Representation theory of compact groups and complex reductive groups, Winter 2011

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Contents

1 Semisimplicity

1.1 Representations

Let G be a group (either a Lie group or a discrete group).

A *representation* of G consists of a finite-dimensional complex vector space V along with a group homomorphism $G \to GL(V)$.

In the case where G is a Lie group, then we ask that the map $G \to GL(V)$ be a smooth map. This is equivalent to asking that for all $v \in V$ and $\alpha \in V^*$, the matrix coefficient

$$G \to \mathbb{C} \quad g \mapsto \langle \alpha, gv \rangle$$

is a smooth (complex-valued) function on G.

A *morphism* ϕ of between representations V, W is a linear map $\phi : V \to W$ such that for all $g \in G$ and $v \in V$, $\phi(gv) = g\phi(v)$ (this condition is called G-equivariance). The vector space of all morphism from V to W is denoted $\mathrm{Hom}_G(V, W)$.

If V is a representation, we can form its dual representation V^*. The action of G on V^* is given by $\langle g\alpha, v \rangle = \langle \alpha, g^{-1}v \rangle$ for $g \in G, \alpha \in V^*, v \in V$ (here $\langle , \rangle$ denotes the canonical bilinear pairing between V^* and V.

If V, W are representations we can form their *direct sum* $V \oplus W$. The action of G on $V \oplus W$ is given by $g(v, w) = (gv, gw)$.

If V, W are representations we can form their *tensor product* $V \otimes W$. The action of G on $V \otimes W$ is given on elementary tensors by $g(v \otimes w) = gv \otimes gw$.

So we can "add" representations and "multiply" them. This suggests that the representations of G form a kind of ring. In fact, we can define a ring $Rep(G)$, called the representation (or Grothendieck) ring, as follows.

We start with the free $\mathbb{Z}$-module with basis given by the isomorphism classes $[V]$ of representations V of G. Then we quotient by the relation $[V \oplus W] = [V] + [W]$. Then we define multiplication by $[V][W] := [V \otimes W]$.

Example 1.1. Suppose that V is a 1-dimensional representation of G. Since linear operators on 1-dimensional vector spaces are just given by multiplicatin by scalars, V is determined by a (smooth) group homomorphism $\rho : G \to \mathbb{C}^\times$. So we will write V as $\mathbb{C}_\rho$.

Note that if ρ_1, ρ_2 are two such homomorphisms, then $\mathbb{C}_{\rho_1} \otimes \mathbb{C}_{\rho_2}$ is again a 1-dimensional representation and is given by the homomorphism $\rho_1\rho_2$ which is defined by $(\rho_1\rho_2)(g) = \rho_1(g)\rho_2(g)$.

1.2 Irreducibles and indecomposables

We have two different notions of simplest possible representations.

A representation which is not isomorphic to a direct sum is called *indecomposable*.

A *subrepresentation* $W \subset V$ is a subspace which is invariant for the action of all elements of G. A representation with no non-trivial subrepresentations is called *irreducible*.

Note that irreducible implies indecomposable, but not the other way around in general.

Example 1.2. Consider $G = \mathbb{Z}$. A representation of $\mathbb{Z}$ is the same thing as a vector space V along with an invertible linear operator $T : V \to V$ (the map $\mathbb{Z} \to GL(V)$ is $n \mapsto T^n$). A subrepresentation is just an invariant subspace for T. It is well-known from linear algebra that for non-diagonalizable T, not every invariant subspace has a complementary invariant subspace.

For example, consider

$$V = \mathbb{C}^2, \ T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Then $\{(x, 0) : x \in \mathbb{C}\}$ is the only invariant subspace, and hence it does not have a complementary invariant subspace.

We have Schur's Lemma.

Lemma 1.3. *If V, W are irreducible representations, then* $\mathrm{Hom}_G(V, W)$ *is 1-dimensional if $V \cong W$ and 0-dimensional otherwise.*

The following simple result is useful for constructing irreducible representations.

Proposition 1.4. *(i) If V is an irreducible representation, then so is V^*.*

(ii) If V is a 1-dimensional representation, then it is irreducible.

(iii) If V is an irreducible representation and W is a 1-dimensional representation, then $V \otimes W$ is an irreducible representation.

All parts follow immediately from the definitions.

In general, if V, W are irreducible representations, each of dimension bigger than 1, then $V \otimes W$ will almost never be irreducible.

1.3 Haar measure and semisimplicity

Let G be a compact connected Lie group.

Theorem 1.5. *There exists an invariant (under right and left multiplication) measure dg on G. This measure is unique once we demand that $\int_G dg = 1$.*

The existence of this measure (called the *Haar measure*) allows us to prove that the category of representations of G is semisimple.

We begin with invariant inner products. A G-invariant inner product on a representation V is a Hermitian inner product $\langle , \rangle$ on V (i.e. a positive definite sesquilinear form) such that $\langle gv, gw \rangle = \langle v, w \rangle$ for all $v, w \in V$ and $g \in G$.

Lemma 1.6. *Every representation V of G admits an G-invariant Hermitian inner product.*

Proof. Choose an Hermitian inner product $\langle , \rangle$ on V. We average this form with respect to the G action by defining $\langle , \rangle_{avg}$ by the formula

$$\langle v, w \rangle_{avg} = \int_G \langle gv, gw \rangle \, dg$$

Then the invariance of the Haar measure shows that $\langle , \rangle_{avg}$ is invariant. The integral of positive numbers is positive, so $\langle , \rangle_{avg}$ is positive definite. □

If G is a finite group, then the above Lemma goes through by replacing the integral with $\frac{1}{|G|} \sum_{g \in G}$. In fact, this is a kind of Haar measure on G, which is just a point measure.

For the remainder of this section, we assume that G is either connected compact or finite. (Of course, we could take G compact, which would cover both these cases, but it is psychologically better to think of them as two separate cases.)

Let V be a representation of G. A subspace $W \subset V$ is called an invariant subspace if $gW \subset W$ for all $g \in G$. An irreducible representation is one with no non-trivial invariant subspaces.

Proposition 1.7. *If W is an invariant subspace, then there exists another invariant subspace U such that $U \oplus W = V$.*

Proof. Choose a G-invariant inner product. Let $U = W^{\perp}$, where the orthogonal complement is taken with respect to the G-invariant inner product. Then by the invariance of the inner product, U is also an invariant subspace. By general results $U \oplus W = V$. □

This immediately implies the following result.

Theorem 1.8. *Every G-representation is equal to a direct sum of irreducible subrepresentations. Equivalently, a representation is indecomposable iff it is irreducible.*

In light of this theorem, we say that every G-representation is *completely reducible* and that the category of G-representations is *semisimple.* (A confusing point: this is not the same thing as G being semisimple.)

Example 1.9. This result certainly fails for non-compact groups. For example, if $G = \mathbb{Z}$, then the Example 1.2 gives an example of representation which is not isomorphic to a direct sum of irreducible representations.

Every G-representation cannot be canonically written as the direct sum of irreducible subrepresentations. However, it can be canonically be written as a direct sum of isotypic components.

Let V be a representation and let W be an irreducible representation. The *W-isotypic component* of V, denoted V_W, is the sum of all subrepresentations isomorphic to W. The following result gives the *isotypic decomposition* of V.

Theorem 1.10. *For each irreducible representation W, there is a canonical isomorphism of G-representations $V_W \cong W \otimes \mathrm{Hom}_G(W, V)$.*

We have

$$V = \bigoplus_W V_W$$

where W ranges over all isomorphism classes of irreducible representations.

Example 1.11. Take $V = \mathbb{C}^2$ with the trivial representation (G is arbitrary). Then we can write $\mathbb{C}^2$ as a direct sum of irreducible representations by picking any two 1-dimensional subspaces L_1, L_2. These lines L_1, L_2 will be subrepresentations and carry the trivial G-action. So $\mathbb{C}^2 = L_1 \oplus L_2$ is a decomposition of $\mathbb{C}^2$ into irreducible subrepresentations. However, this decomposition is of course not unique.

In the isotypic decomposition of $\mathbb{C}^2$, there is just one piece, namely $\mathbb{C}^2$ itself.

If W is a 1-dimensional representation, then V_W is very easy to understand. As remarked earlier, W determines a homomorphism $\rho : G \to \mathbb{C}^\times$ and then

$$V_W = \{v \in V : gv = \rho(g)v \text{ for all } g \in G\}.$$

If W is the trivial representation, then $V_W = V^G$ is the subspace of vectors invariant under G.

To prove Theorem 1.10, we begin with the following useful results.

Lemma 1.12. *Let W be an irreducible representation and let V be a vector space. Define a representation $W \otimes V$ where G acts on the W part. Then every subrepresentation of $W \otimes V$ is of the form $W \otimes U$ for some subspace $U \subset V$. In particular, every irreducible subrepresentation of $W \otimes V$ is isomorphic to W.*

Proof. Let $Y \subset W \otimes V$ be an invariant subspace. For each $\alpha \in V^*$, define a linear map

$$p_\alpha : Y \hookrightarrow W \otimes V \xrightarrow{id \otimes \alpha} W$$

Since W is irreducible, $p_\alpha(Y) = 0$ or $p_\alpha(Y) = W$ for all α.

Then let $A = \{\alpha : p_\alpha(Y) = 0\} \subset V^*$. We leave it to the reader to verify that $Y = W \otimes A^\perp$. $\square$

Proof of Theorem 1.10. We start with the first part.

First, we consider $W \otimes \mathrm{Hom}_G(W, V)$ a G-representation where G acts trivially on $\mathrm{Hom}_G(W, V)$. We define a linear map

$$\Psi : W \otimes \mathrm{Hom}_G(W, V) \to V, \text{ by } w \otimes \phi \mapsto \phi(w).$$

This linear map is easily seen to be G-equivariant, so it is a morphism of G-representations.

By definition, $\operatorname{im} \Psi = V_W$. Now we claim that Ψ is injective and hence an isomorphism onto V_W.

The kernel K of Ψ is a subrepresentation of $W \otimes \mathrm{Hom}_G(W, V)$. Applying Lemma 1.12, K is $W \otimes U$ for some $U \subset \mathrm{Hom}_G(W, V)$. But if $\phi \in \mathrm{Hom}_G(W, V)$ and $W \otimes \phi$ lies in the kernel of Ψ, then $\phi(w) = 0$ for all $w \in W$. This means that $\phi = 0$. Hence we conclude that $K = 0$ and so this map is injective.

Now, to get the direct sum decomposition, we note that by Theorem 1.8, the sum of the subspaces V_W equals V.

We just need to show that we have a direct sum. For this, let $W_1, \ldots, W_k$ be pairwise non-isomorphic irreducible representations and suppose that we already know that $V_{W_1}, \ldots, V_{W_{k-1}}$ give a direct sum. Suppose that

$$W_k \cap V_{W_1} \oplus \cdots V_{W_{k-1}} \neq 0.$$

Then let U be an irreducible subrepresentation of this intersection. Applying the first part and Lemma 1.12, we see that $U \cong W_k$. On the other hand for each $i = 1, \ldots, k-1$, we can take the projection $\pi_i(U) \subset V_{W_i}$ given by the direct sum. Not all these projections can be 0, so let us assume that $\pi_i(U) \neq 0$. Then $U \cong \pi_i(U)$, since U is irreducible. On the other hand, $\pi_i(U)$ is an irreducible subrepresentation of V_{W_i} and so we conclude that $U \cong W_i$. Thus $W_k \cong W_i$, a contradiction. Thus we conclude that we have a direct sum. $\square$

Let V a representation and consider the isotypic decomposition

$$V \cong \bigoplus_W W \otimes \mathrm{Hom}_G(W, V)$$

In the representation ring $Rep(G)$, we see that $[V] = \sum_W \dim \mathrm{Hom}_G(W, V)[W]$. From this we see that the isomorphism classes $[W]$ of irreducible representations forms a $\mathbb{Z}$-basis for $Rep(G)$.

2 Characters

2.1 Definition and basic properties

If V is a representation of G, then we define a smooth function $\chi_V : G \to \mathbb{C}$ called the *character* of V by $\chi_V(g) = \mathrm{tr}(g|_V)$, where tr denotes trace.

These characters are remarkably useful.

Let us investigate the behaviour of our characters under natural operators on representations.

Proposition 2.1. *(i) If V is a representation, then $\chi_{V^*}(g) = \overline{\chi_V(g)}$.*

(ii) If V, W are representations, then $\chi_{V \oplus W} = \chi_V + \chi_W$.

(iii) If V, W are representations, then $\chi_{V \otimes W} = \chi_V \chi_W$.

Proof. For (i), note that $\chi_{V^*}(g) = tr(g^{-1}|_V)$. Now V admits a G-invariant inner product and with respect to that inner product, G acts by unitary operators. Hence $tr(g^{-1}|_V) = \overline{tr(g)|_V}$.

Parts (ii), (iii) follow from standard properties of trace and can be proved by choosing bases. $\square$

Hence the map

$$Rep(G) \to C(G) \quad [V] \mapsto \chi_V$$

is a ring homomorphism.

2.2 Characters and Hom spaces

We begin with the following lemma.

Lemma 2.2. $\dim V^G = \int_G \chi_V(g)\, dg$

Proof. Define a linear operator $A : V \to V$ by $A(v) = \int_G gv\, dg$. A is well-defined since G is compact, matrix coefficients are continuous, and continuous functions on compact sets are integrable.

Note that $A^2 = A$, so that A is a projection operator onto $\{v : Av = v\}$. We claim that this subspace is precisely V^G. Clearly V^G is contained in this subspace.

Suppose that $Av = v$. Applying h to both sides and bringing it under the integral, we obtain

$$\int_G hgv\, dg = hv$$

Using the invariance of the measure, we obtain that $\int_G gv\, dg = hv$ which implies that $v = hv$, as desired.

Hence, we conclude that A is a projection onto its 1-eigenspace V^G (actually, it is easy to see that A is G-equivariant, so it must be the projection given by decomposition into isotypic components).

Since $\int_G \chi_V(g) = tr(A)$, the result follows. □

Characters live inside the space $C(G)$ of continuous functions on G, which embeds into the Hilbert space $L^2(G)$ of square integrable complex valued functions on G. On this Hilbert space, we have a inner product

$$\langle f_1, f_2 \rangle := \int_G \overline{f_1(g)} f_2(g)\, dg.$$

Lemma 2.2 leads to the following useful interpretation of the inner product on characters.

Proposition 2.3. *Let V, W be representations. Then* $\langle \chi_V, \chi_W \rangle = \dim \operatorname{Hom}_G(V, W)$.

Proof. First note that $\operatorname{Hom}(V, W)$ is a representation of G where G acts by $(g\phi)(v) = g\phi(g^{-1}v)$. The space of invariants for this action is $\operatorname{Hom}_G(V, W)$. As a representation of G, we have an isomorphism $\operatorname{Hom}(V, W) \cong V^* \otimes W$.

By Proposition 2.1, $\chi_{V^*\otimes W} = \overline{\chi_V}\chi_W$. Thus applying Lemma 2.2, we see that

$$\dim \operatorname{Hom}_G(V, W) = \dim(V^* \otimes W)^G = \int_G \overline{\chi_V(g)} \chi_W(g)\, dg$$

as desired. □

Theorem 2.4. *(i) The characters of irreducible representations are orthonormal.*

(ii) If V, W are representations, then $V \cong W$ iff $\chi_V = \chi_W$. In words, two representations are isomorphic if and only if they have the same character.

Proof. Part (i) follows immediately from Proposition 2.3 and Schur's Lemma.

For (ii), note that $V \cong W$ iff they have the same size isotypic components. So they are isomorphic iff $\dim \mathrm{Hom}(U, V) = \dim \mathrm{Hom}(U, W)$ for all irreducible representations U. By Proposition 2.3, the size of these hom spaces is determined by the characters. $\square$

Hence the ring homomorphism $Rep(G) \to C(G)$ given by taking characters is injective.

2.3 Characters as class functions

A *class function* on a group G is a function $f : G \to \mathbb{C}$ which is constant on conjugacy classes. In other words, $f(hgh^{-1}) = f(g)$ for all $g, h \in G$. Because trace of a linear operator is invariant under conjugation, we immediately obtain the following.

Lemma 2.5. *A character is a class function.*

Thus, characters live inside the subspace $C(G)^G$ consisting of continuous class functions.

Now suppose that G is finite. Then $C(G)$ is the same thing as the space of all functions on G. It is a vector space of dimension $|G|$. The inner product on this space is given by

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

Consider the subspace $C(G)^G$ of class functions. It has a basis given by the characteristic functions of conjugacy classes. Thus it has dimension equal to the number of conjugacy classes.

Since the characters of irreducible representations are linearly independent and all of them live in $C(G)^G$, we immediately see that the number of irreducible representations of a finite group is less than or equal to the number of conjugacy classes in G. Actually there is equality.

Theorem 2.6. *If G is finite, the characters of irreducible representations form an orthonormal basis for $C(G)^G$ and thus the number of irreducible representations is equal to the number of conjugacy classes of G.*

Before we proceed with the proof, let us define the regular representation of a finite group to be its action on $\mathbb{C}[G]$ by left multiplication. More precisely, if $g \in G$ and $\sum_{h \in G} a_h h \in \mathbb{C}[G]$, then the action is given by

$$g(\sum_{h \in G} a_h h) = \sum_{h \in G} a_h gh$$

Note that $\mathbb{C}[G]$ has a basis given by the group elements of G and this action is just a permutation action with respect to this basis. So it is easy to see that under this representation G is taken to a linearly independent set in $End(\mathbb{C}[G])$. This representation is called the regular representation.

Proof. Suppose that the characters of the irreducible representations do not span $C(G)^G$. Then we can find $f \in C(G)^G$, $f \neq 0$, such that $\langle \chi_V, f \rangle = 0$ for all irreducible representation V.

Fix some irreducible representation V. Let A be the linear operator on V defined by

$$v \mapsto \sum f(g) gv.$$

Then since f is a class function, it is easy to see that A is G-equivariant, so by Schur's lemma, A acts on V by some scalar a. Now

$$a \dim V = \operatorname{tr}(A) = \sum_{g \in G} f(g) tr(g|_V) = \sum_{g \in G} f(g) \chi_V(g) = |G| \langle \chi_{V^*}, f \rangle = 0$$

So $a = 0$. Thus $\sum f(g) g$ acts by 0 in every irreducible representation and hence in every representation of G and in particular in the regular representation.

However, in the regular representation, the group elements give linearly independent endomorphisms. Thus $f(g) = 0$ for all g. So we see that the characters do span after all. $\square$

We can rephrase this result as saying that the map $Rep(G) \otimes_{\mathbb{Z}} \mathbb{C} \to C(G)^G$ is a ring isomorphism (we need to extend scalars from $\mathbb{Z}$ to $\mathbb{C}$, since $Rep(G)$ is only a $\mathbb{Z}$-module).

Eventually, we will prove an analog of this result for compact groups.

2.4 Characters of S_3

Consider $G = S_3$. This is the simplest non-abelian group. In symmetric groups, conjugacy classes are given by cycle type, so S_3 has three conjugacy classes:

$$\{(1)\}, \{(123), (132)\}, \{(12), (23), (13)\}$$

Hence it has three irreducible representations. Let us try to find them.

Two easy 1-dimensional representations are the trivial representation (each group element goes to 1) and the sign representation, which takes even permutations to 1 and odd permutations to -1. It is convenient to represent the characters in a table like this.

	χ_{triv}	χ_{sign}
(1)	1	1
(12)	1	-1
(123)	1	1

We need one more irreducible representation. One obvious representation of S_3 is its action on $\mathbb{C}^3$ by permuting the coordinates, so $\sigma(x_1, x_2, x_3) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ for $\sigma \in S_3$ and $(x_1, x_2, x_3) \in \mathbb{C}^3$. It is easy to see that the character of this representation is given as follows.

$$\chi_{\mathbb{C}^3}((1)) = 3,\ \chi_{\mathbb{C}^3}((123)) = 0,\ \chi_{\mathbb{C}^3}((12)) = 1$$

We can see that this is not an irreducible representation, since $\langle \chi_{\mathbb{C}^3}, \chi_{\mathbb{C}^3} \rangle = 2$. Also we can compute $\langle \chi_{\mathbb{C}^3}, \chi_{triv} \rangle = 1$ to see that $\mathrm{Hom}(\mathbb{C}_{triv}, \mathbb{C}^3)$ is one dimensional and hence $\mathbb{C}^3$ contains one copy of the trivial representation.

Actually this is easy to see, since the subspace $\{(x, x, x) : x \in \mathbb{C}\} \subset \mathbb{C}^3$ is clearly an invariant subspace isomorphic to the trivial representation. We can choose a complementary invariant subspace $W = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$.

Since $\mathbb{C}^3 = \mathbb{C}_{triv} \oplus W$, we see that $\chi_W = \chi_{\mathbb{C}^3} - \chi_{triv}$. Thus χ_W is given as follows.

$$\chi_W((1)) = 2,\ \chi_W((123)) = -1,\ \chi_W((12)) = 0$$

From this, we see that $\langle \chi_W, \chi_W \rangle = 1$ and hence W is an irreducible representation.

Thus, the three irreducible representations are the trivial, the sign, and W. The complete character table is:

	χ_{triv}	χ_{sign}	χ_W
(1)	1	1	2
(12)	1	-1	0
(123)	1	1	-1

3 Tori

3.1 Compact tori

A connected compact abelian group is called a *torus*. The simplest example of a torus is $U(1) = S^1$. More generally, every torus is isomorphic to a product of copies of $U(1)$. Let us formulate a more precise version of this statement.

