Learning seminar on perverse sheaves. Beilinson's theorem and some computations

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These notes consist of two independent parts. In the first part we formulate the decomposition theorem and describe several of its applications in Kazhdan-Lusztig and Springer theories. In the second part we formulate Beilinson's theorem on the derived category of perverse sheaves, and describe some ideas involved in its proof (in very broad strokes).

1 Decomposition theorem and applications

Let X be a variety. A complex $\mathcal{F} \in D^b_c(X)$ is called semisimple, if it is isomorphic to a direct sum of shifts of simple perverse sheaves.

We first state the decomposition theorem in the form in which it is most commonly applied.

Theorem 1 (Decomposition theorem). *Let $f : X \to Y$ be a proper morphism of varieties, and assume that X is smooth. Then the complex $f_*\underline{\mathbb{C}}_X$ is semisimple.*

To state the general result, we will need a notion of a semisimple complex of geometric origin. The class of simple perverse sheaves of geometric origin is the minimal class satisfying the following properties:

- Constant sheaf on a point is in this class.
- If $f : X \to Y$ is a morphism of varieties, and $\mathcal{F}$ is a simple perverse sheaf of geometric origin on X (resp. on Y), irreducible constituents of ${}^p\mathcal{H}^i(T\mathcal{F})$ are of geometric origin, where T is one of $f_!, f_*$ (resp. $f^*, f^!$).
- If $\mathcal{F}, \mathcal{G}$ are simple perverse sheaves of geometric origin, irreducible constituents of ${}^p\mathcal{H}^i(\mathcal{F} \otimes \mathcal{G}), {}^p\mathcal{H}^i(\mathcal{H}om(\mathcal{F}, \mathcal{G}))$ are of geometric origin.

The complex $\mathcal{F} \in D^b_c(X)$ is called semisimple of geometric origin, if it is isomorphic to a direct sum of shifts of simple perverse sheaves of geometric origin.

Theorem 2 (Decomposition theorem, general form). *Let $f : X \to Y$ be a proper morphism of varieties, and let $\mathcal{F} \in D^b_c(X)$ be a semisimple complex of geometric origin. Then $f_*\mathcal{F}$ is a semisimple complex of geometric origin.*

1.1 Schubert and Bott-Samelson varieties

Exposition follows [Ric10], [Spr]. Let G be a semisimple algebraic group, B its Borel subgroup, $T \subset B$ – maximal torus, $\mathcal{B} = G/B$ – flag variety. We consider left B-orbits in $\mathcal{B}$. They are numbered by the elements of the group $W = N_G(T)/T$, where

N_G stands for the normalizer under the adjoint action. Picking a lift $\dot{w}$ of $w \in W$ to G, the orbit $\mathcal{O}_w$, corresponding to w, is $B\dot{w}B$. Varieties $\mathcal{O}_w$ are isomorphic to affine spaces. Let $l(w) = \dim \mathcal{O}_w$. W has a structure of Coxeter group, with the set of simple reflections $S = \{s \in W : \dim \mathcal{O}_s = 1\}$. Bruhat order on W is given by

$$y \leq x \iff \mathcal{O}_y \subset \overline{\mathcal{O}}_x$$

Decomposition $\mathcal{B} = \bigsqcup_{w \in W} \mathcal{O}_w$ gives a stratification of $\mathcal{B}$. We will be working with sheaves constructible in this stratification.

Varieties $X_w = \overline{\mathcal{O}}_w$ are called Schubert varieties.

It will be more convenient for us to work in the following symmetrized setting. Consider the decomposition of the variety $\mathcal{B} \times \mathcal{B}$ into G-orbits with respect to the left diagonal action of G. It is easy to see that they are in bijection with the B-orbits in $\mathcal{B}$,

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} O_w := \bigsqcup_{w \in W} G \cdot (B, \dot{w}B).$$

Denote $\mathfrak{X}_w = \overline{O}_w$. The variety O_w (resp. $\mathfrak{X}_w$) is a locally-trivial bundle over $\mathcal{B}$ with the fiber $\mathcal{O}_w$ (resp. X_w). We have $\mathrm{IC}(X_w)_y[\dim \mathcal{B}] \simeq \mathrm{IC}(\mathfrak{X}_w)_y$, where $\mathcal{F}_y$ denotes the stalk of the complex $\mathcal{F}$ at the stratum numbered by y.

