

**REPRESENTATION THEORY
ASSIGNMENT 4
DUE FRIDAY APRIL 15**

- (1) (a) Use the Weyl character formula to prove the Kostant multiplicity formula

$$\dim V(\lambda)_\mu = \sum_{w \in W} (-1)^{l(w)} kpf(w(\lambda + \rho) - (\mu + \rho))$$

where $\lambda \in X_+$ and μ lies in the convex hull of the Weyl orbit of λ .

(In fact, it is pretty easy to see that this is just a reformulation of the Weyl character formula.)

- (b) Let λ, μ, ν be dominant weights. The tensor product multiplicities $c_{\lambda\mu}^\nu$ are the multiplicities of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. In other words,

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\nu} V(\nu)^{\oplus c_{\lambda\mu}^\nu}$$

Use the Weyl character formula to prove the Steinberg tensor product formula

$$c_{\lambda\mu}^\nu = \sum_{w \in W} (-1)^{l(w)} \dim V(\mu)_{\nu + \rho - w(\lambda + \rho)}$$

(Hint: let $S_{\lambda\mu}^\nu$ denote the right hand side of the above equation. By considering characters, it suffices to show that

$$\sum_{\nu} S_{\lambda\mu}^\nu \chi_\nu = \chi_\lambda \chi_\mu$$

Using the Weyl character formula (as given in equation (4) in the notes), let us write $\chi_\lambda = A_{\lambda+\rho}/A_\rho$, where

$$A_\lambda = \sum_{w \in W} (-1)^{l(w)} e^{w\lambda}$$

Multiplying both sides by A_ρ , it suffices to show that

$$A_{\lambda+\rho} \chi_\mu = \sum_{\nu} S_{\lambda\mu}^\nu A_{\nu+\rho}$$

Prove this last equality by Weyl invariance and by checking the coefficient of $e^{\nu+\rho}$ on both sides.)

- (c) The “leading term” in the Steinberg tensor product formula is $\dim V(\mu)_{\nu-\lambda}$. Prove (without using the Steinberg formula or the Weyl character formula) that

$$c_{\lambda\mu}^\nu \leq \dim V(\mu)_{\nu-\lambda}.$$

(Hint: Note that $c_{\lambda\mu}^{\nu}$ is the dimension of the space of N -invariant vectors of weight ν in $V(\lambda) \otimes V(\mu)$.)

- (2) Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a dominant weight for $GL(n)$, $\lambda_1 \geq \dots \geq \lambda_n$. Let $\mu = (\mu_1, \dots, \mu_{n-1})$ be a dominant weight for GL_{n-1} . We say that $\mu \prec \lambda$ if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$$

Use the Weyl character formula to prove the following

$$V(\lambda)|_{GL_{n-1}} \cong \bigoplus_{\mu \prec \lambda} V(\mu)$$

(Hint: One approach is to use the Weyl character formula to show that the characters of the left and right hand sides are the same.)

- (3) Let $i \in I$. Use the fact that $s_i R_+ = R_+ \setminus \{\alpha_i\} \cup \{-\alpha_i\}$ to prove that $\langle \alpha_i^\vee, \rho \rangle = 1$.
- (4) A dominant weight λ is called minuscule if $\langle \alpha^\vee, \lambda \rangle \leq 1$ for all positive coroots α^\vee .
- (a) Find all minuscule weights of SL_n and SO_{2n} .
- (b) Show that λ is minuscule if and only if the weights of $V(\lambda)$ is the Weyl group orbit of λ .
- (c) Interpret this last statement in terms of the restriction map

$$\Gamma(G/B, L(\lambda)) \rightarrow \Gamma((G/B)^T, L(\lambda))$$

- (5) Print out some paper with the root lattice of GL_3 . (For example you can download some from <http://incompetech.com/graphpaper/triangledots/>.) On this lattice paper draw the weight multiplicity diagram for the representation $V(6, 4, 0)$ (so that the lattice on the paper corresponds to the set $(6, 4, 0) + Q$, where Q is the root lattice). It should look like a hexagon with side lengths 2 and 4, with 1s along the boundary of the hexagon, 2s one step in from the boundary and the rest of the interior lattice point labelled 3.

From part of the paper which you did not use, cut out a unit hexagon. Now place your hexagon on top of your weight multiplicity diagram, so that the vertices of the hexagon line up with points of the lattice. Notice that for almost all positions of the hexagon, the alternating sum of values at the vertices of the hexagon is 0. Explain this fact using the Weyl character formula.