Let T be a torus. Let $\mathfrak{t}$ denote its Lie algebra. There is an exponential map $\mathfrak{t} \to T$. Let Λ denote the kernel of the exponential map.

Proposition 3.1. *The exponential map is a group homomorphism and is surjective. Hence $T \cong \mathfrak{t}/\Lambda$.*

The group Λ is a free abelian group whose rank is equal to the dimension of T (and of $\mathfrak{t}$). We can choose isomorphisms $\mathfrak{t} \cong \mathbb{R}^n, \Lambda \cong \mathbb{Z}^n$, and $T \cong \mathbb{R}^n/\mathbb{Z}^n = U(1)^n$.

Even though all tori are isomorphic to $U(1)^n$, it is usually convenient not to choose such an identification (much like not picking a basis for a vector space).

Example 3.2. Let T be the diagonal unitary matrices of determinant 1. This torus is important because it is a maximal torus of $SU(n)$. From the definition,

$$T = \{(t_1, \dots, t_n) \in U(1)^n : t_1 \dots t_n = 1\}.$$

From this description, we can write

$$\mathfrak{t} = \{(a_1, \dots, a_n) \in \mathbb{R}^n : \sum a_i = 0\} \text{ and } \Lambda = \{(m_1, \dots, m_n) \in \mathbb{Z}^n : \sum a_i = 0\}.$$

Tori are very useful since they are generated by one element in the following sense. We say that $t \in T$ is a *topological generator* of T if the subgroup generated by t, $\{t^n : n \in \mathbb{Z}\}$, is dense in T.

Lemma 3.3. *Every torus has a topological generator.*

3.2 Representations of Tori

Let T be a torus. A *weight* of T is a smooth group homomorphism from T to $U(1)$ and a *coweight* is a smooth group homomorphism from $U(1)$ to T.

A weight of T is the same thing as a 1-dimensional representation of T because every smooth group homomorphism $T \to \mathbb{C}^\times$ must land in $U(1)$, as $U(1)$ is the only non-trivial connected compact subgroup of $\mathbb{C}^\times$.

The set of all (co)weights is called the *(co)weight lattice* and is denoted $X^*(T)$ (resp. $X_*(T)$). The weight and coweight lattice form free abelian groups. There is a perfect pairing defined by composition

$$\langle , \rangle : X^*(T) \otimes_{\mathbb{Z}} X_*(T) \to \mathbb{Z} = X^*(U(1))$$

If $T = U(1)^n$, then any weight is of the form $(t_1, \ldots, t_n) \mapsto t_1^{\mu_1} \ldots t_n^{\mu_n}$ for some $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$. This shows that $X^*(T) = \mathbb{Z}^n$.

In general, $X_*(T)$ is always naturally isomorphic to Λ. This is because given $\mu \in \Lambda$ we can define a map $U(1) = \mathbb{R}/\mathbb{Z} \to T = \mathfrak{t}/\Lambda$ by $[a] \mapsto [a\mu]$. Every group homomorphism $U(1) \to T$ is of this form. This can be seen by considering the map at the level of Lie algebras.

Example 3.4. Let T be the torus from Example 3.2. Then we have $X^*(T) = \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$ because $\mu = (\mu_1, \ldots, \mu_n)$ gives rise to a homomorphism

$$T \to U(1), \quad (t_1, \ldots, t_n) \mapsto t_1^{\mu_1} \cdots t_n^{\mu_n}$$

and this map is trivial if $(\mu_1, \ldots, \mu_n) \in \mathbb{Z}(1, \ldots, 1)$.

Similarly, $X_*(T) = \{(\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n : \mu_1 + \cdots + \mu_n = 0\}$. Such a $(\mu_1, \ldots, \mu_n)$ gives rise to the map

$$U(1) \to T, \quad t \mapsto (t^{\mu_1}, \ldots, t^{\mu_n})$$

Using Schur's Lemma it is easy to see the following.

Proposition 3.5. *Every irreducible representation of an abelian group is 1-dimensional.*

Thus all irreducible representation of a torus T are one-dimensional and they correspond to the set of weights $X^*(T)$.

Hence by the isotypic decomposition, for any representation V of T, we can write V as the direct sum of subspaces

$$V = \bigoplus_{\mu} V_\mu, \text{ where } V_\mu = \{v \in V : tv = \mu(t)v \text{ for all } t \in T\}$$

This is called the weight decomposition — the V_μ are called the weight spaces. A vector $v \in V_\mu$ is a called a weight vector of weight μ.

The character of the 1-dimensional representation given by a weight μ is just μ itself, regarded now as a function on T with values in $\mathbb{C}$. To distinguish the character from the weight, we will write e^μ for the character.

Note that if V is an arbitrary representation of T, then using the above weight decomposition, we see that

$$\chi_V = \sum_{\mu \in X^*(T)} (\dim V_\mu) e^\mu$$

We regard these characters as living in the $\mathbb{Z}$-group algebra of the weight lattice, $\mathbb{Z}[X^*(T)]$ which is a subring of $C(T)$.

Combining these results, we see that

$$Rep(T) \to \mathbb{Z}[X^*(T)] \quad [V] \mapsto \chi_V$$

is a ring isomorphism.

3.3 Complexification of tori

Given a torus T, we will define its complexification $T_\mathbb{C}$. This is a special case of the complexification of compact groups.

The simplest example of this complexification is $U(1)_\mathbb{C} = \mathbb{C}^\times$. Of course, in a sense this is the only example since every torus is a product of copies of $U(1)$. However, we will do things in a more invariant way.

Let us begin by describing what kind of an object a complex torus is.

3.3.1 Complex Lie groups and complex algebraic groups

We will now enter into the world of complex Lie groups and complex algebraic groups.

An n-dimensional *complex manifold* X is a topological space X along with an open cover $\{U_i\}$ and homeomorphisms $U_i \to V_i$, where V_i are open subsets of $\mathbb{C}^n$. We require that the transition functions between these charts are given by holomorphic functions between the open subsets of $\mathbb{C}^n$.

A *complex Lie group* is a group in the category of complex manifolds. By this we mean that it is a complex manifold G along with multiplication maps $G \times G \to G$ and inverse maps $G \to G$ which are holomorphic and make G into a group.

Example 3.6. One simple example of a complex Lie group is $\mathbb{C}^\times$, under multiplication (it has a very simple atlas, since it is already an open subset of $\mathbb{C}$).

Another example is $\mathbb{C}/\mathbb{Z}$, where the group structure is given by addition in $\mathbb{C}$ and the atlas comes by choosing open sets in $\mathbb{C}$ which do not contain two points whose difference is an integer. Note that the map $z \mapsto e^{2\pi i z}$ defines an isomorphism of complex Lie groups between $\mathbb{C}^\times$ and $\mathbb{C}/\mathbb{Z}$.

Another example of a complex Lie group is $\mathbb{C}/\mathbb{Z}^2$. The underlying real manifold is $S^1 \times S^1$. For this reason, $\mathbb{C}/\mathbb{Z}^2$ is sometimes called a complex torus, which is the word which we are reserving for $\mathbb{C}^\times$ and its relatives. In this course, we will not be interested in examples of complex Lie groups whose underlying topological space is compact.

An *affine complex algebraic variety* is a subset X of $\mathbb{C}^n$ which is defined by polynomial equations. Given X, we can consider the ideal $I(X)$ of all polynomials vanishing on X and then form the quotient $\mathcal{O}(X) = \mathbb{C}[x_1, \dots, x_n]/I(X)$, which is the algebra of polynomial functions on X (also called the *coordinate ring* of X).

Conversely, given a finitely generated reduced $\mathbb{C}$-algebra R, we define $\operatorname{Spec} R$ to be the set of $\mathbb{C}$-algebra homomorphisms from R to $\mathbb{C}$. Given an affine complex algebraic variety X, there is a map $X \to \operatorname{Spec} \mathcal{O}(X)$ taking x to the homomorphism given by evaluation at x, $f \mapsto f(x)$. A basic result in algebraic geometry, called Hilbert's nullstellensatz, tells us that this map is a bijection.

A *complex algebraic group* is a group in the category of affine complex algebraic varieties.

Example 3.7. Consider $SL_n(\mathbb{C})$, the group of determinant 1, $n \times n$, complex matrices.

It is an algebraic variety, since it is the subvariety of $\mathbb{C}^{n^2}$ (the matrix entries), given by the polynomial $\det -1$. Note that det is a polynomial in the entries of a matrix.

The group structure on $SL_n(\mathbb{C})$ is given by matrix multiplication which is clearly polynomial in the entries of the matrices. The inverse map $g \mapsto g^{-1}$ is also given by a polynomial in the matrix entries, but this is a bit less obvious.

Every complex algebraic group has an underlying complex Lie group structure, because every smooth complex algebraic variety carries the structure of a complex manifold.

The tangent space at the identity to a complex Lie group or to a complex algebraic group is a complex Lie algebra. The definition of the Lie algebra structure is the definition using left-invariant vector fields, which makes perfect sense for complex Lie groups or complex algebraic groups.

3.3.2 Algebraic varieties and localization

There is one special way to produce algebraic varieties which will be useful for us. If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is a non-constant function, then we can consider its non-vanishing locus $U_f = \{a \in \mathbb{C}^n : f(a) \neq 0\}$. This set naturally carries the structure of an algebraic variety because it is in bijection with

$$\{\big(a, \frac{1}{f(a)}\big) : a \in U_f\} \subset \mathbb{C}^{n+1},$$

which is an algebraic variety because it is defined by the equation $x_{n+1}f(x_1, \ldots, x_n) = 1$.

Note that under this bijection, the coordinate ring of U_f is the quotient of $\mathbb{C}[x_1, \ldots, x_{n+1}]$ by the ideal generated by $x_{n+1}f - 1$. We call this ring, the localization of $\mathbb{C}[x_1, \ldots, x_n]$ at f and think about it as adjoining an inverse for f.

$$O(U_f) = \mathbb{C}[x_1, \ldots, x_{n+1}]/(x_{n+1}f - 1) = \mathbb{C}[x_1, \ldots, x_n][f^{-1}]$$

Example 3.8. Take $G = GL_n = \{A \in \mathrm{End}(\mathbb{C}^n) : A \text{ is invertible }\}$. Note that $\mathrm{End}(\mathbb{C}^n)$ can be identified with $\mathbb{C}^{n^2}$ (by matrix entries) and that A is invertible if and only if $\det(A) \neq 0$. Note that det is a polynomial function of the matrix entries. Thus GL_n is an algebraic group and $\mathcal{O}(GL_n) = \mathbb{C}[x_{ij}, det^{-1}]_{1 \leq i,j \leq n}$.

The simplest example of this is $G = \mathbb{C}^\times$, from which we see that $\mathcal{O}(\mathbb{C}^\times) = \mathbb{C}[x, x^{-1}]$. Some authors write $\mathbb{G}_m$ (the *multiplicative group*) for $\mathbb{C}^\times$ when they think of it as an algebraic group.

3.3.3 Complex tori

Given a torus T, we define its *complexification* $T_\mathbb{C}$ in two ways.

First, $T_\mathbb{C} = \mathfrak{t}_\mathbb{C}/\Lambda$ as a complex Lie group (where $\mathfrak{t}_\mathbb{C} = \mathfrak{t} \otimes_\mathbb{R} \mathbb{C}$). In this approach, the group structure comes from addition in $\mathfrak{t}_\mathbb{C}$.

Second, as a complex algebraic group, we define $T_\mathbb{C}$ as $\operatorname{Spec} \mathbb{C}[X^*(T)]$. So the coordinate ring of $T_\mathbb{C}$ is the complexified group algebra of the weight lattice of T. In this approach, the group structure is slightly more mysterious. It is defined by the equation $\mu(t_1t_2) = \mu(t_1)\mu(t_2)$ for $\mu \in X^*(T)$ and $t_1, t_2 \in T_\mathbb{C}$. This defines the element t_1t_2, because from the algebraic geometry perspective, it is enough to know value of every polynomial function on t_1t_2.

The underlying complex Lie group of the complex algebraic group $T_\mathbb{C}$ is naturally isomorphic to the complex Lie group $T_\mathbb{C}$. To see this, it is enough to define a map $\mathbb{C}$-algebras between $\mathbb{C}[X^*(T)]$ and the algebra of holomorphic functions on $\mathfrak{t}_{\mathbb{C}}/\Lambda$. We do so by assigning to each weight μ the function (also denoted μ) defined by

$$\mu([a]) = e^{2\pi i \mu(a)}$$

where $\mu(a)$ is defined using the emedding $X^*(T) \subset \mathfrak{t}^*$.

$T_{\mathbb{C}}$ is a *complex torus*, which by definition means a complex Lie/algebraic group which is isomorphic to ${\mathbb{C}^{\times}}^n$ for some n.

Example 3.9. (i) Let us take $T = U(1)$, so then $X^*(T) = \Lambda = \mathbb{Z}$ and $\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}$. Thus as a complex Lie group, $T_{\mathbb{C}} = \mathbb{C}/\mathbb{Z}$. On the other hand, using the coordinate ring of the complex algebraic group $T_{\mathbb{C}}$ is given by $\mathcal{O}(T_{\mathbb{C}}) = \mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, x^{-1}]$ and so $T_{\mathbb{C}} = \mathbb{C}^{\times}$. We have already seen that $\mathbb{C}/\mathbb{Z}$ and $\mathbb{C}^{\times}$ are isomorphic as complex Lie groups.

(ii) As a second example, let us take diagonal unitary matrices of determinant 1, from 3.2. Recall that $\mathfrak{t} = \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \sum a_i = 0\}$ and $\Lambda = \{(m_1, \ldots, m_n) : \sum m_i = 0\}$.

Then
$$T_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}/\Lambda = \{(a_1, \ldots, a_n) \in \mathbb{C}^n : \sum a_i = 0\}/\Lambda.$$

Exponentiating each coordinate gives us an isomorphism of complex Lie groups between $T_{\mathbb{C}}$ and

$$\{(t_1, \ldots, t_n) \in (\mathbb{C}^{\times})^n : t_1 \ldots t_n = 1\}$$

On the other hand $\mathbb{C}[X^*(T)] = \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]/(x_1 \ldots x_n)$. This is readily seen to be the algebra of polynomial functions on the above group.

3.4 Representation theory of complex tori

An *algebraic representation* of a complex algebraic group is vector space V and a map of algebraic groups $G \to GL(V)$. This means that the map should be a group homomorphism and also a map of algebraic varieties (i.e. it should be given by polynomials). In particular this means that every matrix coefficient of a representation is a polynomial function on G.

Sometimes people say "rational representation" instead of "algebraic representation". The reason why they say "rational representation" rather than "polynomial representation" is that for a group like GL_n, which is defined by localization, the inverse of the localized function is polynomial function on G. For example the map $g \mapsto \det(g)^{-1}$ is a (1-dimensional) algebraic representation of GL_n.

Example 3.10. The simplest example of a non-algebraic representation is to take $G = \mathbb{C}^\times$ and consider the 1-dimensional representation $\mathbb{C}^\times \to \mathbb{C}^\times$ given by $z \to \overline{z}$. Since complex conjugation is not a polynomial, this is not an algebraic representation (it is however an algebraic representation of the underlying real algebraic group).

From now on, when we talk about a representation of an algebraic group, we will always mean an algebraic representation.

The representation theory of complex tori is very much as that of compact tori. Let $T_\mathbb{C}$ be a complex torus, the complexification of a compact torus T. Given a weight $\mu \in X^*(T)$, we can construct a map $T \to \mathbb{C}^\times$ in one of the following two equivalent ways (depending on which definition of $T_\mathbb{C}$ we take).

In the first approach, we think of $T_\mathbb{C} = \mathfrak{t}_\mathbb{C}/\Lambda$, and we write $\mu \in \mathrm{Hom}(\Lambda, \mathbb{Z})$ (using the identification of $X_*(T)$ and Λ and the pairing between $X_*(T)$ and $X^*(T)$). Then we define

$$T_\mathbb{C} \to \mathbb{C}^\times, \text{ by } [a] \mapsto e^{2\pi i \mu(a)}.$$

This is a well-defined map since if $a \in \Lambda$, then $\mu(a)$ is an integer and so $e^{2\pi i\mu(a)} = 1$. (This the same function on $T_\mathbb{C}$ we defined in section 3.3.3 when we were showing that weights give holomorphic functions on $T_\mathbb{C}$).

In the second approach, we just note that since we defined $\mathcal{O}(T_\mathbb{C}) = \mathbb{C}[X^*(T)]$, μ is an invertible element of $\mathcal{O}(T_\mathbb{C})$ and hence defines a map from $T_\mathbb{C}$ to $\mathbb{C}^\times$. This map is a group homomorphism by the definition of the group structure on $T_\mathbb{C}$.

Actually, in practice, we don't really think of it in either of these two ways.

Example 3.11. Return to our favourite torus $T_\mathbb{C} = \{(t_1, \dots, t_n) \in {\mathbb{C}^\times}^n : t_1 \dots t_n = 1\}$. Recall that the weight lattice is $X^* = \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1)$. Given $\mu = (\mu_1, \dots, \mu_n) \in X^*$, we get a group homomorphism by $(t_1, \dots, t_n) \mapsto t_1^{\mu_1} \dots t_n^{\mu_n}$ as before.

Lemma 3.12. *The above construction gives an isomorphism of abelian groups $X^*(T) \cong \mathrm{Hom}(T_\mathbb{C}, \mathbb{C}^\times)$. Similarly, we have an isomorphism $X_*(T) \cong \mathrm{Hom}(\mathbb{C}^\times, T_\mathbb{C})$. Here* Hom *denotes either as complex Lie groups or as complex algebraic groups (the homomorphisms are the same).*

Because of these isomorphisms, we will write $X^*(T_{\mathbb{C}}) = X^*(T)$ and $X_*(T_{\mathbb{C}}) = X_*(T)$.

4 Maximal tori and characters

For this section, fix a compact connected Lie group G.

4.1 Maximal tori

A *maximal torus* T in a compact group G is a Lie subgroup $T \subset G$ which is a torus and which is not contained in another torus.

We have the following fundamental result. Fix a maximal torus $T \subset G$.

Theorem 4.1. *Every element of G is conjugate into T.*

Example 4.2. Take $G = U(n)$. Then a maximal torus T for G is given by the diagonal unitary matrices. Then Theorem 4.1 is equivalent in this case to the statement that all unitary matrices are unitarily diagonalizable.

Using the existence of topological generators for tori, we immediately conclude the following.

Corollary 4.3. *Every torus in G is conjugate into T. In particular, all maximal tori are conjugate.*

In other words, if T' is another maximal torus in G, then there exists $g \in G$, such that $gT'g^{-1} = T$.

From this corollary, we conclude that all maximal tori have the same dimension. The dimension of the maximal torus is called the *rank* of G and is denoted ℓ.

Another corollary of Theorem 4.1 will be quite useful to us. It is also proven using topological generators for tori.

Corollary 4.4. *If T is a maximal torus, then*

$$T = Z_G(T) := \{g \in G : gtg^{-1} = t, \text{ for all } t \in T\}.$$

4.2 Restriction of representations to maximal tori

For us, the main point of maximal tori is that we can use them to study representations of G. We fix a maximal torus T and write $X = X^*(T)$ for its weight lattice.

Let V be a representation of G. We can regard V as a representation of T and then form its weight decomposition.

$$V = \bigoplus_{\mu \in X} V_\mu$$

and we can consider its character χ_V^T as a T-representation. As we discussed above $\chi_V^T = \sum_{\mu \in X} (\dim V_\mu) e^\mu$.

Theorem 4.5. *Let V, W be representations of G. Then $V \cong W$ as G-representations iff $\chi_V^T = \chi_W^T$. Equivalently, $V \cong W$ iff $\dim V_\mu = \dim W_\mu$ for all μ.*

Proof. Suppose that $\chi_V^T = \chi_W^T$. This means that $\chi_V(t) = \chi_W(t)$ for all $t \in T$. But since the character of a representation is a class function, this implies that $\chi_V(gtg^{-1}) = \chi_W(gtg^{-1})$ for all $g \in G$. Every element of G is of this form by Theorem 4.1. Hence $\chi_V = \chi_W$ and so $V \cong W$ by Theorem 2.4. □

Yet another equivalent formulation of Theorem 4.5 is to say that two representations of G are isomorphic iff their restriction to T is isomorphic. Put another way, restriction gives us a injective ring homomorphism $Rep(G) \to Rep(T)$.

Since the character of V is completely determined by its restriction to T, which is in turn determined by the dimension of the weight spaces, from now on, if V is a representation of G, then we will think of its character as

$$\chi_V = \sum_{\mu \in X^*(T)} (\dim V_\mu) e^\mu$$

which we regard as a element of $\mathbb{Z}[X]$.

4.3 The Weyl group

Let $N_G(T)$ denote the normalizer of T in G. So

$$N_G(T) = \{g \in G : gtg^{-1} \in T, \text{ for all } t \in T\}.$$

Inside $N_G(T)$ we have T which is a normal subgroup. We define the *Weyl group* W by $W = N_G(T)/T$. Note that $N_G(T)$ acts on T by conjugation and T acts trivially, so W acts on T. In fact, by Lemma 4.4, we see that this is a faithful action (i.e. no non-identity element of W acts trivially on T).

Example 4.6. Take $G = U(n)$ and T to be the diagonal matrices. There is a map $S_n \to N_G(T)$ taking a permutation w to a permutation matrix P_w. We claim this this map induces an isomorphism $S_n \to N_G(T)/T$.

It is easy to see that the map is injective, since every non-identity permutation matrix acts non-trivially on T by conjugation.

