 6$ first.
6. Prove the a subgroup of index 2 is always normal.
7. Find two different decomposition series of D_{12} that do not produce the same composition factors in the same order. How many different decomposition series does D_{12} have?
8. Can you express Q_8 as a semi-direct product?
9. Find a presentation for S_4. (Warning: proving that you have a complete set of relations may be tedious)
10. Let G be a nonabelian simple group. Prove that $|G|$ is divisible by at least two prime numbers.

11. Let p be a prime. Let $\sigma \in S_p$ be a p-cycle and let $\tau \in S_p$ be a transposition. Prove that σ and τ generate S_p. What if p is not prime?

12. Let G be a finite group and let H be a proper subgroup. Prove that $\bigcup_{g \in G} gHg^{-1} \neq G$.

13. Let G be a finite group. Prove that G acts transitively on $\{1, \ldots, n\}$ if and only if G contains a subgroup H such that $[G : H] = n$.

Fields

1. Let K be a finite field. Prove the product of all the non-zero elements of K is equal to -1.

2. Give an example of a homomorphism $\phi : F \to F$ from a field to itself which is not an automorphism.

3. Determine the splitting field of the polynomial $x^p - x - 1$ over $\mathbb{F}_p$. Show directly that its Galois group is cyclic.

4. Determine $[\mathbb{Q}(\sqrt{7 + 4\sqrt{3}}) : \mathbb{Q}]$.

5. Let K/F be an extension of degree 2. Suppose that the characteristic of F is not 2. Show that $K = F(\alpha)$ for some $\alpha \in K$ such that $\alpha^2 \in F$. What happens if the characteristic of F is equal to 2? Find an example of a degree 3 extension K/F such that K is not equal to $F(\alpha)$ for any $\alpha \in K$ such that $\alpha^3 \in F$.

6. Let $\Phi_n(x) \in \mathbb{Q}[x]$ denote the nth cyclotomic polynomial. Show that $\Phi_{2^k}(x) = x^{2^{k-1}} + 1$.

7. For each $n \geq 1$, find a finite extension K/F such that $K \neq F(\alpha_1, \ldots, \alpha_n)$ for any $\alpha_1, \ldots, \alpha_n \in K$.

8. Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Let α, β be two roots of $f(x)$ (in some bigger field). Prove that there exists an isomorphism $F(\alpha) \cong F(\beta)$ (you may not use Theorem A). Give an example to show that the irreducibility of $f(x)$ is a necessary assumption.

9. Let F be an algebraically closed field of characteristic p, and let $K = F(x, y)$. Let a and b satisfy $a^p = x$ and $b^p = y$. Prove that $K(a, b)/K$ is a finite extension with infinitely many intermediate fields.

Rings

1. Let $f(x), g(x) \in \mathbb{Z}[x]$. Prove that $f(x)$ and $g(x)$ are relatively prime in $\mathbb{Q}[x]$ if and only if the ideal generated by $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$ contains a nonzero integer.

2. Let R be any ring with 1. Let $a \in R$ and suppose that $a^n = 0$ for some n. Prove that $1 + a$ is a unit in R.

3. Let R be a UFD in which every non-zero prime ideal is maximal. Prove that R is a principal ideal domain.

4. Let X be any set. We denote its power set (that is, the set of subsets of X) by $\mathcal{P}(X)$. We define two operations in $\mathcal{P}(X)$; given $A, B \subseteq X$:

$$\begin{aligned} A \setminus B &= \{x \in A \mid x \notin B\} \\ A \triangle B &= (A \setminus B) \cup (B \setminus A) \end{aligned}$$

   Notice that $(\mathcal{P}(X), \triangle, \cap)$ is an commutative ring with identity for any set X. (You do not need to prove this, but I recommend that you persuade yourself that this is true.)

   (a) Assume that X is finite. Then $(\mathcal{P}(X), \triangle)$ is a finite abelian group. By the Fundamental Theorem of Finite Abelian Groups, we know that it has to be isomorphic to a direct product of finite cyclic groups. Find positive integers $n_1, \ldots, n_r$ such that
$$(\mathcal{P}(X), \triangle) \cong Z_{n_1} \times \cdots \times Z_{n_r}$$
   and explicitly construct one such isomorphism.

   (b) Find all zero-divisors and all units of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (c) Describe all the principal ideals of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (d) Assume X is finite. Prove that every ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$ is principal.

   (e) Construct an explicit example of a set X and a non-principal ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (f) Assume X is finite. Find all prime ideals of the ring $(\mathcal{P}(X), \triangle, \cap)$.

   (g) Assume X is infinite. Prove that there exists a non-principal prime ideal of the ring $(\mathcal{P}(X), \triangle, \cap)$.

5. Factor 4004 into primes in $\mathbb{Z}[i]$.

6. Prove that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is an integral domain but not a field.