To see that it is surjective, let $g \in N_G(T)$. Then we can rephrase the condition of lying in the normalizer as saying that for all $t \in T$, there exists $t' \in T$ such that $tg = gt'$. Since tg amounts to multiplying the rows by (non-zero) scalars and gt' amounts to multiplying the columns by (non-zero) scalars, we see that g can only have one non-zero entry in each row and coloumn. From this it follows that $g \in S_n T$.

Hence $S_n \to W$ is an isomorphism.

Lemma 4.7. *The Weyl group is finite.*

Before proceeding with this lemma, let us note the following general fact.

Proposition 4.8. *If G is a Lie group and H is a closed Lie subgroup, then G/H naturally carries the structure of a manifold of dimension $\dim G - \dim H$. If H is a normal closed Lie subgroup, then G/H is a Lie group. If G is compact, then G/H is compact.*

Proof of Lemma 4.7. Note that $N_G(T)$ is a closed Lie subgroup, since it is a subgroup which is topogically closed (conjugating T to T is a closed condition). So from the Proposition, we see that W is compact. Hence to show that W is finite, it suffices to show that it is discrete.

Consider the action of W on T. Since it acts on T, it acts linearly on $\mathfrak{t}$ and preserves the lattice Λ. Hence we get a group homomorphism $W \to GL(\Lambda)$ where $GL(\Lambda)$ denotes the $\mathbb{Z}$-linear automorphisms of Λ (this is the same thing as the linear operators on $\mathfrak{t}$ which take Λ isomorphically to Λ).

Since $GL(\Lambda)$ is discrete, the connected component of the identity in W must be taken to the identity in $GL(\Lambda)$. Thus if $w \in W$ is in the connected component of the identity, it acts trivially on Λ. Since it acts trivially on Λ and Λ spans $\mathfrak{t}$ (over $\mathbb{R}$), w acts trivially on $\mathfrak{t}$ and hence it acts trivially on $T = \mathfrak{t}/\Lambda$.

Thus by the remarks at the beginning of this section, we see that w is the identity element of W and thus W is discrete. ☐

Because W acts on T, it acts $\mathbb{Z}$-linearly on the weight and coweight lattices $X_*(T)$ and $X^*(T)$. The argument in the proof of the above Proposition shows that W acts faithfully on the lattice Λ. Since Λ is isomorphic to

$X_*(T)$ and is dual to $X^*(T)$, W acts faithfully on the weight and coweight lattices as well.

Example 4.9. For $G = U(n)$, the action of S_n on $X^*(T) = \mathbb{Z}^n$ is given by

$$w(\mu_1, \ldots, \mu_n) = (\mu_{w(1)}, \ldots, \mu_{w(n)})$$

4.4 The Weyl group and characters

As before, let G be a compact Lie group, let T be its maximal torus, and let X be its weight lattice. Let W be the Weyl group and let $\mathbb{Z}[X]^W$ denote the subalgebra of Weyl invariants. So an element of $\mathbb{Z}[X]^W$ is a linear combination $\sum_{\mu \in X} a_\mu e^\mu$, with $a_{w\mu} = a_\mu$ for all $\mu \in X, w \in W$.

We begin with the following relatively straightforward observation.

Proposition 4.10. *If V is a representation of G, then $\chi_V \in \mathbb{Z}[X]^W$.*

Proof. We have already observed that χ_V is a class function on G (when regarded as a function on G). Thus when we restrict χ_V to T, it must be invariant under the action of $N_G(T)$. This immediately implies the desired result. □

Here is another way to think about this proof. For each $w \in W$, we pick some $g \in N_G(T) \subset G$ such that $[g] = w$. Let $v \in V_\mu$ be a weight vector of weight μ. Then because $g \in N_G(T)$, $g^{-1}tg \in T$ and we have

$$tgv = gg^{-1}tgv = g(w\mu)(t)v = (w\mu)(t)gv$$

using the definition of the action of W on X. Hence $g(V_\mu) \subset V_{w\mu}$ and using g^{-1}, it is easy to see that g gives an isomorphism from V_μ to $V_{w\mu}$. Thus, $\dim V_\mu = \dim V_{w\mu}$ and hence the character $\chi_V = \sum_\mu (\dim V_\mu) e^\mu$ is W-invariant.

(In the case of $G = U(n)$, we can actually choose a lift $W \to G$ which is a group homomophism, by assigning to each permutation $w \in S_n = W$, its permutation matrix. In general, it is not possible to choose such a lift.)

From the proposition, we see that there is an injective ring homomorphism $Rep(G) \to \mathbb{Z}[X]^W$. We will soon see that it is an isomorphism.

4.5 Representations of $U(n)$

Let us carry out an extended example when $G = U(n)$. In this case, we can do a complete analysis of the situation. We take T to be the diagonal unitary matrices. So $X = \mathbb{Z}^n$ and $W = S_n$. We will write

$$\mathbb{Z}[X] = \mathbb{Z}[e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1}]$$

where $e_i = e^{(0,\dots,1,\dots,0)}$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in X$, following our usual notation, we write

$$e^\lambda = e_1^{\lambda_1} \dots e_n^{\lambda_n}$$

First, let us consider $\mathbb{Z}[X]^W$, which we can think of as the ring of Laurent polynomials in the variable $e_1, \dots, e_n$ which are invariant under the symmetric group S_n. This ring, or more precisely the $n \to \infty$ limit, is called the *ring of symmetric functions*.

Let

$$X_+ = \{(\lambda_1, \dots, \lambda_n) \in X : \lambda_1 \geq \dots \geq \lambda_n\}.$$

It is called the set of *dominant weights*. For $\lambda \in X_+$, let us write

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu.$$

It is immediate from the definition that the set $\{m_\lambda\}_{\{}\lambda \in X_+\}$ forms a basis for $\mathbb{Z}[X]^W$. m_λ is called a *monomial symmetric function*.

Define a partial order on $X = \mathbb{Z}^n$ by $(\lambda_1, \dots, \lambda_n) \geq (\mu_1, \dots, \mu_n)$ if

$$\lambda_1 \geq \mu_1, \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2, \dots, \lambda_1 + \dots \lambda_n = \mu_1 + \dots + \mu_n.$$

We will now define other bases for $\mathbb{Z}[X]^W$ which will also be labelled by dominant weights and which will be upper-triangular with respect to this partial order. The motivation for our constructions in the following idea.

Let I be a partially ordered set, possibly infinite, but with the property that for each $i \in I$, $\{j \in I : j \leq i\}$ is finite. Let V be a free abelian group with basis $\{v_i\}_{i \in I}$. Let $\{w_i\}_{i \in I}$ be another set of vectors in V labelled by I. We say that $\{w_i\}_{i \in I}$ is uni-upper triangular with respect to $\{v_i\}_{i \in I}$ if for all $i \in I$, we have

$$w_i - v_i \in \operatorname{span}(w_j : j < i)$$

Lemma 4.11. *Under the above setup, $\{w_i\}_{i \in I}$ forms a $\mathbb{Z}$-basis for V as well.*

For $k = 1, \dots, n$, let $p_k(e_1, \dots, e_n)$ denote the kth elementary symmetric function. So

$$p_1 = e_1 + \dots + e_n,\ p_2 = e_1 e_2 + e_1 e_3 + \dots + e_{n-1} e_n,\ \dots,\ p_n = e_1 \dots e_n$$

and, in general, p_k is the sum over all k-element subsets of $\{1, \dots, n\}$. Clearly $p_k \in \mathbb{Z}[X]^W$, for $k = 1, \dots, n$. Also note that p_n is invertible in $\mathbb{Z}[X]$ and $p_n^{-1} \in \mathbb{Z}[X]^W$.

Let $\omega_i = (1, \dots, 1, 0, \dots, 0)$. If we take $\lambda \in X_+$, we can write $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$, with $m_i \in \mathbb{N}$, for $1 \le i \le n$ and $m_n \in \mathbb{Z}$. Given $\lambda \in X_+$, let us define $p_\lambda = p_1^{m_1} \dots p_n^{m_n}$. ($p_\lambda$ is sometimes called a *elementary symmetric function.*)

Theorem 4.12. *The set $\{p_\lambda\}_{\lambda\in X_+}$ forms a basis for $\mathbb{Z}[X]^W$. Hence*

$$\mathbb{Z}[X]^W \cong \mathbb{Z}[p_1, \dots, p_{n-1}, p_n, p_n^{-1}].$$

Proof. Note that $p_\lambda = m_\lambda + v$, where v lies in the span of the set $\{m_\mu\}_{\mu<\lambda}$ (for example, see Lemma 4.15 below). The result follow from Lemma 4.11. □

Now, let us try to find representations of $U(n)$ for which these p_k are the characters.

First, a general construction. If V is a representation of a group G, then $V^{\otimes k}$ is a representation of G. There is also an action of S_k on $V^{\otimes k}$ which commutes with the action of G on $V^{\otimes k}$.

Hence the symmetric power $\mathrm{Sym}^k V := (V^{\otimes k})^{S_k}$ carries an action of G. Similarly

$$\Lambda^k V := \{v \in V^{\otimes k} : \sigma v = \mathrm{sign}(\sigma)v\}$$

is a representation of G.

More generally, if W is an irreducible representation of S_k, then $\mathrm{Hom}_{S_k}(W, V^{\otimes k})$ carries a representation of G. If W is the trivial representation of S_k, then $\mathrm{Hom}_{S_k}(W, V^{\otimes k}) = \mathrm{Sym}^k V$, while if W is the sign representation of S_k, then $\mathrm{Hom}_{S_k}(W, V^{\otimes k}) = \Lambda^k V$

We will specialize this to the case where $V = \mathbb{C}^n$ with the usual action of $U(n)$. So we have an action of $U(n)$ on $\Lambda^k\mathbb{C}^n$, for $k = 1, \dots, n$.

Lemma 4.13. *The character of $\Lambda^k\mathbb{C}^n$ is p_k.*

Proof. Let $v_1, \dots, v_n$ be the standard basis for $\mathbb{C}^n$. There is a basis for $\Lambda^k\mathbb{C}^n$ consisting of $v_{i_1} \wedge \cdots \wedge v_{i_k}$ where $i_1 < \cdots < i_k$. The result follows immediately. □

The representation $\Lambda^n\mathbb{C}^n$ is 1-dimensional and is given by the group homomorphism $\det : U(n) \to U(1)$. Its dual representation is given by $1/\det$ and has character p_n^{-1}.

Corollary 4.14. *The map $Rep(G) \to \mathbb{Z}[X]^W$ is an isomorphism.*

Proof. We already knew it was injective and since it hits all the generators $p_1, \dots, p_n, p_n^{-1}$, it must be surjective. □

Let us use this result to get a description of the irreducible representations.

We say that a representation V of $U(n)$ has *highest weight* $\lambda \in X_+$, if

(i) $\dim V_\lambda \neq 0$, and

(ii) For all $\mu \in X$ such that $\dim V_\mu \neq 0$, $\mu \leq \lambda$.

Note that a representation can only have one highest weight (of course, it may not have any highest weight). We will also say that V is of highest weight λ.

Lemma 4.15. *If V_1 and V_2 are representations of highest weights λ_1 and λ_2, then $V_1 \otimes V_2$ has highest weight $\lambda_1 + \lambda_2$.*

Proof. Note that

$$(V_1 \otimes V_2)_\mu = \bigoplus_{\mu_1+\mu_2=\mu} (V_1)_{\mu_1} \otimes (V_2)_{\mu_2}$$

Now if $\mu_1 \leq \lambda_1$ and $\mu_2 \leq \lambda_2$, then $\mu_1 + \mu_2 \leq \lambda_1 + \lambda_2$. The result follows. $\square$

We are now in a position to prove our decisive result.

Theorem 4.16. *For each $\lambda \in X_+$, there exists a unique irreducible representation $V(\lambda)$ of $U(n)$ of highest weight λ. These are all non-isomorphic and they are all the irreducible representations of $U(n)$.*

Proof. Let $\lambda \in X_+$ and let us write $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$. Then we can consider the representation

$$W = (\mathbb{C}^n)^{\otimes m_1} \otimes (\Lambda^2\mathbb{C}^n)^{\otimes m_2} \otimes \cdots \otimes (\Lambda^n\mathbb{C}^n)^{\otimes m_n}$$

of $U(n)$. By the above results, the character of W is p_λ.

By Lemma 4.15, W has highest weight λ. In particular, the dimension of its λ-weight space is 1-dimensional. Hence there exists a unique irreducible subrepresentation $V(\lambda)$ whose λ-weight space is 1-dimensional. Since W has highest weight λ, so hence $V(\lambda)$ does as well.

If $\lambda \neq \mu$, then $V(\lambda)$ and $V(\mu)$ will have different highest weights, so they will be different representations.

To see that these are all the irreducible representations, it is enough to see that the set $\{\chi_\lambda := \chi_{V(\lambda)}\}_{\lambda \in X_+}$ forms a basis for $\mathbb{Z}[X]^W$. To see this, we again appeal to Lemma 4.11. $\square$

Later, we will see a few different ways of computing the characters χ_λ (equivalently, understanding the weight decomposition of $V(\lambda)$).

Example 4.17. If $\lambda = (k, 0, \ldots, 0) = k\omega_1$, then $V(\lambda) = \mathrm{Sym}^k \mathbb{C}^n$. This can be proven by noting that $\mathrm{Sym}^k \mathbb{C}^n$ is irreducible[1] and is of highest weight $k\omega_1$.

Example 4.18. Consider $U(3)$. Let us think about $V(2,1,0)$. Consider $\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3$. This representation has the following weight diagram.

It contains one copy of the determinant representation, since there is a non-zero map $\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3 \to \Lambda^3\mathbb{C}^3$ which is the determinant representation. So we can split

$$\mathbb{C}^3 \otimes \Lambda^2\mathbb{C}^3 = W \oplus \mathbb{C}_{det}$$

This representation W has the weight diagram.

It is not that hard to see that W is irreducible and hence $W = V(2,1,0)$.

5 Complexification of compact groups

5.1 Complexification in general

5.1.1 Vector spaces

We begin with vector spaces. If V is a real vector space, then we can construct $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$. We write elements of $V_\mathbb{C}$ as $v_1 + iv_2 = v_1 \otimes 1 + v_2 \otimes i$ for $v_1, v_2 \in V$. $V_\mathbb{C}$ will carry a conjugate-linear map $\sigma : V_\mathbb{C} \to V_\mathbb{C}$, which is an involution ($\sigma^2 = id$). We can recover V from $V_\mathbb{C}$ and σ by setting $V = V_\mathbb{C}^\sigma$.

Finally, note that $V_\mathbb{C}$ can also be defined by the following universal property. Let W be a complex vector space. Every $\mathbb{R}$-linear map $T : V \to W$ extends uniquely to a $\mathbb{C}$-linear map $T_\mathbb{C} : V_\mathbb{C} \to W$ by setting $T_\mathbb{C}(v_1 + iv_2) = T(v_1) + iT(v_2)$. Another way to say the same thing is that complexification is the left adjoint functor to the forgetful functor from complex vector spaces to real vector spaces. In an equation,

$$\mathrm{Hom}_\mathbb{R}(V, W) = \mathrm{Hom}_\mathbb{C}(V_\mathbb{C}, W).$$

[1] I don't see a complete elementary way to prove that $\mathrm{Sym}^k \mathbb{C}^n$ is irreducible, but there should be one.

5.1.2 Lie algebras

Now suppose that $\mathfrak{g}$ is a real Lie algebra. Then we can construct a complex Lie algebra $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. The Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ comes from the Lie bracket on $\mathfrak{g}$ by extending it to be complex linear. As with vector spaces, conjugate-linear σ and the universal property. In particular, every representation of $\mathfrak{g}$ on a complex vector space V extends uniquely to a representation of $\mathfrak{g}_{\mathbb{C}}$ on V.

Example 5.1. (i) One simple example is to take $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$. Then we see $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$.

(ii) Another straightforward example is $\mathfrak{g} = \mathfrak{so}(n)$, the Lie algebra of $n \times n$ skew-symmetric real matrices. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_n(\mathbb{C})$, the $n \times n$ skew-symmetric complex matrices because every complex skew-symmetric matrix can be written as $A + iB$, where A, B are both real skew-symmetric matrices.

Another way to think about the same thing, is to describe $\mathfrak{so}(n)$ as the subspace of $\mathfrak{gl}_n(\mathbb{R})$ defined by the linear equation $A + A^{tr} = 0$. So its complexification is the subspace of $\mathfrak{gl}_n(\mathbb{C})$ defined by the same equation, which is of course $\mathfrak{gl}_n(\mathbb{C})$.

(iii) A more complicated example is $\mathfrak{g} = \mathfrak{u}(n)$, the Lie algebra of skew-Hermitian $n \times n$ complex matrices. Note that $\mathfrak{u}(n)$ is a real Lie algebra, not a complex Lie algebra. The complexification of $\mathfrak{u}(n)$ is $\mathfrak{gl}_n(\mathbb{C})$. This can be see in two different ways.

First of all, we can embed $\mathfrak{u}(n)$ into $\mathfrak{gl}_n(\mathbb{C})$ in the obvious way. Under this embedding $i\mathfrak{u}(n)$ becomes the set of all Hermitian matrices. Every complex matrix can be written uniquely as the sum of a Hermitian and a skew-Hermitian matrix, so $\mathfrak{u}(n) \oplus i\mathfrak{u}(n) = \mathfrak{gl}_n(\mathbb{C})$.

Another way to think about this is to think of an element $\mathfrak{u}(n)$ a sum $A + iB$, where A, B are real matrices. Then,

$$\mathfrak{u}(n) = \{(A, B) \in M_n(\mathbb{R}) \times M_n(\mathbb{R}) : A + iB = -(A^{tr} - iB^{tr})\}$$

We can rewrite this as the space of pairs of real matrices (A, B) as $A = -A^{tr}$ and $B = B^{tr}$. Hence

$$\mathfrak{u}(n)_{\mathbb{C}} = \{(A, B) \in M_n(\mathbb{C}) \times M_n(\mathbb{C}) : A = -A^{tr}, B = B^{tr}\}$$

With this description, the map $\mathfrak{u}(n)_{\mathbb{C}} \to \mathfrak{gl}_n(\mathbb{C})$ taking $(A, B) \mapsto A + iB$ is an isomorphism of complex Lie algebras.

Notice that in the example above, $\mathfrak{u}(n)$ and $\mathfrak{gl}_n(\mathbb{R})$ are non-isomorphic real Lie algebras (for $n \geq 2$), but their complexifications are isomorphic.

5.1.3 Affine varieties

Now let $X \subset \mathbb{R}^n$ be a real affine algebraic variety. We let $I(X) \subset \mathbb{R}[x_1, \ldots, x_n]$ denote the ideal of polynomials vanishing on X. We define the *complexification* $X_{\mathbb{C}}$ of X to be the set of $x \in \mathbb{C}^n$ such that $f(x) = 0$ for all $f \in I(X)$. For example, if $I(X)$ is generated by just one polynomial f, then $X_{\mathbb{C}}$ will be the subvariety of $\mathbb{C}^n$ defined by the same polynomial.

Conversely if we fix a complex affine variety Y, then any real variety X such that $X_{\mathbb{C}} \cong Y$ is called a *real form* of Y.

Example 5.2. Let us take $X = U(1)$ to be a circle in $\mathbb{R}^2$, defined by the equation $x^2 + y^2 = 1$. So $f = x^2 + y^2 - 1$ and $X_{\mathbb{C}} \subset \mathbb{C}^2$ is the locus $x^2 + y^2 = 1$. However, $x^2 + y^2 = (x + iy)(x - iy)$, so applying a linear change of coordinates in $\mathbb{C}^2$, we see that

$$X_{\mathbb{C}} = \{(u, v) \in \mathbb{C}^2 : uv = 1\}$$

Note that $X_{\mathbb{C}}$ is the same thing as $\mathbb{R}^{\times}_{\mathbb{C}} = \mathbb{C}^{\times}$, where as usual, we regard $\mathbb{R}^{\times}$ as the affine variety

$$\mathbb{R}^{\times} = \{(u, v) \in \mathbb{R}^2 : uv = 1\}.$$

So $U(1)$ and $\mathbb{R}^{\times}$ an example of a pair of non-isomorphic real varieties whose complexifications are isomorphic.

The following result will be of importance to us.

Lemma 5.3. *Let X be a real affine variety and $X_{\mathbb{C}}$ its complexification. If $f \in \mathcal{O}(X_{\mathbb{C}})$ and f vanishes on X, then $f = 0$. (In other words, X is Zariski-dense in $X_{\mathbb{C}}$.)*

Proof. It suffices to show that $\mathcal{O}(X_C) = \mathcal{O}(X) \otimes_{\mathbb{R}} \mathbb{C}$. To see this, note that $\mathbb{R}[x_1, \ldots, x_n] \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[x_1, \ldots, x_n]$, so the above statement is equivalent to showing that $I(X_{\mathbb{C}}) = I(X) \otimes_{\mathbb{R}} \mathbb{C}$. By the definition, we have containment, $I(X) \otimes_{\mathbb{R}} \mathbb{C} \subset I(X_{\mathbb{C}})$, so by the Nullstellensatz, it suffices to show that $I(X) \otimes_{\mathbb{R}} \mathbb{C}$ is a radical ideal.

Suppose that $a, b \in \mathbb{R}[x_1, \ldots, x_n]$ and $(a + ib)^k \in I(X) \otimes_{\mathbb{R}} \mathbb{C}$ for some k. Then multiplying by $(a - ib)^k$, we see that $(a^2 + b^2)^k \in I(X) \otimes_{\mathbb{R}} \mathbb{C}$. Hence $(a^2 + b^2)^k \in I(X)$. Since $I(X)$ is radical, $a^2 + b^2 \in I(X)$. But if $a^2 + b^2 \in I(X)$, then $a^2(x) + b^2(x) = 0$ for all $x \in X$. Hence $a(x) = b(x) = 0$

for all $x \in X$ (since $a(x)$ and $b(x)$ are real numbers). Thus $a, b \in I(X)$ and we are done[2]. $\square$

If $X \subset \mathbb{R}^n$ is a real affine variety, then $X_{\mathbb{C}} \subset \mathbb{C}^n$ carries an involution $\sigma : X_{\mathbb{C}} \to X_{\mathbb{C}}$ which is defined by $\sigma(x_1, \ldots, x_n) = (\overline{x_1}, \ldots, \overline{x_n})$, using complex conjugation of each coordinate. Since the polynomials defining $X_{\mathbb{C}}$ have real coefficients, if $x \in X_{\mathbb{C}}$, then $\sigma(x) \in X_{\mathbb{C}}$. By construction, we can recover X from $X_{\mathbb{C}}$ by looking at the fixed points of σ.

Example 5.4. Take X to be the circle in $\mathbb{R}^2$ as above. As above, we identify $X_{\mathbb{C}} = \mathbb{C}^\times$. Then $\sigma(z) = \overline{z^{-1}}$. So $\sigma(z) = z$ if and only if $|z| = 1$ as expected.

The complexification of algebraic varieties satisfies a universal property similar to the one for vector spaces. Let X be a real affine variety and $X_{\mathbb{C}}$ be its complexification. Let Y be a complex affine variety. Then a real-algebraic map from X to Y extends uniquely to a complex algebraic map from $X_{\mathbb{C}}$ to Y. Moreover all complex algebraic maps from $X_{\mathbb{C}}$ to Y arise this way. In an equation,

$$\mathrm{Hom}_{\mathbb{R}-\text{varieties}}(X, Y) = \mathrm{Hom}_{\mathbb{C}-\text{varieties}}(X_{\mathbb{C}}, Y). \tag{1}$$

To see this, let recall that if $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{C}^m = \mathbb{R}^{2m}$, then a morphism F of real varieties from X to Y is given by $2m$-polynomials $\{f_j, g_j\}_{j=1\ldots m}$ in n variables with real coefficients, which map X into Y. Then we can define $F_{\mathbb{C}} : X_{\mathbb{C}} \to Y$, by using $\{f_j + i g_j\}_{j=1,\ldots,m}$, which gives us m polynomials in n variables with complex coefficients, which maps $X_{\mathbb{C}}$ into Y. Note that this univeral property implies that the complexification of X is independent of the embedding of X into affine space.

We can also think about this by examining coordinate rings. Let $A = \mathcal{O}_{\mathbb{C}}(Y)$ and $B = \mathcal{O}_{\mathbb{R}}(X)$. The coordinate ring of X when regarded as a real variety is $(A \otimes \bar{A})^\sigma$, where $\bar{A}$ denotes the same $\mathbb{C}$-algebra as A but with scalar multiplication twisted by complex conjugation and where σ denotes the map which permutes tensors. Then the equivalent statement to (1) is

$$\mathrm{Hom}_{\mathbb{R}-\text{algebras}}((A \otimes \bar{A})^\sigma, B) = \mathrm{Hom}_{\mathbb{C}-\text{algebras}}(A, B \otimes_{\mathbb{R}} \mathbb{C})$$

This last statement can be verified directly by noting that there are two (vector space) embeddings of A into $(A \otimes \bar{A})^\sigma$ and these two embeddings can be used to define a map from A to $B \otimes_{\mathbb{R}} \mathbb{C} = B \oplus iB$.

Finally, we note the following fact. Suppose we have an embedding map $X \to Y$ of affine varieties. The functoriality of complexification gives us a

[2] Thanks to David Speyer for this argument.

map $X_{\mathbb{C}} \to Y_{\mathbb{C}}$. We claim that this gives an embedding $X_{\mathbb{C}} \subset Y_{\mathbb{C}}$. To see this, embed Y into some affine space $\mathbb{R}^n$. This gives an embedding of X into $\mathbb{R}^n$ and we can define the complexification of X with respect to this embedding. Thus we get $X_{\mathbb{C}} \subset Y_{\mathbb{C}} \subset \mathbb{C}^n$ as desired.

5.1.4 Algebraic groups

Now, let G be a real algebraic group. Then we complexify G as a real variety to obtain a complex variety $G_{\mathbb{C}}$. The group structure on G gives us a group structure on $G_{\mathbb{C}}$. There are a couple of ways to see that the group structure survives the complexification. The first is to just note that the group multiplication is given by polynomial functions and these polynomial functions continue to make sense after the complexification. Another way is to note that the group structure on G provides $\mathcal{O}(G)$ with the structure of a Hopf algebra and so its complexification $\mathcal{O}(G_{\mathbb{C}}) = \mathcal{O}(G) \otimes_{\mathbb{R}} \mathbb{C}$ also carries the structure of a Hopf algebra.

Example 5.5. (i) The important example of $U(1)$ was already mentioned above. More generally, if T is any torus, then its complexification $T_{\mathbb{C}}$ defined in section 3.3 agrees with the above definition.

To make this precise, we have to first give T the structure of a real algebraic group. One way to do this is to choose an isomorphism of Lie groups $T \to U(1)^n$ and then define a real algebraic group structure on T using the real algebraic group structure on $U(1)^n$. The resulting structure on T is independent of the isomorphism with $U(1)^n$, because all Lie group automorphisms of $U(1)^n$ are real algebraic.

An alternate approach to define $\mathcal{O}(T)$ is as follows. First we define $\mathcal{O}(T_{\mathbb{C}}) = \mathbb{C}[X]$ as before and then we define $\sigma : \mathcal{O}(T_{\mathbb{C}}) \to \mathcal{O}(T_{\mathbb{C}})$ by $\mu \mapsto -\mu$. Then we set $\mathcal{O}(T) := \mathcal{O}(T_{\mathbb{C}})^{\sigma}$. Then we check that there is a bijection between $\operatorname{Spec}_{\mathbb{R}} \mathcal{O}(T)$ (the set of maximal ideals of $\mathcal{O}(T)$ whose residue field is $\mathbb{R}$) and T.

(ii) Let us now take $G = SO(n)$. We view $SO(n)$ as the variety in $M_n(\mathbb{R})$ ($n \times n$ matrices) defined by the equations $AA^{tr} = 1$ and $\det A = 1$. Its complexification is hence given by the same conditions in $M_n(\mathbb{C})$ and thus is $SO_n(\mathbb{C})$.

More generally, if V is any real vector space with a non-degenerate symmetric bilinear form $\langle, \rangle$, then its automorphisms $SO(V, \langle, \rangle)$ is a real algebraic group. Note that $SO(V, \rangle, \langle)$ is determined up to isomorphism by the signature of the bilinear form. We can complexify the pair $V, \langle , \rangle$ to obtain $V_{\mathbb{C}}, \langle , \rangle_{\mathbb{C}}$ and then we have $SO(V)_{\mathbb{C}} \cong SO(V_{\mathbb{C}})$. Note that $SO(V_{\mathbb{C}}) \cong SO_n(\mathbb{C})$ where $n = \dim_{\mathbb{R}} V$, since $V_{\mathbb{C}}, \langle , \rangle_{\mathbb{C}}$ is isomorphic to $\mathbb{C}^n$ with its standard bilinear form.

(iii) Now, we consider $G = U(n)$. We regard $U(n)$ as a variety in $M_n(\mathbb{R})^2$ defined by the equation $(A + iB)(A + iB)^* = 1$. This is equivalent to $AA^{tr} + BB^{tr} = I$ and $AB^{tr} = BA^{tr}$. Hence

$$U(n)_{\mathbb{C}} = \{(A, B) \in M_n(\mathbb{C})^2 : AA^{tr} + BB^{tr} = I, AB^{tr} = BA^{tr}\}$$

with the group structure given by a slightly strange expression.

We claim that the map

$$U(n)_{\mathbb{C}} \to GL_n(\mathbb{C}),\ (A, B) \mapsto A + iB$$

is an isomorphism of complex algebraic groups. To see this, note that the inverse is given by

$$g \mapsto \left(\frac{1}{2}(g + (g^{tr})^{-1}), \frac{1}{2i}(g - (g^{tr})^{-1})\right)$$

Thus the complexification of $U(n)$ is $GL_n(\mathbb{C})$, as is to be expected from looking at the Lie algebra.

Note that $GL_n(\mathbb{C})$ is also the complexification of $GL_n(\mathbb{R})$. So $U(n)$ and $GL_n(\mathbb{R})$ are two real forms of $GL_n(\mathbb{C})$ and they are non-isomorphic for all $n \geq 1$.

As in the previous sections, we note that $G_{\mathbb{C}}$ enjoys a universal property. Let H be a complex algebraic group. Then every map of real algebraic groups from G to H extends uniquely to a map of complex algebraic groups from $G_{\mathbb{C}}$ to H and all maps of complex algebraic groups from $G_{\mathbb{C}}$ to H arise this way.

This universal property is particularly useful in the case where $H = GL(V)$ for some complex vector space V.

Proposition 5.6. *Let G be a real algebraic group. A real algebraic representation of G on a complex vector space V extends uniquely to an algebraic representation of $G_{\mathbb{C}}$ on the same vector space.*

Also as in the previous sections, the complexification $G_{\mathbb{C}}$ carries an involution σ whose fixed point set is G. In the algebraic group setting, σ is a group homomorphism.

Example 5.7. If $G = U(n)$, then $U(n)_{\mathbb{C}} = GL_n(\mathbb{C})$ carries the involution $\sigma(g) = (g^*)^{-1}$.

On the other hand, if we regard $GL_n(\mathbb{C})$ as the complexification of the real algebraic group $GL_n(\mathbb{R})$, then it carries the involution $\sigma(g) = \bar{g}$.

5.2 Complex reductive groups

If G is a complex algebraic group, then we will study its complex algebraic representations. We say that G is *reductive* if it is connected and if every representation is isomorphic to a direct sum of irreducible subrepresentations (in other words, the category is semisimple). Another way to formulate this is to say that every invariant subspace of a representation has a complementary invariant subspace.

Recall that we proved that the (smooth) representations of a compact Lie group were semisimple. We can harness this fact as follows.

Theorem 5.8. *Let K be a real algebraic group, whose underlying Lie group is compact and connected. Then $G = K_{\mathbb{C}}$ is reductive.*

The theorem follows from the following Lemma, which is useful in its own right.

Lemma 5.9. *Let K be a real algebraic group and let G be its complexification. Let V be a representation of G and let W be a K-invariant subspace. Then W is G-invariant.*

Proof. The fact that W is K-invariant is equivalent to the vanishing of some matrix coefficients. These matrix coefficients are polynomial functions on G which vanish on K. Hence they also vanish on G. Thus W is also G-invariant as desired. □

There is a Lie algebra version of this lemma which has a simpler proof.

Lemma 5.10. *Let $\mathfrak{k}$ be a real Lie algebra and let $\mathfrak{g}$ be its complexification. Let V be a representation of $\mathfrak{g}$ and let W be a $\mathfrak{k}$-invariant subspace. Then W is $\mathfrak{g}$-invariant.*

Proof. Every element of $\mathfrak{g}$ is of the form $X + iY$ for $X, Y \in \mathfrak{k}$. Since W is $\mathfrak{k}$-invariant, it is invariant under X and Y and hence under $X + iY$. □

Proof of Theorem 5.8. Let V be a representation of G. Let $W \subset V$ be a G-invariant subspace. Then W is a K-invariant subspace, so there exists a complementary K-invariant subspace W'. By Lemma 5.9, W' is G-invariant.

Hence every G-invariant subspace of V has a complementary G-invariant subspace. □

Theorem 5.8 has some "converses" which are also true, but which we will not prove.

(i) Every complex reductive group G is the complexification of a unique real algebraic group whose underlying Lie group is compact and connected.

(ii) Every compact connected Lie group has the structure of a real algebraic group, in a unique way.

Accepting these converses, Theorem 5.8 gives a bijection between the isomorphism classes of compact connected Lie groups and the isomorphism classes of complex reductive groups. In fact, I think that this is an equivalence of categories.

From this point on, we will let K denote a real algebraic group whose underlying Lie group is compact, connected and $G = K_{\mathbb{C}}$ will denote its complexification. We will refer to K as a compact Lie group and G as a complex reductive group.

Example 5.11. Let us list some pairs of a compact Lie group K and the resulting complex reductive group G.

(i) We can take $K = T$, a torus. Then $G = T_{\mathbb{C}}$.

(ii) We can take $K = U(n)$. Then $G = GL_n(\mathbb{C})$.

(iii) As a slight modification, take $K = SU(n)$, then $G = SL_n(\mathbb{C})$ (note that this example is defined by imposing one equation on the previous example).

(iv) Another good family of examples is $K = SO(n)$. Then $G = SO_n(\mathbb{C})$.

(v) A more "exotic" example is to take $K = Sp(n)$, the group of $n \times n$ unitary quaternion matrices. Then $G = Sp_{2n}(\mathbb{C})$, the symplectic group (i.e. the automorphisms of $\mathbb{C}^{2n}$ preserving the standard symplectic form).

5.3 Lie algebras of algebraic groups

Let G be an algebraic group (either real or complex). Then it has a *Lie algebra* $Lie(G)$, which is defined using the left-invariant vector fields on the group, much as with Lie groups. Just as with Lie groups, the space of left invariant vector fields is isomorphic to the tangent space at the identity.

There is a nice way to find the tangent space of an algebraic variety, which is particularly useful for finding the Lie algebra of an algebraic group. We start with varieties.

Let X be an affine variety over a field k. Then for any k-algebra R, we can consider the R-points of X as follows by defining $X(R)$ to be the subset of R^n defined by the polynomials in $I(X)$ (this is the same as what we did to define the complexification of a real variety). Another way to say the same thing is to define $X(R)$ as the set of k-algebra homomorphisms from $\mathcal{O}(X)$ to R.

Now, let $R = k[\varepsilon]/\varepsilon^2$. There is a map of k-algebras from R to k given by sending ε to 0. So for any variety X, we get $X(R) \to X(k)$.

Lemma 5.12. *For any point $x \in X(k)$, the preimage of x in $X(R)$ is T_xX.*

Proof. To prove this lemma, we have to pick a definition of the tangent space. Let us pick a "differential geometry" definition. Suppose that $I(X)$ is generated by the polynomials $f_1, \ldots, f_m$. Namely, we will think of T_xX as the intersections of the kernels of the linear maps $d_xf_1, \ldots, d_xf_m$. So $(y_1, \ldots, y_n) \in T_xX$ if $d_xf_i(y_1, \ldots, y_n) = 0$ for all i.

On the other hand $(x_1 + y_1\varepsilon, \ldots, x_n + y_n\varepsilon) \in X(R)$ if and only if $f_i(x_1 + y_1\varepsilon, \ldots, x_n + y_n\varepsilon) = 0$ for all i. But using basic calculus, we see that these two conditions are the same. □

So if G is an algebraic group, its Lie algebra is given by the fibre of $G(R)$ over $e \in G(k)$. Let us illustrate this in some examples.

Example 5.13. (i) Let us take $G = U(n)$ viewed as a real algebraic group. Then $T_e(G)$ consists of those $I + A\varepsilon$ such that $(I + A\varepsilon)(I + A\varepsilon)^* = 1$. Expanding this out using $\varepsilon^2 = 0$, we find that $A + A^* = 0$. So $\mathfrak{g} = \mathfrak{u}(n)$ is the Lie algebra of $n \times n$ complex skew-Hermitian matrices.

(ii) Also another example, let us take $G = SL_n(\mathbb{C})$ (or $SL_n(\mathbb{R})$). Then $T_e(G)$ consists of those $I + A\varepsilon$ such that $\det(I + A\varepsilon) = 1$. Expanding this out gives $\mathrm{tr}(A) = 0$.

One advantage of this approach is that it allows us to show that taking the Lie algebra commutes with complexification.

Proposition 5.14. *Let G be a real algebraic group. Then $Lie(G)_{\mathbb{C}} = Lie(G_{\mathbb{C}})$.*

Proof. Let $R = \mathbb{R}[\varepsilon]/\varepsilon^2$ and let $R_{\mathbb{C}} = \mathbb{C}[\varepsilon]/\varepsilon^2$. Let us pick an embedding $G \subset \mathbb{R}^n$. If $e + a\varepsilon$ and $e + b\varepsilon$ are two points in R^n which lie in $G(R)$. Then we can regard $a + ib$ as a point in $R^n_{\mathbb{C}}$ and it is easy to see that $e + (a + ib)\varepsilon \in G_{\mathbb{C}}(R)$. This shows that $Lie(G)_{\mathbb{C}} \subset Lie(G_{\mathbb{C}})$. The converse is also easy. $\square$

5.4 The root datum

In this section, we write $X = X^*(T)$ for the weight lattice and $X^\vee = X_*(T)$ for the coweight lattice of the maximal torus.

Let $\mathfrak{g}$ be the Lie algebra of G. By the above results, $\mathfrak{g}$ is the complexification of $\mathfrak{k}$. We have an *adjoint action* of G on $\mathfrak{g}$, defined as follows. For each $g \in G$, we have a conjugation by g map

$$G \to G, \; h \mapsto ghg^{-1},$$

taking e to e. Taking the derivative at e this gives a map $T_eG \to T_eG$ which we denote $X \mapsto gXg^{-1}$ for $X \in \mathfrak{g} = T_e(G)$. This is the adjoint action.

When $G = GL_n(\mathbb{C})$ and so $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, gXg^{-1} is given by conjugation of matrices. Since the adjoint action is compatible with embeddings of groups, if $G \subset GL_n(\mathbb{C})$, then the adjoint action of G on $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ is also given by conjugation.

We can consider the weight decomposition of $\mathfrak{g}$ for the adjoint action. Let R be the set of non-zero weights of this representation. These are called the *roots* of the group G. Hence we can write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

The zero weight space $\mathfrak{g}_0 = \mathfrak{t}_{\mathbb{C}}$, the Lie algebra of the maximal torus. Since the torus T acts by Lie algebra automorphisms, we see that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for any two roots α, β.

Theorem 5.15. *For each root $\alpha \in R$, the following holds*

(i) $\mathfrak{g}_\alpha$ *is 1-dimensional.*

(ii) $-\alpha$ *is also a root, but no other $\mathbb{Q}$-multiple of α is a root.*

(iii) $\mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}$ *forms a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.*

(iv) *There exists a unique map $\psi_\alpha : SL_2(\mathbb{C}) \to G$, which induces the above isomorphism on the level of Lie algebras (this map is either 1-1 or is a 2-1 cover).*

We restrict ψ_α to the maximal torus $\mathbb{C}^\times$ of $SL_2(\mathbb{C})$. By construction, $\psi_\alpha(\mathbb{C}^\times) \subset T_\mathbb{C}$. Hence it gives a coweight of T, which we denote by $\alpha^\vee$, and call a *coroot*. We have $\langle \alpha^\vee, \alpha \rangle = 2$.

Example 5.16. (i) Take $G = GL_n(\mathbb{C})$. Then $R = \{e_i - e_j\}_{i \neq j}$ and if $\alpha = e_i - e_j$, then $\mathfrak{g}_\alpha$ consists of those matrices with an entry in the ith row and jth column and zeros elsewhere.

For each $\alpha = e_i - e_j$, the resulting map $SL_2(\mathbb{C}) \to GL_n(\mathbb{C})$ consists of embedding of 2×2 matrices into the i, j rows and i, j columns (putting 1s on the diagonal away from these rows/columns). Hence $\alpha^\vee = e_i - e_j$.

(ii) Another simpler example is to take $G = T_\mathbb{C}$ a torus. Then there are no roots $R = \emptyset$.

(iii) A third example is $G = SL_2(\mathbb{C})$. Then $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the conjugation action. We can identify X with $\mathbb{Z}$ by choosing the isomorphism $\mathbb{C}^\times \cong T$ given by $t \mapsto \left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right]$. Then with this identification the roots R are$\{2, -2\}$ and the coroots are $\{1, -1\}$.

(iv) A fourth example is $G = PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/\mathbb{C}^\times = SL_2(\mathbb{C})/\{\pm I\}$. Then $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and the conjugation action of $SL_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})$ descends to the adjoint representation of $PGL_2(\mathbb{C})$. We choose an isomorphism $\mathbb{C}^\times \cong T$ by $t \mapsto \left[\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and hence identify X with $\mathbb{Z}$. With this identification, the roots are $\{1, -1\}$. For $\alpha = 1$, the map ψ_α is the 2-1 cover $SL_2(\mathbb{C}) \to PGL_2(\mathbb{C})$. On the maximal torus of $SL_2(\mathbb{C})$, this map becomes

$$\left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} t^2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$$

and so the coroots are $\{2, -2\}$.

For each root α, we define the *reflection* in α by

$$s_\alpha : X \to X, \ \mu \mapsto \mu - \langle \alpha^\vee, \mu \rangle \alpha \tag{2}$$

Recall that we have a faithful action of the Weyl group on X. Hence we can identify W with its image in $GL(X)$. Note that since $\mathfrak{g}$ is a representation of G, the set of roots is invariant under W (just as the set of weights of any representation is W-invariant.)

Theorem 5.17. *For each root α, $s_\alpha \in W$ and it is represented by $\psi_\alpha(\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right])$. Moreover the s_α generate W.*

Example 5.18. Continuing with the example of GL_n, we have that if $\alpha = e_i - e_j$ is a root, then s_α is the transposition $(i\,j)$ in $S_n = W$, which as we mentioned acts on $X = \mathbb{Z}^n$ by permuting the coordinates.

The collection $(X, R, X^\vee, R^\vee)$ is called the *root datum* of G (or of K). It determines G up to isomorphism.

The root datum of a reductive group is an example of an abstract root datum which is defined as follows.

A *root datum* is a 4-tuple $(X, R, X^\vee, R^\vee)$ with the following structure/axioms.

(i) $X, X^\vee$ are finite rank free $\mathbb{Z}$-modules with a perfect pairing $\langle , \rangle$ between them.

(ii) $R \subset X$, $R^\vee \subset X^\vee$ are finite subsets, such that if $\alpha \in R$, then $-\alpha \in R$, but no other $\mathbb{Q}$-multiple of α is in R.

(iii) There is a bijection $\alpha \mapsto \alpha^\vee$ between R and $R^\vee$, such that $\langle \alpha^\vee, \alpha \rangle = 2$.

(iv) We define s_α as in (2). Then, $s_\alpha(R) \subset R$ and $s_\alpha(R^\vee) \subset R^\vee$.

An important result is that the root datum determines the group.

Theorem 5.19. *The above method of associated a root datum to a reductive group gives a bijection between isomorphism classes of complex reductive groups (and hence compact connected Lie groups) and isomorphism classes of root data.*

There is also the related concept of root system.

A *root system* is a 4-tuple $(V, R, V^*, R^\vee)$ with the following structure/axioms.

(i) V, V^* are dual finite dimensional $\mathbb{R}$-vector spaces.

(ii) $R \subset V$, $R^\vee \subset V^*$ are finite subsets, such that if $\alpha \in R$, then $-\alpha \in R$, but no other $\mathbb{R}$-multiple of α is in R. Moreover, we require that R spans V and $R^\vee$ spans V^*.

(iii) There is a bijection $\alpha \mapsto \alpha^\vee$ between X and $X^\vee$, such that $\langle \alpha^\vee, \alpha \rangle = 2$.

(iv) We define s_α as in (2). Then, $s_\alpha(R) \subset R$.

A root datum $(X, R, X^\vee, R^\vee)$ has underlying *root system*, defined as follows. Let $Q = \mathbb{Z}R$ be the *root lattice*, which is the $\mathbb{Z}$-span of the roots inside X (similarly we define the coroot lattice $Q^\vee$). Then $(Q \otimes_{\mathbb{Z}} \mathbb{R}, R, Q^\vee \otimes_Z \mathbb{R}, R^\vee)$ is a root system.

5.5 Centre of the group and semisimple groups

Given K, G, we define

$$Q_{\mathbb{R}}^{\perp} = \{x \in X^{\vee} \otimes_{\mathbb{Z}} \mathbb{R} : \langle x, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}$$
$$Q_{\mathbb{C}}^{\perp} = \{x \in X^{\vee} \otimes_{\mathbb{Z}} \mathbb{C} : \langle x, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}$$

be the "perp" (or dual) of the root lattice.

We have the following nice result describing the centre $Z(G)$ of G.

Theorem 5.20. *There is are isomorphisms (as real/complex Lie groups)*

$$Z(K) \cong Q_{\mathbb{R}}^{\perp}/X^{\vee}$$
$$Z(G) \cong Q_{\mathbb{C}}^{\perp}/X^{\vee}$$

Proof. We will prove the result for K. First, by the fact that the maximal torus is its own centralizer, $Z(K) \subset T$. Examining the adjoint representation, we see that the subgroup of T which acts trivially on $\mathfrak{g}$ is $\bigcap_{\alpha \in R} \ker(\alpha)$, where each α is thought of as a map $T \to U(1)$. Hence we see that $Z(K) \subset \bigcap_{\alpha} \ker(\alpha)$. In fact, there is equality here, because if $t \in \bigcap_{\alpha}$, then the closed subgroup of elements commuting with t will have Lie algebra $\mathfrak{k}$ and hence must be K.

So $Z(K) = \bigcap_{\alpha \in R} \ker(\alpha)$. If we think of $T = X^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}/X^{\vee}$ then the result follows. □

A companion result, which we will not prove is the following.

Theorem 5.21. *The fundamental groups of K, G are given as*

$$\pi_1(K) = \pi_1(G) \cong X^{\vee}/Q^{\vee}$$

The rough idea is to note that every element of $X^{\vee}$ gives us a loop in T since $X^{\vee} = \mathrm{Hom}(U(1), T)$. In fact, $X^{\vee} = \pi_1(T)$. However, some of these loops are contractible in K. In particular, if $\alpha^{\vee}$ is a coroot, then the corresponding loop in T is contractible in K, since it factors through $SU(2) \to K$ and $SU(2)$ is simply-connected.

Some elementary reasoning shows the following.

Lemma 5.22. *The following are equivalent.*

(i) Q and X have the same rank.

(ii) $Z(K)$ and $Z(G)$ are finite and they are equal.

(iii) $\pi_1(K)$ is finite.

When the above equivalent conditions hold, then we say that G (or K) is *semisimple*. In this case, we write $X_{\text{ab}}^{\vee}$ for $Q_{\mathbb{R}}^{\perp}$ and we call it the *absolute coweight lattice*. Similarly,

$$X_{\text{ab}} := (Q^{\vee})^{\perp} = \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} : \langle \alpha^{\vee}, x \rangle \in \mathbb{Z} \text{ for all } \alpha^{\vee} \in R^{\vee}$$

is called the *absolute weight lattice*. We have containments

$$Q \subset X \subset X_{\text{ab}}, \quad Q^{\vee} \subset X^{\vee} \subset X_{\text{ab}}^{\vee}$$

Also when G is semisimple, then $Z(G) \cong X_{\text{ab}}^{\vee}/X^{\vee} \cong X/Q$, since a finite abelian group is isomorphic to its Pontryagin dual (here the Pontryagin dual of H is $\mathrm{Hom}(H, U(1))$).

If $Z(G) = \{1\}$ (equivalently $X = Q$ or $X^{\vee} = X_{\text{ab}}$), then we say that G (or K) is of *adjoint type*. If $\pi_1(G) = \{1\}$ (equivalently $X^{\vee} = Q$ or $X = X_{\text{ab}}$), then we say that G (or K) is *simply-connected*.

If $(V, R, V^*, R^{\vee})$ is an root system, then a semisimple root datum having this underlying root system is determined by choosing the lattice X between Q and X_{ab}. There are only finitely many such choices and they correspond to subgroups of the finite abelian group X_{ab}/Q. There are two natural extremal choices. The first is the adjoint type, where we take $X = Q$ and end up with $(Q, R, X_{\text{ab}}^{\vee}, R^{\vee})$. The second is the simply-connected type, where we take $X = X_{\text{ab}}$ and end up with $(X_{\text{ab}}, R, Q, R^{\vee})$.

Example 5.23. (i) If $G = PGL_2(\mathbb{C})$, then $Q = \mathbb{Z} = X$ (since 1 is a root) and so $Z(G) = \{1\}$. So $PGL_2(\mathbb{C})$ is of adjoint type.

(ii) If $G = SL_2(\mathbb{C})$, then $Q = 2\mathbb{Z}$ and $X = \mathbb{Z}$ and so $Z(G) = \{\pm 1\}$. So $SL_2(\mathbb{C})$ is simply-connected.

(iii) If $G = SL_n(\mathbb{C})$, then we have $X = \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$ and $Q = \{(a_1, \ldots, a_n) \in X : \sum a_i = 0\}$. So we see that G is semisimple. We have $X^{\vee} = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \sum a_i = 0\} = Q^{\vee}$. So G is simply-connected. The centre of G is isomorphic to X/Q which is $\mathbb{Z}/n$.

(iv) If $G = SO_{2n}(\mathbb{C})$, then one can show that there are proper containments $Q \subset X \subset X_{\text{ab}}$, with $X_{\text{ab}}/X = \mathbb{Z}/4$. So $SO_{2n}(\mathbb{C})$ is semisimple, but neither of adjoint type nor simply-connected.

(v) If $G = GL_n(\mathbb{C})$, then $Q = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n : \sum a_i = 0\}$ and

$$Q^{\perp} = \{(b_1, \ldots, b_n) \in \mathbb{C}^n : b_i - b_j \in \mathbb{Z}, \text{ for all } i, j\}$$

We see that $Q^\perp/\mathbb{Z}^n \cong \mathbb{C}/\mathbb{Z}$ via the map $(b_1, \ldots, b_n) \mapsto b_1$. We can also see directly that $Z(GL_n(\mathbb{C})) = \mathbb{C}^\times$ since it consists of multiples of the identity matrix. So $GL_n(\mathbb{C})$ is not semisimple.

Let $(X = X_{\text{ab}}, R, X^\vee = Q^\vee, R^\vee)$ be a simply connected root datum. If we choose some lattice $Q \subset X' \subset X$, then we can define $G' = G/((X')^\vee/Q^\vee)$ and produce a new group whose root datum is $(X', R, (X')^\vee, R^\vee)$. The map $G \to G'$ is a finite cover and gives an isomorphism on Lie algebras. These G' are all the reductive groups whose Lie algebra is $\mathfrak{g}$.

We say a complex Lie algebra is *semisimple* if it is the Lie algebra of a semisimple reductive group. We can summarize the above paragraph as follows.

Theorem 5.24. *There is a bijection between isomorphism classes of complex semisimple Lie algebras and isomorphism classes of root systems, which is compatible with the map from a semisimple group to its Lie algebra and the map from a root datum to a root system.*

$$\begin{array}{ccc} \text{semisimple reductive groups} & = & \text{root data} \\ \downarrow & & \downarrow \\ \text{semisimple Lie algebras} & = & \text{root systems} \end{array}$$

5.6 Positive systems

A coweight μ is called *regular* if its stabilizer in the Weyl group is trivial (equivalently, it does not lie on any root hyperplanes $\ker \alpha \subset X^\vee$). A subset R_+ of R is called a *positive system* or *set of positive roots* if there exists a regular coweight μ such that

$$R_+ = \{\alpha \in R : \langle \mu, \alpha \rangle > 0\}.$$

Example 5.25. Take $G = GL_n(\mathbb{C})$. Then $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n$ is regular iff $\mu_i \neq \mu_j$ for all $i \neq j$. Let us choose μ such that $\mu_1 > \mu_2 > \cdots > \mu_n$. Then the set of positive roots is $R_+ = \{e_i - e_j : i < j\}$.

Fix a positive system R_+. A *simple root* is a positive root α which is not a sum of two positive roots.

Theorem 5.26. *(i)* $R_+ \sqcup -R_+ = R$.

(ii) Every positive root can be written uniquely as a positive linear combination of simple roots.

(iii) The number of simple roots is the same as the rank of Q.

So if G is semisimple, then the number of simple roots is l, the dimension of the maximal torus.

We will write the simple roots as $\{\alpha_i\}_{i\in I}$. If $\alpha = \sum_{i\in I} n_i\alpha_i$ is a positive root, then the sum $\sum_{i\in I} n_i$ is called the *height* of a positive root α.

The *Cartan matrix* is the matrix (with rows and columns indexed by I) given by $a_{ij} = \langle \alpha_i^\vee, \alpha_j\rangle$.

Example 5.27. If $G = GL_n$, then the simple roots are $\alpha_i = e_i - e_{i+1}$. The Cartan matrix is given by

$$a_{ij} = \begin{cases} 2 \text{ if } i = j \\ -1 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise} \end{cases}$$

Proposition 5.28. *W is generated by the reflections corresponding to simple roots $s_i = s_{\alpha_i}$. The relations among these generators are given as follows.*

(i) $s_i^2 = e$.

(ii) $s_is_j = s_js_i$ *if* $a_{ij} = a_{ji} = 0$.

(iii) $s_is_js_i = s_js_is_j$ *if* $a_{ij} = a_{ji} = -1$.

(iv) $s_is_js_is_j = s_js_is_js_i$ *if* $\{a_{ij}, a_{ji}\} = \{-1, -2\}$.

(v) $s_is_js_is_js_is_j = s_js_is_js_is_js_i$ *if* $\{a_{ij}, a_{ji}\} = \{-1, -3\}$.

The *Dynkin diagram* is used to record the Cartan matrix.

Example 5.29. Take $G = GL_n$ as before. Then s_i is the transposition $(i\,i+1)$. These neighbouring transpositions generate $W = S_n$.

6 Flag varieties and the Borel-Weil theorem

6.1 Coadjoint orbits

Our goal is to study flag varieties for complex reductive groups. Let us begin by viewing them as coadjoint orbits of compact groups.

The adjoint action of K on $\mathfrak{k}$ gives rise to a coadjoint action of K on $\mathfrak{k}^*$. The orbit of $X \in \mathfrak{k}^*$ under this action is called the *coadjoint orbit* through X. (Actually for compact groups, the representations $\mathfrak{k}$ and $\mathfrak{k}^*$ are isomorphic, since we can always choose K-invariant inner product on $\mathfrak{k}$, so it doesn't make much difference whether we speak of adjoint or coadjoint orbits. However, for arbitrary Lie groups, adjoint and coadjoint orbits are different and coadjoint orbits behave better.)

We say that $X \in \mathfrak{t}$ is *regular* if $\langle X, \alpha \rangle \neq 0$ for all roots α. Under this hypothesis, we can easily see that centralizer of X in $\mathfrak{g}$ is $\mathfrak{t}$. Hence the stabilizer T' of X in K (for the adjoint representation) will have T as the connected component of the identity. I believe that T' is actually connected, so that in fact $T' = T$. However, I'm not sure how to prove this. On the other hand, if we take $X \in \mathfrak{t}$ such that X generates a dense 1-parameter subgroup of T, then the stabilizer of X in K is definitely T (because T is its own centralizer). However, this is a stronger condition than X being regular.

Choose $X \in \mathfrak{t}$ such that the stabilizer of X is T. Then the (co)adjoint orbit through X is isomorphic to K/T. Note that K/T is a compact manifold of real dimension

$$\dim K - \dim T = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} T_{\mathbb{C}} = \dim \mathfrak{g} - \dim \mathfrak{t}_{\mathbb{C}} = 2|R_+|$$

6.1.1 Regular coadjoint orbits for $U(n)$

. Take $K = U(n)$. Recall that $\mathfrak{u}(n)$ is the vector space of skew-Hermitian matrices and we will somewhat arbitrary think of $\mathfrak{u}(n)^*$ as the vector space of Hermitian matrices. Let $\lambda \in \mathfrak{u}(n)^*$ be a diagonal matrix with distinct real entries $\lambda_1, \dots \lambda_n$. Then the G orbit through λ is exactly the set $\mathcal{H}_\lambda$ of all Hermitian matrices whose eigenvalues are $\lambda_1, \dots, \lambda_n$. Such a matrix is determined by the eigenspaces $L_1, \dots, L_n$ corresponding to these eigenvalues. Hence we obtain an isomorphism between $\mathcal{H}_\lambda$ and the space of all orthogonal decompositions of $\mathbb{C}^n$ into lines $L_1, \dots, L_n$.

Example 6.1. Take $n = 2$, then $K/T = \mathbb{CP}^1$, the manifold of all lines in $\mathbb{C}^2$.

Now, let us think about it from a different perspective. A *flag* in $\mathbb{C}^n$ is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ of $\mathbb{C}^n$ with $\dim V_i = i$. Note that an orthogonal decomposition $L_1, \dots, L_n$ determines a flag with $V_i = L_1 \oplus \cdots \oplus L_i$. Conversely, from a flag $V_0, \dots, V_n$ we can recover an orthogonal decomposition by setting L_i to be the orthogonal complement to V_{i-1} inside V_i. Let $Fl(\mathbb{C}^n)$ denote the set of all flags in $\mathbb{C}^n$. Hence we have a bijection $K/T \cong Fl(\mathbb{C}^n)$.

Note that a flag (unlike an orthogonal decomposition) is defined in complex linear algebra terms (no reference to a sesquilinear form) and so $Fl(\mathbb{C}^n)$

carries an action of $G = GL_n(\mathbb{C})$. It is easy to see that this action is transitive and is compatible with the above action of $U(n)$ on the decompositions of $\mathbb{C}^n$ into orthogonal lines. The *standard flag* is the flag

$$0 \subset \text{span}(v_1) \subset \text{span}(v_1, v_2) \subset \cdots \subset \text{span}(v_1, \dots, v_n) = \mathbb{C}^n$$

where $v_1, \dots, v_n$ is the standard basis for $\mathbb{C}^n$. The stabilizer of the standard flag in $\mathbb{C}^n$ is the group of invertible upper triangular matrices, which is denoted B. So we have a bijections $G/B \cong Fl(\mathbb{C}^n) \cong K/T \cong \mathcal{H}_\lambda$.

6.1.2 The moment map image

Return to the general case, with K a compact group and $\lambda \in \mathfrak{t}^*$, such that the stabilizer of λ is T. Let $\mathcal{H}_\lambda$ be the coadjoint orbit through λ.

We begin with the following observation about fixed points.

Lemma 6.2. *The set of fixed points for the T-action on K/T is the Weyl group $W = N(T)/T$.*

Proof. Suppose that $[k]$ is a fixed point. Then $[tk] = [k]$ for all $t \in T$. So $k^{-1}tk \in T$ for all $t \in T$. Hence k lies in the normalizer of T. $\square$

So, if we view $\mathcal{H}_\lambda = K/T$, then the fixed points will be given by $w\lambda$ for $w \in W$.

Example 6.3. When $K = U(n)$, $W = S_n$. The T-fixed points acting on the set of orthogonal decompositions $E_1 \oplus \cdots \oplus E_n = \mathbb{C}^n$ consists of those $(L_1, \dots, L_n)$ where each L_i is a coordinate line. So there is some permutation w such that $L_i = E_{w(i)}$ where E_j denotes the jth coordinate line in $\mathbb{C}^n$.

The advantage of viewing K/T as $\mathcal{H}_\lambda$ is that it allows us to consider the moment map. For our purposes, we define the *moment map* as the restriction of the projection $\pi : \mathfrak{k}^* \to \mathfrak{t}^*$ to $\mathcal{H}_\lambda$.

Note that for each $w \in W$, $\pi(w\lambda) = w\lambda$.

The coadjoint orbit $\mathcal{H}_\lambda$ is a symplectic manifold and $\pi : \mathcal{H}_\lambda \to \mathfrak{t}^*$ is a moment map in the sense of symplectic geometry. Applying the Atiyah-Guillemin-Sternberg convexity theorem from symplectic geometry, we obtain the following result.

Theorem 6.4. *The moment map image $\pi(\mathcal{H}_\lambda)$ is the convex hull of the set $\{w\lambda\}_{w \in W}$.*

So this moment map image looks something like a weight diagram. We will explain this coincidence later.

Example 6.5. Take $K = U(n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ regular. Then $\mathcal{H}_\lambda$ is the set of Hermitian matrices with eigenvalues $\lambda_1, \dots, \lambda_n$. The map $\pi : \mathcal{H}_\lambda \to \mathbb{R}^n$ is the map which takes a Hermitian matrix to its diagonal entries. Note that the image of π lands in the affine subspace given by the sum of the coordinates being equal to $\lambda_1 + \cdots + \lambda_n$ (since that is the trace of any matrix in $\mathcal{H}_\lambda$).

6.2 Borel subalgebras

Our goal is now to generalize the $K/T = G/B$ result to arbitrary compact groups. We begin with the construction of the Lie algebra of the Borel subgroup.

The *Borel subalgebra* $\mathfrak{b}$ and the *nilpotent subalgebra* $\mathfrak{n}$ of $\mathfrak{g}$ are defined as

$$\mathfrak{b} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha, \ \ \mathfrak{n} = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha.$$

Example 6.6. Take $G = GL_n(\mathbb{C})$ and choose the positive system

$$R_+ = \{e_i - e_j\}_{i<j}$$

The Borel subalgebra $\mathfrak{b}$ of $\mathfrak{gl}_n(\mathbb{C})$ consists of upper triangular matrices and the nilpotent subalgebra $\mathfrak{n} \subset \mathfrak{b}$ consists of strictly upper triangular matrices.

We will need some results from the elementary theory of Lie algebras. A Lie algebra is called *solvable* if there exists a filtration $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ such that $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$. A Lie algebra is called *nilpotent* if there exists a filtration $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$ such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$.

Proposition 6.7. *The Borel subalgebra $\mathfrak{b}$ is solvable and the nilpotent subalgebra $\mathfrak{n}$ is nilpotent.*

Proof. For the purposes of this proof, we set $\mathfrak{g}_0 = \mathfrak{t}_\mathbb{C}$ and declare $ht(0) = 0$.

An appropriate filtration is to set

$$\mathfrak{b}_k = \bigoplus_{\alpha \in R_+ \cup \{0\}, ht(\alpha) \geq k} \mathfrak{g}_\alpha$$

for $k \geq 0$. Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ and $ht(\alpha + \beta) = ht(\alpha) + ht(\beta)$, we see that $[\mathfrak{g}_k, \mathfrak{g}_k] \subset \mathfrak{g}_{k+1}$. □

The important result about solvable algebras is the following result which is known as Lie's theorem.

Theorem 6.8. *Let V be a finite-dimensional representation of a solvable Lie algebra $\mathfrak{b}$. Then there exists an eigenvector for the action of $\mathfrak{b}$. (In other words, there exists a non-zero vector $v \in V$ and a linear map $\lambda : \mathfrak{b} \to \mathbb{C}$ such that $Xv = \lambda(X)v$ for all $X \in \mathfrak{b}$.)*

Proof. Since $\mathfrak{b}$ is solvable, $[\mathfrak{b}, \mathfrak{b}] \neq \mathfrak{b}$. Let $\mathcal{H} = \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$; it is an abelian Lie algebra. Let $\mathfrak{a}$ be the preimage in $\mathfrak{b}$ of any codimension 1 subspace of $\mathcal{H}$. Then $\mathfrak{a}$ is solvable and it is also a Lie ideal in $\mathfrak{b}$.

By induction there is an eigenvector v for the action of $\mathfrak{a}$. Let $\lambda : \mathfrak{a} \to \mathbb{C}$ be the associated eigenvalue. Let $W \subset V$ be the λ-eigenspace for $\mathfrak{a}$.

We claim that W is $\mathfrak{b}$-invariant. Fix $Y \in \mathfrak{b}$. If $X \in \mathfrak{a}$ and $w \in W$,

$$XYw = YXw + [X, Y]w = \lambda(X)Yw + \lambda([X, Y])w.$$

So we need to show that $\lambda([X, Y]) = 0$. To see this, let $U = \operatorname{span}(w, Yw, Y^2w, \dots)$. This U is Y invariant and $\mathfrak{a}$ invariant. In fact, any $X \in \mathfrak{a}$ acts on U upper-triangularly with respect to the given basis with constant diagonal entries given by $\lambda(X)$. So the trace of X acting on U is $\lambda(X) \dim U$. Apply this reasoning to $[X, Y] \in \mathfrak{a}$. Since the trace of $[X, Y]$ is 0, this implies that $\lambda([X, Y]) = 0$ as desired.

Now pick $Y \in \mathfrak{b} \smallsetminus \mathfrak{a}$. Then $YW \subset W$ and so Y has an eigenvector in W. This will also be an eigenvector for $\mathfrak{a}$ and hence it will be an eigenvector for all of $\mathfrak{b}$, since $\mathfrak{a}$ is codimension 1 in $\mathfrak{b}$. □

From the previous theorem, we can immediately deduce the following corollary.

Corollary 6.9. *Let V be a finite-dimensional representation of a solvable Lie algebra $\mathfrak{b}$. There exists a basis for V such that every element of $\mathfrak{b}$ is represented by upper-triangular matrices.*

Note that this corollary implies that the only irreducible representations of a solvable Lie algebra are 1-dimensional.

In the case of our Borel subalgebra, we get a strengthening of the above theorem. First, note that $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ since if $X \in \mathfrak{g}_\alpha$, there exists $H \in \mathfrak{t}$ such that $\alpha(H) \neq 0$ and so $[H, X] = \alpha(H)X$ is a non-zero multiple of X, which implies that $X \in [\mathfrak{b}, \mathfrak{b}]$. Combined with the observation that the commutator of upper triangular matrices is strictly upper triangular, we obtain the following result.

Corollary 6.10. *Let V be a finite-dimensional representation of the Borel subalgebra $\mathfrak{b}$. Then there exists a basis for V such that every element of $\mathfrak{b}$*

is represented by upper-triangular matrices and every element of $\mathfrak{n}$ is represented by strictly upper triangular matrices. In particular, all elements of $\mathfrak{n}$ act nilpotently.

6.3 Iwasawa decomposition

We begin with the Iwasawa decomposition (also known as KAN decomposition) of $GL_n(\mathbb{C})$. The algebraic subgroup of upper triangular matrices in $GL_n(\mathbb{C})$ is called the *Borel subgroup* and is denoted B. The algebraic subgroup of upper triangular matrices in $GL_n(\mathbb{C})$ with 1s on the diagonal is called the *unipotent subgroup* and is denoted N.

Theorem 6.11. *Let $G = GL_n(\mathbb{C}), K = U(n)$ and let N be as above. Let T be the usual maximal torus of diagonal matrices and let A be the Lie subgroup of $T_{\mathbb{C}}$ consisting of diagonal matrices whose entries are positive real numbers.*

Then every element of g can be uniquely factored as kan with $k \in K$, $a \in A$, and $n \in N$. Moreover, multiplication gives a diffeomorphism $K \times A \times N \to G$.

Proof. Given a matrix $g \in GL_n(\mathbb{C})$ we get an ordered basis by looking at its columns. Applying the Gram-Schmidt process to this basis gives us an orthonormal basis, which gives us a unitary matrix k. When we apply Gram-Schmidt, we are effectively taking g and multiplying on the right by AN (N adds vectors to earlier vectors and A scales the vectors to make them unit length) and obtaining k. This shows existence.

The uniqueness follows from the fact that $AN \cap K = \{1\}$ and $A \cap N = \{1\}$.

The multiplication map is a diffeomorphism, since the Gram-Schmidt process gives a smooth inverse. □

Now, we want to do this for an arbitrary group. Let K be a compact Lie group and G be its complexification, as usual. We need to define N and A. A is easy to define since it just has to be a real Lie subgroup, while N is harder since it is an algebraic subgroup.

Let A be the Lie subgroup of $T_{\mathbb{C}}$ corresponding to the Lie algebra $i\mathfrak{t}$. If we think of $T_{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}/\Lambda$, then $A = i\mathfrak{t}/\Lambda$. On the other hand if we think of $T_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[X^*(T)]$, then

$$A = \{t \in T_{\mathbb{C}} : \mu(t) \in \mathbb{R}_{>0} \text{ for all } \mu \in X^*(T)\}.$$

The following result gives the Iwasawa decomposition for G.

Theorem 6.12. *There exists an algebraic subgroup B of G whose Lie algebra is $\mathfrak{b}$ and an algebraic subgroup N whose Lie algebra is $\mathfrak{n}$. B is the semidirect product of $T_{\mathbb{C}}$ and N, $B = T_{\mathbb{C}} \ltimes N$.*

Every element of G can be written uniquely as a product kan with $k \in K, a \in A, n \in N$. The multiplication map $K \times A \times N \to G$ is a diffeomorphism.

As before B is called the *Borel subgroup* of G and N the *unipotent subgroup.*

We begin with the following preliminary. Let G be a complex algebraic group and let $\mathfrak{g}$ be its Lie algebra. From the theory of Lie groups, we have an exponential map $\mathfrak{g} \to G$, but this map is not in general a map of algebraic varieties and hence not a map of algebraic group. However, for the unipotent subgroup of $GL_n(\mathbb{C})$, we do have an algebraic exponential map. This leads to the following result.

Lemma 6.13. *Let N denote the unipotent subgroup of $GL_n(\mathbb{C})$ and $\mathfrak{n}$ the nilpotent Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$. If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{n}$, then there exists an algebraic subgroup A of N whose Lie algebra is $\mathfrak{a}$.*

Proof. Define

$$\exp : \mathfrak{n} \to N, \quad X \mapsto I + X + \cdots + \frac{1}{(n-1)!} X^{n-1}$$

(note that $X^n = 0$, so this is actually the restriction of the usual exponential map). This is a polynomial map and an inverse is given by

$$\log : N \to \mathfrak{n}, \quad g \mapsto (g-1) - \frac{1}{2}(g-1)^2 + \cdots \pm \frac{1}{n-1}(g-1)^{n-1}$$

So we have an isomorphism of algebraic varieties.

Now let $\mathfrak{a} \subset \mathfrak{n}$ be a Lie subalgebra. Since exp is an isomorphism of varieties, then $A := \exp(\mathfrak{a})$ is a subvariety of N. So it remains to show that it is a subgroup. There are two possible approaches to showing this. One is to note that the exp map we have defined coincides with the usual exp map from Lie groups. Then a standard result from Lie groups shows that there exists a Hausdorf neighbourhood U of the identity in N such that if $g, h \in A$ and $g, h \in U$, then $gh \in A$. So consider the map $m : A \times A \to N$ given by multiplication. The intersection $V = U \cap A \times U \cap A$ is Hausdorf open in $A \times A$ and hence Zariski dense in $A \times A$. Since $V \subset m^{-1}(A)$, and $m^{-1}(A)$ is Zariski-closed, we conclude that $A \times A = m^{-1}(A)$ as desired.

Alternatively, we can give a purely algebraic proof using the Baker-Campbell-Hausdorff formula which shows that for $X, Y \in \mathfrak{n}$,

$$\exp(X)\exp(Y) = \exp(X + Y - \frac{1}{2}[X, Y] + \dots)$$

. $\square$

Proof. To prove this theorem, we will need to assume that K admits a faithful finite-dimensional algebraic representation, so it is actually an algebraic subgroup of $U(n)$ for some n. This is possible to prove. First, by considering $L^2(K)$ (which is a faithful, infinite-dimensional, unitary representation) and doing a little bit of functional analysis, it is possible to construct a finite-dimensional faithful representation. Next, we would have to show that it is algebraic.

Anyway, with the assumption, let us choose such a representation. Because it is a faithful representation of K, it will be a faithful representation of G.

This will also be a representation of $\mathfrak{b}$, the Borel subalgebra. By Corollary 6.10, we can choose a basis $v_1, \dots, v_n$ such that $\mathfrak{b}$ acts by upper-triangular matrices and $\mathfrak{n}$ acts by strictly upper-triangular matrices. Applying the Gram-Schmidt process, we can choose this basis to be orthonormal.

Note that $T \subset K$ acts semisimply and hence $\mathfrak{t}$ does as well. Since $\mathfrak{t} \subset \mathfrak{b}$, we see that $\mathfrak{t}$ must act by diagonal matrices. Thus when we choose the maximal torus T_n of $U(n)$ using the above basis, we see that T embeds into T_n and $T_{\mathbb{C}}$ embeds in $T_{\mathbb{C},n}$.

Let N_n denote the unipotent subgroup of $GL_n(\mathbb{C})$ and $\mathfrak{n}_n$ the nilpotent Lie algebra of $\mathfrak{gl}_n(\mathbb{C})$. Since $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{n}_n$, Lemma 6.13 shows us that there is an algebraic subgroup $N \subset N_n$ whose Lie algebra is $\mathfrak{n}$. Now $T_{\mathbb{C},n}$ normalizes N, so $T_{\mathbb{C}}$ does as well. Because $T_{\mathbb{C}}$ normalizes N, $B = T_{\mathbb{C}}N$ is a subgroup of G and is a semidirect product. It is an algebraic subgroup since it is the product of $T_{\mathbb{C}}$ and N.

Now, we must show the decomposition. First, note that the map $K \times A \times N \to G$ is a diffeomorphism onto its image, since $K \subset U(n)$, $A \subset A_n$ and $N \subset N_n$ and we have the Iwasawa decomposition (Theorem 6.11) for $GL_n(\mathbb{C})$.

It remains to show that it is surjective. To see this, note that

$$\dim_{\mathbb{R}}(K \times A \times N) = \dim_{\mathbb{R}}(K) + \dim_{\mathbb{R}}(T) + \dim_{\mathbb{R}}(N) = \dim_{\mathbb{C}} G + l + 2m$$

where $m = |R_+|$ and $l = \dim_{\mathbb{R}} T$. Since $l + 2m = \dim_{\mathbb{C}} \mathfrak{g}$, we see that $\dim_{\mathbb{R}}(K \times A \times N) = \dim_{\mathbb{R}} G$.

Hence it follows that the image is all of G (note that since the multiplication is a diffeomorphism, the image is closed in G). $\square$

6.4 Flag varieties

Our goal now is to use the above results to define flag varieties for any reductive group. We begin with some algebraic geometry background.

Up until now, we have only been speaking about affine varieties. We will now need to expand our attention to projective varieties. Let $\mathbb{P}^n = \mathbb{CP}^n$ denote complex projective space. It is the set of lines in $\mathbb{C}^{n+1}$ or equivalently non-zero elements of $\mathbb{C}^{n+1}$ modulo the equivalence relation of scaling. A point in $\mathbb{P}^n$ is written as $[x_0, \ldots, x_n]$. If f is a homogeneous polynomial in $n+1$ variables, then we can consider its vanishing set $V(f)$ in $\mathbb{P}^n$, which is defined as

$$V(f) = \{[x_0, \ldots, x_n] : f(x_0, \ldots, x_n) = 0\}.$$

A *projective variety* is the simultaneous vanishing set of a collection of homogeneous polynomials. A *quasi-projective variety* X is a Zariski-open subset of a projective variety (this means that there are two projective varieties Y, Z with $Z \subset Y$ and $X = Y \smallsetminus Z$). Note that a quasi-projective variety X has a Hausdorff topology inherited from the topology of $\mathbb{P}^n$. If X is smooth as a variety, then X acquires the structure of a smooth manifold with respect to the Hausdorff topology. A general result from complex algebraic geometry tell us that a quasi-projective variety is projective if and only if it is compact in the Hausdorff topology.

Let G be a connected algebraic group and let H be a closed subgroup. A *quotient* G/H is a complex variety X with a point $x \in X$ such that G acts on X (i.e. there is an action such that $G \times X \to X$ is a morphism of algebraic varieties) which is universal in the sense that for all varieties Y with G-action and point $y \in Y$ whose stablizer contains H, there is a unique G-equivariant map $X \to Y$, taking x to y.

The following result gives us the existence of quotients.

Theorem 6.14. *Let G be an algebraic group and let H be a closed subgroup. Then a quotient G/H exists as a smooth quasi-projective variety. Its underlying set is the right H-cosets in G.*

Now, we define the *flag variety* of G as the quotient G/B. By the above theorem it has the structure of a quasi-projective variety.

Theorem 6.15. *There exists a diffeomorphism $G/B \cong K/T$. The flag variety is compact and is a projective variety.*

Proof. We define a map $K/T \to G/B$ using the inclusion of K into G. The map is well-defined since $T \subset B$. We define a map backwards $G/B \to K/T$ using the Iwasawa decomposition. So if $g = kan \in G$, then $g \mapsto k$. This map is well-defined since $B = TAN$ (we already know that $B = T_{\mathbb{C}}N$ and we can see that $T_{\mathbb{C}} = TA$ by examining the case $T = U(1)^n$). This gives us the diffeomorphism $G/B \to K/T$.

Since K is compact, K/T is compact and so G/B is compact in the Hausdorff topology. So it is a projective variety. □

Example 6.16. The flag variety of $GL_n(\mathbb{C})$ has already been discussed. Let us consider as a second example, $G = SO_{2n}(\mathbb{C})$. An *isotropic flag* in $\mathbb{C}^{2n}$ is a flag

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n \subset V_{-n} \subset \cdots V_{-1} \subset \mathbb{C}^{2n}$$

such that for all i, $V_i^{\perp} = V_{-i}$.

We leave it as an exercise to check that the flag variety for $SO_{2n}(\mathbb{C})$ embed into the set of orthogonal flags in $\mathbb{C}^{2n}$. In fact the set of all orthogonal flags in $\mathbb{C}^{2n}$ is disconnected as a topological space and the flag variety for $SO_{2n}(\mathbb{C})$ is one connected component.

Recall that we already showed that if $\mathcal{H}$ was a regular coadjoint orbit, then $\mathcal{H} \cong K/T$. Thus we have shown that all regular coadjoint orbits are isomorphic to G/B.

More generally, a subgroup $P \subset G$ is called a *parabolic subgroup* if G/P is projective. There is a nice combinatorial theory of parabolic subgroups, but we will not develop it during this course. We will content ourselves with stating the following result which partially explains the importance of the Borel subgroup.

Theorem 6.17. *A subgroup $P \subset G$ is parabolic if and only if it contains a conjugate of B. Hence any projective variety with a transitive action of G is isomorphic to a quotient of the flag variety.*

6.5 Borel-Weil theorem

Recall that we showed that taking characters gives an injective map $Rep(K) \to \mathbb{Z}[X]^W$. Moreover, we showed that there was an isomorphism between $Rep(G)$ and $Rep_{alg}(K)$, the representation ring of algebraic representations of K. So our situation is as follows.

$$Rep(G) \subseteq Rep(K) \subseteq \mathbb{Z}[X]^W$$

Our goal now is to show that these two inclusions are equalities.

In the case of $K = U(n)$, we proved that these were equalities by explicitly constructing representations $V(\lambda)$ for every dominant weight λ. We will now do this for every group.

6.5.1 Dominant weights

We begin by defining the set of *dominant weights*, X_+, as

$$X_+ := \{\lambda \in X : \langle \alpha^\vee, \lambda \rangle \geq 0 \text{ for all positive coroots } \alpha^\vee\}$$

Then we define an ordering $\leq$ on X and X_+ by $\lambda \geq \mu$ if $\lambda - \mu$ lies in the root lattice Q.

For each $\lambda \in X_+$, we define the *monomial symmetric function* by

$$m_\lambda = \sum_{\mu \in W\lambda} e^\mu.$$

Lemma 6.18. *The monomial symmetric functions gives a basis $\Big(m_\lambda\Big)_{\lambda \in X_+}$ for $\mathbb{Z}[X]^W$.*

Proof. The statement is equivalent to showing that there is exactly one dominant weight in every Weyl orbit on the weight lattice.

Suppose that A is a Weyl orbit on the weight lattice. Choose $\mu \in A$ such that μ is maximal with respect to $\leq$. Then $s_\alpha \mu \leq \mu$ for all positive coroots α. Hence $\langle \alpha^\vee, \mu \rangle \geq 0$ for all positive roots $\alpha^\vee$. Thus μ is dominant. So every Weyl orbit contains a dominant weight.

Now, suppose that $\lambda \in X_+$. We would like to show that if $w \in W$ and $w\lambda$ is dominant, then $w\lambda = \lambda$. We will content ourselves to discussing the case when λ is regular (i.e. $\langle \alpha^\vee, \lambda \rangle \neq 0$ for all coroots $\alpha^\vee$). Then it follows from the fact that the Weyl group acts simply-transitively on the set of Weyl chambers. ☐

We say that a representation V of G has *highest weight* $\lambda \in X$, if

(i) $V_\lambda \neq 0$, and

(ii) For all $\mu \in X$ such that $V_\mu \neq 0$, $\mu \leq \lambda$.

In this circumstance, we call V a highest weight representation.

Lemma 6.19. *If V has highest weight λ, then λ is dominant.*

Proof. Let $\alpha \in R_+$ be a positive root. By Proposition 4.10 the character of V is invariant under W. Thus, since $V_\lambda \neq 0$, we see that $V_{s\alpha\lambda} \neq 0$. Since λ is the highest weight, this means that $\lambda \geq s_\alpha \lambda$. Hence $\lambda - s_\alpha \lambda \in Q_+$.

Now, $\lambda - s_\alpha \lambda = \langle \alpha^\vee, \lambda \rangle \alpha$ and hence we conclude that $\langle \alpha^\vee, \lambda \rangle \geq 0$. Since this holds for all positive roots, λ is dominant. □

6.5.2 Highest weight vectors

Let V be a representation of G and let $v \in V$ be a non-zero weight vector. We say that v is a *highest weight vector* if $gv = v$ for all $g \in N$.

Lemma 6.20. *v is a highest weight vector iff $Yv = 0$ for all $Y \in \mathfrak{n}$.*

Proof. Consider the exponential map as in the proof of Theorem 6.13. □

Let us now tie together the notion of highest weight representation and highest weight vector.

Lemma 6.21. *Let V be a representation of highest weight λ and let $v \in V_\lambda$ be a non-zero vector. Then v is a highest weight vector.*

Proof. Let $Y \in \mathfrak{g}_\alpha$ for some positive root α. Then $Yv \in V_{\lambda+\alpha}$. But since V has highest weight λ, $V_{\lambda+\alpha} = 0$. Thus, $Yv = 0$. Hence $Yv = 0$ for all $Y \in \mathfrak{n}$ as desired. □

The same method of proof allows us to prove the following result.

Lemma 6.22. *If V is a non-zero representation of G, then V contains a highest weight vector whose weight is dominant.*

Proof. Choose λ to be a maximal element of the set of weights of V. By this we mean, choose λ such that $V_\lambda \neq 0$ and if $\mu > \lambda$, then $V_\mu = 0$. (Since $\leq$ is a partial order, this is not the same thing as λ being the highest weight of the representation.) Then we pick $v \in V_\lambda$ non-zero and proceed as in the proof of the previous Lemma.

Note that λ is neccesarily dominant, since if not, then there exists a positive root α such that $\langle \alpha^\vee, \lambda \rangle < 0$, which implies that $s_\alpha \lambda > \lambda$, which is a contradiction since $V_{s_\alpha \lambda} \neq 0$ (by the fact that the character is Weyl-invariant). □

Corollary 6.23. *Let V be a representation. If $\dim V^N = 1$, then V is irreducible.*

Proof. Suppose that $V = V_1 \oplus V_2$ for two non-trivial subrepresentations V_1, V_2. Then $\dim V^N = \dim V_1^N + \dim V_2^N \geq 2$, a contradiction. □

The converse of this theorem is true as well, but we are not yet in a position to prove it.

If V is a vector space, then we write $\mathbb{P}(V)$ for the *projective space* of 1-dimensional subspaces of V. If $W \subset V$ is a subspace, then $\mathbb{P}(W) \subset \mathbb{P}(V)$ is a called a *projective subspace* of $\mathbb{P}(V)$.

Theorem 6.24. *Let V be a representation with a highest weight vector v. Then there is a map $G/B \to \mathbb{P}(V)$ taking $[g]$ to $[gv]$.*

Moreover, if V is irreducible, then the image of G/B in $\mathbb{P}(V)$ lies in no proper projective subspace.

Proof. We observe that G acts on $\mathbb{P}(V)$. The stabilizer of $[v]$ contains N since v is a highest weight vector. Also every element of T acts on v by a scalar and hence acts trivially on $[v]$. Thus, the stabilizer of $[v]$ contains B and so we have a map $G/B \to \mathbb{P}(V)$. □

Thus, for every representation, we can choose a highest weight vector and get a map from G/B to a projective space. So this motivates us to study maps from G/B to projective space.

6.5.3 Line bundles on projective varieties

We will need to review some results concerning line bundles on projective varieties.

A *line bundle* on a complex variety X is a variety L with a map $\pi : L \to X$, such that X admits a cover by open affine varieties U_α, along with isomorphisms $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$, compatible with the projections to U_α, such that the transition functions are linear along the fibres.

L is a called the *total space* of the line bundle. Each fibre $L_x := \pi^{-1}(x)$ carry the structure of a 1-dimensional complex vector space. Natural operations on 1-dimensional vector spaces go over to operations on line bundles as follows. If L is a line bundle, then there is a *dual line bundle* L^* whose fibre at x is L_x^*. If L, L' are two line bundles, then $L \otimes L'$ is the line bundle whose fibre at x is $L_x \otimes L'_x$.

A *section* of L is a map of varieties $s : X \to L$, such that $\pi \circ s$ is the identity. In other words, a section is a way of picking an element $s(x) \in L_x$ in each fibre. The set of all sections $\Gamma(X, L)$ forms a complex vector space.

We will now study the relationship between line bundles and maps to projective space. First, let us discuss line bundles on projective space. Let

V be a vector space. On $\mathbb{P}(V)$, we have two natural line bundles $\mathcal{O}(-1)$ and $\mathcal{O}(1)$, which are defined as follows. The fibre of $\mathcal{O}(-1)$ at a point $\ell \in \mathbb{P}(V)$ (here ℓ is a 1-dimensional subspace of V) is ℓ itself. So the total space of $\mathcal{O}(-1)$ is described as follows.

$$\mathcal{O}(-1) = \{(v, \ell) : v \in V, \ell \in \mathbb{P}(V), \text{ and } v \in \ell\}$$

The line bundle $\mathcal{O}(1)$ is defined as the dual of $\mathcal{O}(-1)$, so its fibre at ℓ is ℓ^*.

The following basic result from algebraic geometry will be quite important to us.

Proposition 6.25. *The spaces of sections of* $\mathcal{O}(1)$ *and* $\mathcal{O}(-1)$ *are given as follows.*

$$\Gamma(\mathbb{P}(V), \mathcal{O}(1)) = V^*, \quad \Gamma(\mathbb{P}(V), \mathcal{O}(-1)) = 0$$

Proof. We will content ourselves with explaining how to construct the map from V^* to $\Gamma(\mathbb{P}(V), \mathcal{O}(1))$. Given $\alpha \in V^*$, we get a section s_α of $\mathcal{O}(1)$ whose value at $\ell \in \mathbb{P}(V)$ is $\alpha|_\ell$, the restriction of α to $\ell \subset V$. It is not hard to check that this is an isomorphism. $\square$

Suppose that X is a projective variety and we are given a map $\rho : X \to \mathbb{P}(V)$. Then we can consider the pullback of $\mathcal{O}(1)$ under this map, which we denote by $\mathcal{O}_{X,\rho}(1)$ (or just $\mathcal{O}(1)$ for short). By definition, the fibre of $\mathcal{O}_{X,\rho}(1)$ at x is the fibre of $\mathcal{O}(1)$ at $\rho(x)$. We can pullback sections of $\mathcal{O}(1)$ to get a map $V^* \to \Gamma(X, \mathcal{O}(1))$.

We recall the following terminology from algebraic geometry. A *linear system* on X is a triple (L, V, i) where L is a line bundle on X, V is a vector space and $i : V^* \to \Gamma(X, L)$. A linear system is called *base-point free* if for all $x \in X$, there exists $\alpha \in V^*$ such that $i(\alpha)(x) \neq 0$.

We can now state the main result of this section.

Theorem 6.26. *Fix a vector space* V *and a projective variety* X*. There is a bijection between maps* $\rho : X \to \mathbb{P}(V)$ *and base-point free linear systems* (L, V, i) *on* X *(up to isomorphisms of* L*).*

Proof. Given $\rho : X \to \mathbb{P}(V)$ we have already seen how to produce the map $i : V^* \to \Gamma(X, \mathcal{O}(1))$. It is base-point free, since for all $x \in X$, $\rho(x)$ is a line in V and hence there exists $\alpha \in V^*$ which pairs non-trivially with this line.

Conversely, given (L, V, i), we can define a map $\rho : X \to \mathbb{P}(V)$ as follows. We think of points in $\mathbb{P}(V)$ as hyperplanes in V^* and define,

$$\rho(x) = \{\alpha \in V^* : i(\alpha)(x) = 0\}$$

Since our linear system is base-point free, $\rho(x)$ is actually a hyperplane in V^* for each $x \in X$.

It is easy to see that these constructions are inverse to each other. $\square$

A map $\rho : X \to \mathbb{P}(V)$ is called *non-degenerate* if its image is not contained in any projective hyperplane in $\mathbb{P}(V)$ (equivalently any proper projective subspace).

Proposition 6.27. *If $\rho : X \to \mathbb{P}(V)$ is non-degenerate, then $i : V^* \to \Gamma(X, \mathcal{O}(1))$ is an inclusion.*

Proof. Let $\alpha \in V^*$. If α gives 0 in $\Gamma(X, \mathcal{O}(1))$, then α restricts to 0 on $\rho(x) \subset V$ for all $x \in X$ and so $\rho(x) \in \alpha^\perp$ for all $x \in X$. Since ρ is non-degenerate, this implies that $\alpha = 0$. $\square$

6.5.4 Line bundles on flag varieties

We have seen that if V is an irreducible highest weight representation of G, then we get a non-degenerate map $G/B \to \mathbb{P}(V)$. By the above results, these correspond to base-point free linear systems on G/B with injective i. So in order to construct these representations, we should start by looking for line bundles on G/B. Actually these line bundles will be G-equivariant.

Let X be a projective variety with an action of a group G. A G-*equivariant line bundle* on X is a line bundle L with an action of G on the total space of L, which is linear on fibres and compatible with the action of G on X. In other words, for each $g \in G$ and $x \in X$, we are given a linear map $L_x \to L_{gx}$. In particular, the fibre L_x carries an action of the stabilizer of x in G.

If L is a G-equivariant, then there is an action of G on the space of sections $\Gamma(X, L)$, given by

$$(g \cdot s)(x) = gs(g^{-1}x)$$

for $g \in G, s \in \Gamma(X, L)$, and $x \in X$. A *G-equivariant linear system* is a linear system (L, V, i) where the line bundle L is G-equivariant and the vector space V carries a compatible action of G. We can extend Theorem 6.26 to the equivariant setting as follows.

Theorem 6.28. *Fix a vector space V and a projective variety X, both of which carry actions of a group G. There is a bijection between G-equivariant maps $\rho : X \to \mathbb{P}(V)$ and G-equivariant base-point free linear systems (L, V, i) on X (up to isomorphisms of L).*

In the setting of Theorem 6.24, the projective space $\mathbb{P}(V)$ will carry a G action coming from the G action on V and the map $G/B \to \mathbb{P}(V)$ will be G-equivariant.

Now, suppose that X has a transitive action of G. We pick $x \in X$ and let H be the stabilizer of x in G. Thus, $X = G/H$. Suppose that L is a G-equivariant line bundle on X. Then the fibre L_x is a 1-dimensional representation of H.

Proposition 6.29. *The above construction gives an equivalence of categories between the category of G-equivariant line bundles on G/H and the category of 1-dimensional representations of H.*

Proof. We have already seen how to go from a G-equivariant line bundle on X to a 1-dimensional representation of H. Conversely, given a 1-dimensional representation W of H, we define a line bundle L whose total space is

$$L = G \times_H W := \{[g, w]_H\}$$

where $[g, w]_H$ denotes the equivalence class for the equivalence relation given by the diagonal H action. In other words, $[g, w]_H = [gh^{-1}, hw]_H$ for all $h \in H$.

There is a map $L \to G/H$ given by $[g, w]_H \mapsto [g]$. L is a G-equivariant line bundle on G/H where G acts on L via $g'[g, w] = [g'g, w]$. □

The above results suggest that we focus on the 1-dimensional representations of B.

Let $\lambda \in X$ be a weight. So λ defines a 1-dimension representation $\mathbb{C}(-\lambda)$ of T. We can extend this to a 1-dimensional representation of B by using the map $B \to B/N = T$. These are actually all the 1-dimensional representations of B, since $N = [B, B]$ (in fact, these are all the irreducible representations of B). We let $L(\lambda) = G \times_B \mathbb{C}(-\lambda)$ denote the associated line bundle on G/B.

Example 6.30. Let $G = GL_n$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a dominant weight. Recall that a point in the flag variety G/B is a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$. The fibre of $L(\lambda)$ at this point $V_\bullet$ is

$$(V_1/V_0)^{\otimes -\lambda_1} \otimes \cdots \otimes (V_n/V_{n-1})^{\otimes -\lambda_n}.$$

To see this, just note if $V_\bullet = E_\bullet$ is the standard flag, then B acts on this line with via the representation $-\lambda$.

We are now in a position to state the Borel-Weil theorem.

Theorem 6.31. *If λ is a dominant weight, then $V(\lambda) := \Gamma(X, L(\lambda))^*$ is an irreducible representation of G with highest weight λ. These are all the irreducible representations of G.*

If λ is not dominant, then $\Gamma(X, L(\lambda))^* = 0$

Example 6.32. Take $G = SL_2$, so $G/B = \mathbb{P}^1$. The line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$ carries a natural G-equivariant structure and hence it corresponds via 6.29 to the 1-dimensional representation of B on the line over the point $[1, 0] \in \mathbb{P}^1$. This line is precisely the span of $(1, 0)$ and hence it is the representation of B coming from the weight 1. Thus $\mathcal{O}(-1) = L(-1)$.

More generally $L(n) = \mathcal{O}(n)$, the nth tensor power of $\mathcal{O}(1)$. A generalization of Proposition 6.25 tells us that $\Gamma(\mathbb{P}^1, \mathcal{O}(n)) \cong Sym^n C^2$, the space of homogeneous polynomials in two variables of degree n.

Thus, in the case of SL_2, the Borel-Weil theorem is telling us that the irreducible representation of SL_2 are $Sym^n \mathbb{C}^2$.

We begin by proving part of the theorem.

Theorem 6.33. *Let V be an irreducible representation of G. Then there exists a dominant weight λ such that $V \cong V(\lambda)$.*

Proof. By Lemma 6.22, V has a highest weight vector v. Let λ be the weight of v. Hence we get a non-degenerate G-equivariant map

$$G/B \to \mathbb{P}(V), \ [g] \mapsto [gv]$$

from the flag variety to projective space (Theorem 6.24). Let L denote the pullback of $\mathcal{O}(1)$ under this map. Hence we get an injective G-equivariant map $V^* \to \Gamma(G/B, L)$ (Proposition 6.27).

L is a G-equivariant line bundle and so it is determined by the action of B on the fibre at [1] (Proposition 6.29). By the definition of $\mathcal{O}(1)$, the fibre $L_{[1]}$ is $\mathrm{span}(v)^*$ and hence it is the representation $\mathbb{C}(-\lambda)$ of B. Thus $L = L(\lambda)$. Collecting this altogether we obtain a injective G-equivariant map

$$V^* \to \Gamma(G/B, L(\lambda)) = V(\lambda)^*$$

or equivalently a surjective G-equivariant map

$$V(\lambda) \to V$$

Since $V(\lambda)$ is irreducible, this map is an isomorphism. □

Another way to prove this result is to think about characters and to note that the characters χ_λ of $V(\lambda)$ give a basis for $\mathbb{Z}[X]^W$.

6.6 Bruhat decomposition

In order to prove the Borel-Weil theorem, we will first need to understand the Bruhat decomposition.

First, we introduce some notation. Let N_- be the opposite unipotent subgroup of G. It is constructed just like N, except that its Lie algebra is $\mathfrak{n}_- = \oplus_{\alpha \in R_-} \mathfrak{g}_\alpha$. Then for $w \in W$, we set $N_w = N \cap wNw^{-1}$.

Note that the Lie algebra of N_w is

$$\mathfrak{n}_w = \mathfrak{n} \cap w\mathfrak{n}_- w^{-1} = \oplus_{\alpha \in R_+ \cap wR_-} \mathfrak{g}_\alpha$$

We define the *length* $l(w)$ of w to be the size of the set $R_+ \cap wR_-$. This is the same thing as the minimal length of an expression $w = s_{i_1} \dots s_{i_k}$ where each s_{i_j} is a simple reflection.

The Weyl group has a unique longest element, denoted w_0, which has the property that $w_0 R_- = R_+$. So $N_{w_0} = N$.

We have the exponential map $exp : \mathfrak{n} \to N$ which is an isomorphism of varieties and it restricts to give an isomorphism $\mathfrak{n}_w \to N_w$. In particular, we see that N_w is isomorphic as a variety to $\mathbb{C}^{l(w)}$.

In order to state the Bruhat decomposition, let us pick a representative $\tilde{w}$ for each $w \in W$.

Theorem 6.34. *Every element of g can be written as $g = b\tilde{w}b'$ for $b \in N_w$, $b' \in B$ and $w \in W$. This expression is unique in two ways.*

First, if

$$b_1 \tilde{w}_1 b_1' = b_2 \tilde{w}_2 b_2'$$

for $b_1, b_2 \in B$, then $w_1 = w_2$.

Second, if

$$b_1 \tilde{w} b_1' = b_2 \tilde{w} b_2'$$

for $b_1, b_1' \in N_w$,$b_2, b_2' \in B$, then $b_1 = b_1'$ and $b_2 = b_2'$.

Proof. We will prove this theorem for GL_n only. Let us start by giving a desription of N_w. We have that N_w consists of those upper-triangular matrices with 1s on the diagonal, whose i, j entry is 0 if $w^{-1}(i) < w^{-1}(j)$.

Now take $g \in GL_n$. We begin by looking at the first column of g. Take the lowest non-zero entry, say that it is in the jth row. By multiplying g on the right by an element of T we make this non-zero entry equal to 1. Then by multiplying g on the right by an element of$N$, we perform rightward column operations to make the rest of the j row equal to 0. Then we add the $j$row to earlier rows by left multiplying by an element of N and make the rest of the first column equal to 0. Continuing in this way, we reduce our element of G to a permutation matrix $\tilde{w}$.

This shows us that $G = \sqcup NwB$. To get the more refined statement, we just note that we can restrict the row operations that we perform to those that lie in N_w. $\square$

The geometric counterpart of the Bruhat decomposition is the decomposition of the flag variety into Schubert cells. Recall that for each $w \in W$, we can consider $w \in G/B$ and that these are precisely the T-fixed point on the flag variety. Now given $w \in W$, we call $X_w^0 = Bw$, the *Schubert cell* for w. Its closure $X_w = \overline{X_w^0}$ is called the *Schubert variety* for w.

As a consequence of the Bruhat decomposition, we have the following result.

Theorem 6.35. *There is a decomposition $G/B = \sqcup X_w^0$. Moreover, $N_w \cong X_w^0$ and in particular, $X_w^0 \cong \mathbb{C}^{l(w)}$.*

As a consequence of this theorem, $X_{w_0}^0$ is an open dense subvariety of G/B. It is called the *big cell* and it is isomorphic to N.

6.7 Schubert cells and varieties for GL_n

In the case of $G = GL_n$, it is possible to give a linear algebra description of the Schubert cells. Let $V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$) and $W_0 \subset W_1 \subset \cdots \subset W_n = \mathbb{C}^n$ be two flags. Then we can consider the matrix $A = (\dim V_i \cap W_j)_{1 \leq i,j \leq n}$ where we record all dimensions of intersections between the two flags.

Lemma 6.36. *For each pair of flags $V_\bullet, W_\bullet$, there exists a permutation w such that the matrix of intersection dimensions is given by $A_{ij} = \#$ of $k < j$ such that $w(k) < i$. In other words, A_{ij} equals the number of 1 left and right from (i,j) in the permutation matrix for w.*

Under the circumstance of the Lemma, we say that $V_\bullet$ and $W_\bullet$ are in relative position given by the permutation w.

Let us write $E_\bullet$ for the standard flag and $wE_\bullet$ for the w-permuted standard flag. So

$$wE_1 = \operatorname{span}(e_{w(1)}) \subset wE_2 = \operatorname{span}(e_{w(1)}, e_{w(2)}) \subset \cdots \subset \mathbb{C}^n$$

From the definition, we immediately see that $E_\bullet$ and $wE_\bullet$ are in relative position w.

Proposition 6.37. *X_w^0 consists of those flags $V_\bullet$ such that $E_\bullet$ and $V_\bullet$ are in relative position w.*

Proof. First we will show that if $V_\bullet$ lies in X_w^0, then it is in relative position w with respect to $E_\bullet$. To see this, we know that $V_\bullet = bwE_\bullet$ for some $b \in B$. Then

$$\dim E_i \cap V_j = \dim E_i \cap bwE_j = \dim bE_i \cap bwE_j = \dim E_i \cap wE_j$$

since E_i is invariant under b, and b is invertible. Thus we see that $E_\bullet$ and $V_\bullet$ are in relative position w.

Conversely, suppose that $V_\bullet$ has relative position w with respect to $E_\bullet$. Then since $G/B = \sqcup_{w\in W} X_w^0$, we see that $V_\bullet \in X_u^0$ for some u. By the above analysis, we see that $E_\bullet$ and $V_\bullet$ are in relative position u. Hence $u = w$ and we are done. $\square$

We can also give a linear algebra description of the closure of a Schubert cell.

Proposition 6.38.

$$X_w = \{V_\bullet \in G/B : \dim E_i \cap V_j \geq A_{ij}^w\}$$

where

$$A_{ij}^w = \#\ \textit{of}\ k < j\ \textit{such that}\ w(k) < i$$

6.8 Proof of the Borel-Weil theorem

We begin with the following lemma.

Lemma 6.39. *Let $w \in W$ and let $\lambda \in X$. The action of T on the fibre of $L(\lambda)$ over the point $w \in G/B$ is by weight $-w^{-1}\lambda$.*

In particular, T acts on the fibre over w_0 by $\lambda^* := -w_0\lambda$. Note that λ^* is dominant if and only if λ^* is dominant.

Proof. By definition the fibre over w is $\{[\tilde{w}, s] : s \in \mathbb{C}(-\lambda)\}$. Hence for $t \in T$,

$$t[\tilde{w}, s] = [t\tilde{w}, s] = [\tilde{w}(w^{-1}tw), s] = [\tilde{w}, (-\lambda)(w^{-1}tw)s] = (-w^{-1}\lambda)(t)[\tilde{w}, s]$$

as so the result follows. $\square$

Example 6.40. Consider the case $G = GL_n$. Then the point w in G/B is the flag $wE_\bullet$. Hence applying Example 6.30, the fibre of $L(\lambda)$ at $wE_\bullet$ is

$$\operatorname{span}(e_{w(1)})^{-\lambda_1} \otimes \cdots \otimes \operatorname{span}(e_{w(n)})^{-\lambda_n}$$

on which T acts by

$$(t_1, \dots, t_n) \mapsto t_{w^{-1}(1)}^{-\lambda_1} \cdots t_{w^{-1}(n)}^{-\lambda_n}$$

which is the weight $-w\lambda$.

Given a section $s \in \Gamma(G/B, L(\lambda))$, we can restrict s to the big cell $X_{w_0}^0 \cong N$. This idea will be the key to everything which follows.

Proposition 6.41. $\dim \Gamma(G/B, L(\lambda))^N \leq 1$. *Moreover, if* $s \in \dim \Gamma(G/B, L(\lambda))^N$, *then* s *has weight* λ^*.

Proof. Let $s \in \Gamma(G/B, L(\lambda))^N$. Then s is determined by its restriction to the big cell $X_{w_0}^0$, since the big cell is dense. Since N acts transitively on the big cell, s is determined by its restriction to any one point in $X_{w_0}^0$. In other words, for any $x \in X_{w_0}^0$, the map

$$\Gamma(G/B, L(\lambda))^N \to L(\lambda)_x$$

is injective. Since the right hand side is one-dimensional, the left hand side is at most 1-dimensional.

Moreover, if we choose $x = w_0$, then since the above restriction map is T-equivariant, we see that s has weight λ^*. $\square$

From this, we apply Corollary 6.23 to obtain the following.

Corollary 6.42. *If* $\Gamma(G/B, L(\lambda)) \neq 0$, *then* $\Gamma(G/B, L(\lambda))$ *is an irreducible representation.*

It remains to show that the space of sections in non-zero if and only if λ is dominant. Let us first see why λ being dominant is a necessary condition.

From the proof of 6.41, we have a T-equivariant inclusion

$$\Gamma(G/B, L(\lambda)) \hookrightarrow \Gamma(X_{w_0}^0, L(\lambda))$$

Since $X_{w_0}^0$ is an affine space, the right hand side is very easy to understand. (Note that the above map is not G-equivariant, since G does not act on the right hand side, as it does not preserve $X_{w_0}^0$. On the other hand, it is possible to define a $\mathfrak{g}$-action on $\Gamma(X_{w_0}^0, L(\lambda))$ such that this map is $\mathfrak{g}$-equivariant.)

Proposition 6.43. *As a T-representation, $\Gamma(X^0_{w_0}, L(\lambda))$ is isomorphic to $Sym(n^*) \otimes \mathbb{C}(\lambda^*)$*

Proof. We can use the transitive N action on $X^0_{w_0}$ to trivialize the line bundle $L(\lambda)$ over $X^0_{w_0}$, as follows. We have an isomorphism of T-equivariant line bundles

$$N \times \mathbb{C}(\lambda^*) \to L(\lambda)|_{X^0_{w_0}}$$

by $(n, s) \mapsto ns$.

Thus we can compute the space of sections of the trivial line bundle on N (which is just $\mathcal{O}(N)$) and then tensor with the 1-dimensional T-representation $\mathbb{C}(\lambda^*)$.

Since $N \cong \mathfrak{n}$, we see that $\mathcal{O}(N) \cong \mathrm{Sym}(n^*)$ and so the result follows. $\square$

In particular, the weights of $\Gamma(X^0_{w_0}, L(\lambda))$ are all of the form $\lambda^* - \mu$, where $\mu \in Q_+$. In fact we see that $\dim \Gamma(X^0_{w_0}, L(\lambda))_{\lambda^*-\mu}$ is the number of ways to write μ as a sum of positive roots. This number is called the *Kostant partition function* of μ and is denote $\mathrm{kpf}(\mu)$.

Because of the inclusion

$$\Gamma(G/B, L(\lambda)) \hookrightarrow \Gamma(X^0_{w_0}, L(\lambda))$$

we see that the weights of $\Gamma(G/B, L(\lambda))$ are all of the form $\lambda^* - \mu$ for $\mu \in Q_+$.

Corollary 6.44. *If $\Gamma(G/B, L(\lambda)) \neq 0$, then it has highest weight λ^*. If λ is not dominant, then $\Gamma(G/B, L(\lambda)) = 0$.*

Proof. Suppose that $\Gamma(G/B, L(\lambda)) \neq 0$, then it has a non-zero N-invariant vector, which is of weight λ^* by Proposition 6.41. Combining this with the above description of the weights of $\Gamma(G/B, L(\lambda))$, we see that $\Gamma(G/B, L(\lambda))$ has highest weight λ^*. Hence by Lemma 6.19, we conclude that λ^* (and hence λ) is dominant. $\square$

From the above analysis, we also see that

$$\dim \Gamma(G/B, L(\lambda))_{\lambda^*-\mu} \leq \mathrm{kpf}(\mu)$$

for $\mu \in Q_+$.

Dualizing, we see that the weights of $\Gamma(G/B, L(\lambda))^*$ are all of the form $\mu + w_0\lambda$ for $\mu \in Q_+$ and that

$$\dim \Gamma(G/B, L(\lambda))^*_{\mu+w_0\lambda} \leq \mathrm{kpf}(\mu).$$

Since the dimensions of the weight spaces of G-representations are W-invariant, we see that all the weights of $\Gamma(G/B, L(\lambda))^*$ are of the form $\lambda - \mu$ for $\mu \in Q_+$ and that

$$\dim \Gamma(G/B, L(\lambda))^*_{\lambda-\mu} \leq \mathrm{kpf}(\mu).$$

More generally, we get a bound on the weight spaces for each $w \in W$ as follows.

$$\dim \Gamma(G/B, L(\lambda))^*_{\nu} \leq \mathrm{kpf}(\mu), \text{ if } \nu = w\lambda - w\mu \text{ and } \mu \in Q_+$$

6.8.1 Existence of a section

To complete the proof of the Borel-Weil theorem, it suffices to prove that if λ is dominant, then $\Gamma(G/B, L(\lambda)) \neq 0$.

More precisely, we will prove the following statement.

Theorem 6.45. *Let λ be a dominant weight. Let s be a non-zero N-invariant section of $L(\lambda)$ over $X^0_{w_0}$. Then s extends to a section of $L(\lambda)$ over all of G/B.*

Proof. We begin with the case of $G = SL_2$, so $G/B = \mathbb{P}^1$. We think of $\mathbb{P}^1$ as $0 \cup \mathbb{C}^\times \cup \infty$ and let us set things us so that $\infty, 0$ are T-fixed points, with $\infty = 1$, $0 = w_0$ as Weyl group elements.

We write $\lambda = n \in \mathbb{N}$. Then $L(\lambda) = \mathcal{O}(n)$.

We can think of the line bundle $\mathcal{O}(n)$ being glued from trivial line bundles over $\mathbb{P}^1 \setminus \infty$ and $\mathbb{P}^1 \setminus 0$. More explicitly, we write

$$\Gamma(\mathbb{P}^1 \setminus \infty, \mathcal{O}(n)) = \mathbb{C}[z], \quad \Gamma(\mathbb{P}^1 \setminus 0, \mathcal{O}(n)) = z^n \mathbb{C}[z^{-1}]$$

The section s above can be then taken to be the constant polynomial $1 \in \mathbb{C}[z]$. Thus it extends over $\mathbb{P}^1$.

Now, our strategy is to reduce to the SL_2 case. We will need the following result from algebraic geometry, which is sometimes known as Hartog's theorem.

Theorem 6.46. *Let X be a normal irreducible variety and let L be a line bundle on X. Let $U \subset X$ be an open subset of X such that $\dim(X \setminus U) \leq \dim X - 2$. Let $s \in \Gamma(U, L)$ be a section of L over U. Then s extends to a section of L over X.*

We apply this theorem where

$$U = X^0_{w_0} \cup \bigcup_{i \in I} X^0_{s_i w_0}.$$

Thus it suffices to check that s extends to a section of $L(\lambda)$ over $X^0_{s_i w_0}$ for each i.

Fix $i \in I$. Consider $Y := X^0_{w_0} \cup X^0_{w_0 s_i}$. Recall that we have a root SL_2 subgroup $\Psi_i : SL_2 \to G$. Note that $\Psi_i(N)$ (which is the same thing as N_{s_i}) stabilizes $s_i w_0$. This allows us to define a map

$$\mathbb{P}^1 = SL_2/B \to Y, \text{ by } [g] \mapsto \psi(g) w_0 s_i$$

The image of $\mathbb{P}^1$ under this map (which we will denote by $\mathbb{P}^1_i$) is precisely the stabilizer for the action of $N_{w_0 s_i}$ on Y.

Since $X^0_{w_0}$ is an orbit of N and $X^0_{w_0 s_i}$ is an orbit of $N_{w_0 s_i}$, we see that there is an isomorphism of varieties

$$N \times_{N_{s_i}} \mathbb{P}^1_i \to Y$$

given by $[n, a] \mapsto na$.

Now, we have our line bundle $L(\lambda)$ on Y and our N-invariant section s over $X^0_{w_0}$. We can restrict the line bundle $L(\lambda)$ to $\mathbb{P}^1_i$ and restrict our section s to $\mathbb{P}^1_i \smallsetminus \infty$. By the equivariance and the above isomorphism, it suffices to check that s extends over ∞ in $\mathbb{P}^1$.

The restriction of $L(\lambda)$ to $\mathbb{P}^1_i$ is SL_2-equivariant (via Ψ_i) and thus is determined by the action of the Borel $B \subset SL_2$ on the fibre over $w_0 s_i$. By Lemma 6.39, $\mathbb{C}^\times \subset B$ acts on the fibre by the weight

$$\langle \alpha_i^\vee, -(w_0 s_i)^{-1} \lambda \rangle = -\langle \alpha_i^\vee, \lambda^* \rangle,$$

since $\Psi|_{\mathbb{C}^\times}$ is given by the coweight $\alpha_i^\vee$. Since λ^* is dominant, $-\langle \alpha_i^\vee, \lambda^* \rangle \leq 0$ and thus $L(\lambda)|_{\mathbb{P}^1_i}$ is the line bundle $\mathcal{O}(k)$ for some $k \geq 0$. Since $s|_{\mathbb{P}^1 \smallsetminus 0}$ is N invariant (for $N \subset SL_2$), this section extends by the SL_2 version of this theorem, proved above. Thus s extends over all of Y and so the result follows. $\square$

7 The Weyl character formula

Now that we have defined the irreducible representations $V(\lambda)$, the next step is to discuss their characters. Our first main result in this direction is the Weyl character formula.

7.1 The formula

There are a few equivalent ways to formulate the Weyl character formula. Let χ_λ denote the character of $V(\lambda)$. Define $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ to be half the sum of the positive roots.

Theorem 7.1.

$$\chi_\lambda = \sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}\frac{1}{\prod_{\alpha\in R_+}1-e^{-\alpha}}$$

Here $\frac{1}{1-\prod_{\alpha\in R_+}e^{\alpha}}$ is an infinite sum which is equal to the character of $\operatorname{Sym}\mathfrak{n}^* = \mathcal{O}(N)$, which entered our story earlier.

Let WCF_λ denote the right hand side of the Weyl character formula.

We can rewrite WCF_λ as

$$WCF_\lambda = \sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}\sum_{\mu\in Q_+}\operatorname{kpf}(\mu)e^{-\mu} \tag{3}$$

which "converges" in the sense that for any ν the coefficient of e^ν is given by a finite sum over the Weyl group.

It is not immediately clear that this expression is W-invariant however. To see this, we use the following lemma.

Lemma 7.2.

$$e^\rho\prod_{\alpha\in R_+}1-e^{-\alpha}$$

transforms by $(-1)^{l(w)}$ under the action of w.

Proof. We can write

$$\begin{aligned} e^\rho\prod_{\alpha\in R_+}1-e^{-\alpha} &= \prod_{\alpha\in R_+}e^{\alpha/2}\prod_{\alpha\in R_+}1-e^{-\alpha} \\ &= \prod_{\alpha\in R_+}e^{\alpha/2}-e^{-\alpha/2} \end{aligned}$$

If $w\in W$, we see that

$$w(\prod_{\alpha\in R_+}e^{\alpha/2}-e^{-\alpha/2}) = \prod_{\alpha\in R_+}e^{w\alpha/2}-e^{-w\alpha/2} = (-1)^{l(w)}\prod_{\alpha\in R_+}e^{\alpha/2}-e^{-\alpha/2}$$

and so the result follows. ∎

From the Lemma, we deduce the following corollaries.

Corollary 7.3. *(i) WCF_λ is Weyl invariant.*

(ii) WCF_λ is supported on the convex hull of $W\lambda$ and the coefficient of e^λ is 1. In other words, we have

$$WCF_\lambda = m_\lambda + \sum_{\mu \in X_+, \mu < \lambda} a_{\lambda\mu} m_\mu,$$

for some $a_{\lambda\mu} \in \mathbb{Z}$.

(iii)

$$\prod_{\alpha \in R_+} 1 - e^{-\alpha} = \sum_{w \in W} (-1)^{l(w)} e^{w\rho - \rho}$$

Proof. (i) Since WCF_λ is the ratio of two expressions which both transform by $(-1)^{l(w)}$ under a Weyl group element w, we see that WCF_λ is Weyl-invariant.

(ii) From (i), WCF_λ can be written as a sum of the monomial symmetric functions m_μ, where μ ranges over X_+. From (3), we see that the coeffiecient of e^λ in WCF_λ is 1 and that the coeffiecient of e^μ is 0, unless $\mu \leq \lambda$. The result follows.

(iii) By definition WCF_0 is the ratio of the left hand side and the right hand side. However, by (ii), $WCF_0 = 1$.

□

Using part (iii) of the above corollary, we can rewrite the Weyl character formula as

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{l(w)} e^{w\rho}} \tag{4}$$

There are various different proofs of the Weyl character formula. We will give some details about a few of them.

7.2 Compact groups proof

Recall that we defined the inner product $\langle , \rangle$ on the space $C^\infty(K)$ of complex-valued smooth functions on a compact group K by

$$\langle f_1, f_2 \rangle = \int_K \overline{f_1} f_2 dg$$

If V, W were irreducible representations of a compact group K, then we saw that their characters χ_V, χ_W were orthonormal with respect to this inner product. We also saw that since χ_V, χ_W were class functions, they were completely determined by their restriction to the maximal compact $T \subset K$. However, we did not previously study the relationship between the inner product and the restriction to T. This relationship follows from the Weyl integration formula.

Theorem 7.4. *Let f be a class function on K. Then*

$$\int_K f dg = \frac{1}{|W|} \int_T f|_T \prod_{\alpha \in R} 1 - e^{\alpha} dt$$

Proof. We give a sketch of the proof.

Define a map

$$\pi : T \times K/T \to K,\ (t, [k]) \mapsto ktk^{-1}$$

Note that π is generically a $|W| : 1$ cover, so

$$\int_K f dg = \frac{1}{|W|} \int_T \pi^*(f) \pi^*(dg)$$

where $\pi^*(f)$ denotes the pullback of f to $T \times K/T$, and $\pi^*(dg)$ denotes the pullback of the top form dg to $T \times K/T$.

Since f is a class function, $\pi^*(f)$ is just the function on $T \times K/T$ which is the restriction of f to T (and constant along the K/T factor). Similarly $\pi^*(dg)$ is constant along the K/T factor. In fact at a point $(t, [k])$,

$$\pi^*(dg) = \det(T_{t,[1]}\pi) dt d[k],$$

where

$$T_{t,[1]}\pi : T_t T \oplus T_{[1]} K/T \to T_t K$$

To compute this determinant, let us identify $T_t T$ with $\mathfrak{t}$ and $T_t K$ with $\mathfrak{k}$ by left multiplication. Then we have

$$T_{t,[1]}\pi : \mathfrak{t} \oplus \mathfrak{k}/\mathfrak{t} \to \mathfrak{k}$$

and it makes sense to talk about its determinant as a number, since it is a map $\det(\mathfrak{k}) = \det(\mathfrak{t}) \otimes \det(\mathfrak{k}/\mathfrak{t}) \to \det(\mathfrak{k})$.

Now, to understand $T_{t,[1]}$, we let $X \in \mathfrak{t}$ and $Y \in \mathfrak{k}/\mathfrak{t}$ and compute

$$\pi(t + \varepsilon t X, 1 + \varepsilon Y) = (1 + \varepsilon Y)(t + \varepsilon t X)(1 - \varepsilon Y) = t + \varepsilon t(X + Y - t^{-1} Y t)$$

and thus

$$T_{t,[1]}\pi(X,Y) = X + Y - t^{-1}Yt$$

To compute $\det(T_{t,[1]}\pi)$, we complexify $\mathfrak{k}$ to $\mathfrak{g}$ and then with respect to the decomposition $\mathfrak{g} = \mathfrak{t}_{\mathbb{C}} \oplus (\mathfrak{n} \oplus \mathfrak{n}_-)$, $T_{t,[1]}\pi$ is given by the matrix

$$\begin{bmatrix} I & 0 \\ 0 & I - ad_t \end{bmatrix}$$

where ad_t denotes the adjoint action of t on $\mathfrak{g}$. Thus we conclude that

$$\det(T_{t,[1]}\pi) = \prod_{\alpha \in R} \det(1 - \alpha(t))$$

which gives the desired result. $\square$

Motivated by this theorem, we introduce an inner product $\langle , \rangle_K$ on the W-invariant smooth functions on T by

$$\langle f_1, f_2 \rangle_K = \int_T \overline{f_1} f_2 \prod_{\alpha \in R} 1 - e^{\alpha} dt$$

and we note that by the above reasoning, $\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu}$.

Proposition 7.5. *The characters χ_λ of the irreducible representations are uniquely determined by the following facts.*

(i) $\chi_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu}$ *for* $a_{\lambda\mu} \in \mathbb{Z}$.

(ii) $\{\chi_\lambda\}$ *forms an orthonormal set with respect to* $\langle , \rangle_K$.

Proof. This follows from the Gram-Schmidt process applied to $\{m_\lambda\}_{\lambda \in X_+}$. $\square$

Thus in order to prove the Weyl character formula, it suffices to prove the following result.

Lemma 7.6. *The set $\{WCF_\lambda\}_{\lambda \in X_+}$ forms an orthonormal set with respect to $\langle , \rangle_K$. In other words, $\langle WCF_\lambda, WCF_\mu \rangle_K = \delta_{\lambda\mu}$*

Proof. We apply the Weyl integration formula to get

$$\langle WCF_\lambda, WCF_\mu \rangle_K = \frac{1}{|W|} \int_T \overline{WCF_\lambda} WCF_\mu \prod_{\alpha \in R} 1 - e^{\alpha} dt$$

Since $\overline{1 - e^{\alpha}} = 1 - e^{-\alpha}$, the "extra factor" coming from the Weyl integration formula cancels the denominators in WCF_λ and WCF_μ and we deduce that

$$\begin{aligned}\langle WCF_\lambda, WCF_\mu\rangle_K &= \frac{1}{|W|}\int_T \sum_{w\in W}(-1)^{l(w)}\overline{e^{w(\lambda+\rho)-\rho}}\sum_{w'\in W}(-1)^{l(w')}e^{w'(\mu+\rho)-\rho}dt \\ &= \frac{1}{|W|}\sum_{w,w'}(-1)^{l(w)+l(w')}\langle e^{w(\lambda+\rho)-\rho}, e^{w'(\mu+\rho)-\rho}\rangle_T\end{aligned}$$

Now, we use the fact that the e^ν are the characters of the irreducible representation of T and hence they form an orthonormal basis with respect to $\langle,\rangle_T$. Hence we see that

$$\langle WCF_\lambda, WCF_\mu\rangle_K = \frac{1}{|W|}\sum_{w,w'}(-1)^{l(w)+l(w')}\delta_{w(\lambda+\rho),w'(\mu+\rho)}$$

Suppose that $w(\lambda+\rho) = w'(\mu+\rho)$ for some w, w'. Since $\lambda+\rho$ and $\mu+\rho$ are both dominant, we see that this forces $\lambda = \mu$. Also since $\lambda+\rho$ is dominant regular (i.e. $\langle\alpha^\vee, \lambda\rangle > 0$ for all $\alpha \in R_+$), it has no stabilizer in W and thus $w = w'$.

Hence we deduce that $\langle WCF_\lambda, WCF_\mu\rangle_K = 1$ if $\lambda = \mu$ and is 0 otherwise. $\square$

7.3 Topological proof

For this proof, we will first need an extension of the Borel-Weil theorem. If L is a line bundle on a complex projective variety X, in addition to the space of sections $\Gamma(X, L)$, we also have the higher cohomology groups $H^i(X, L)$ for $i = 1, \ldots, \dim X$. These are defined as the derived functor of the global sections functor and they measure the failure of local sections to glue together to form global sections. We have $H^0(X, L) = \Gamma(X, L)$.

Bott's extension of the Borel-Weil theorem is the following result.

Theorem 7.7. *Let λ be a dominant weight. Then $H^i(G/B, L(\lambda)) = 0$ for $i > 0$.*

This is quite useful, since information about higher cohomology groups is often useful, in particular, it is useful in the following result, called the Atiyah-Bott fixed point theorem.

Theorem 7.8. *Let X be a smooth complex projective variety and let L be a line bundle on X. Let $f : X \to X$ be an automorphism of X with finitely* *many fixed points X^f. Assume that there is a lift of f to L and choose such a lift. Then*

$$\sum_{i=0}^{\dim X} (-1)^i \operatorname{tr}(f|_{H^i(X,L)}) = \sum_{x \in X^f} \frac{\operatorname{tr}(f|_{L_x})}{\det(1 - T_x f)}$$

where $T_x f : T_x X \to T_x X$ is the derivative of f at $x \in X$.

This theorem is very powerful since it relates global information about the action of f on $H^i(X, L)$ to local information at each fixed point.

We will now apply these results to prove the Weyl character formula.

Proof. Let λ be a dominant weight. By the Borel-Weil theorem and Bott's extension, we have that $H^0(G/B, L(\lambda))^* = V(\lambda)$ and $H^i(G/B, L(\lambda)) = 0$ if $i > 0$.

Choose $t \in T$ such that the t-fixed points are the same as the T-fixed points, which are the elements of the Weyl group by Lemma 6.2. Applying the Atiyah-Bott fixed point theorem gives us that

$$\operatorname{tr}(t|_{V(\lambda)^*}) = \sum_{w \in W} \frac{\operatorname{tr}(t|_{L(\lambda)_w})}{\det(1 - T_w t)}$$

We know from Lemma 6.39 that the action of t on $L(\lambda)_w$ is by the scalar $(-w^{-1}\lambda)(t) = (w^{-1}\lambda)(t^{-1})$.

To understand $\det(1 - T_w t)$, let us note that we have a natural identification $T_w G/B = \mathfrak{g}/w\mathfrak{b}w^{-1}$, since wBw^{-1} is the stabilizer of w in G. Since

$$\mathfrak{g} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \text{ and } w\mathfrak{b}w^{-1} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in wR_+w^{-1}} \mathfrak{g}_\alpha$$

there is a T-equivariant isomorphism

$$\mathfrak{g}/w\mathfrak{b}w^{-1} = \oplus_{\alpha \in wR_-w^{-1}} \mathfrak{g}_\alpha$$

Hence we see that

$$\det(1 - T_w t) = \prod_{\alpha \in wR_-w^{-1}} 1 - \alpha(t) = \prod_{\alpha \in wR_+w^{-1}} 1 - \alpha(t^{-1})$$

Note that $\operatorname{tr}(t|_{V(\lambda)}) = \operatorname{tr}(t^{-1}|_{V(\lambda)^*})$. Putting this all together and replacing w with w^{-1}, we see that

$$\operatorname{tr}(t|_{V(\lambda)}) = \sum_{w \in W} \frac{(w\lambda)(t)}{\prod_{\alpha \in w^{-1}R_+w} 1 - \alpha(t)}$$

The RHS is almost the Weyl character formula. We just need to massage the denominator a little bit.

Take $\alpha \in w^{-1}R_+w \cap R_+$. Then

$$\frac{1}{1-\alpha(t)} = -\frac{\alpha(t)^{-1}}{1-\alpha(t)^{-1}}$$

Hence

$$\frac{1}{\prod_{\alpha \in w^{-1}R_+w} 1-\alpha(t)} = (-1)^{l(w)} \frac{\prod_{\alpha \in w^{-1}R_+w \cap R_+}(-\alpha)(t)}{\prod_{\alpha \in R_-} 1-\alpha(t)}$$

Finally note that $\sum_{\alpha \in w^{-1}R_+w \cap R_+} -\alpha = w\rho - \rho$, which completes the proof.

$\square$