Example. $G = SL(n)$, $\mathcal{B}$ – variety of flags of vector spaces

$$\mathbb{C}^n = V_n \supset V_{n-1} \supset \cdots \supset V_0 = \{0\}, \dim V_i = i,$$

$W = S_n$ – symmetric group. $(V_\bullet, V'_\bullet) \in O_w$ if there exists a basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$ with

$$V_i = \mathrm{span}(e_1, \ldots, e_i), V'_i = \mathrm{span}(e_{w(1)}, \ldots, e_{w(i)}).$$

Consider the variety

$$\tilde{\mathfrak{X}}_{\underline{w}} = \{(x_1, \ldots, x_{k+1}) \in \mathcal{B}^{k+1} : (x_l, x_{l+1}) \in O_{s_{i_l}}\}.$$

It is called the Bott-Samelson veriety. First, observe that $\tilde{\mathfrak{X}}_{\underline{w}}$ is smooth, given by iterated fibrations with smooth base and fiber $\mathbb{P}^1$, and there is a proper map

$$\pi_{\underline{w}} : \tilde{\mathfrak{X}}_{\underline{w}} \to \mathfrak{X}_w, (x_1, \ldots, x_{k+1}) \mapsto (x_1, x_{k+1}).$$

In case $\underline{w}$ is a reduced expression, $\pi_{\underline{w}}$ is a resolution of singularities, called the Bott-Samelson (or Demazure) resolution.

Example. Let $G = SL(3)$. In this case $W = S_3$, generated by two reflections $s_1, s_2, W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$. We have $v < w$ iff $l(v) < l(w)$ in this case.

Let $\underline{w} = s_1s_2$. The Bott-Samelson variety $\tilde{\mathfrak{X}}_{\underline{w}}$ consists of triples of flags $(V_\bullet^{(1)}, V_\bullet^{(2)}, V_\bullet^{(3)})$ satisfying $V_2^{(1)} = V_2^{(2)}, V_1^{(2)} = V_1^{(3)}$. So $V_\bullet^{(2)}$ is completely determined by $V_\bullet^{(1)}, V_\bullet^{(3)}$ and $\pi_{\underline{w}}$ is an isomorphism. In particular, $\mathfrak{X}_w$ is smooth.

Similarly, $\mathfrak{X}_{s_2s_1}$ is also smooth and is isomorphic to the corresponding Bott-Samelson variety.

Let $\underline{w} = s_1s_2s_1$. $\tilde{\mathfrak{X}}_{\underline{w}}$ consists of tuples of flags $(V_\bullet^{(1)}, V_\bullet^{(2)}, V_\bullet^{(3)}, V_\bullet^{(4)})$ satisfying

$$V_2^{(1)} = V_2^{(2)}, V_1^{(2)} = V_1^{(3)}, V_2^{(3)} = V_2^{(4)}.$$

By the discussion above, $V_\bullet^{(2)}$ is completely determined by $V_\bullet^{(1)}, V_\bullet^{(3)}$, so this simplifies to triples $(V_\bullet^{(1)}, V_\bullet^{(2)}, V_\bullet^{(3)})$ with

$$V_1^{(2)} \subset V_2^{(1)}, V_2^{(2)} = V_2^{(3)}.$$

We again get that $V_2^{(2)}$ is determined by $V_\bullet^{(3)}$. Since $V_1^{(2)} \subset V_2^{(1)} \cap V_2^{(3)}$, it is also determined, if $V_2^{(1)} \neq V_2^{(3)}$. Otherwise, when $V_2^{(1)} = V_2^{(3)}$, which is the same as saying $(V_\bullet^{(1)}, V_\bullet^{(3)}) \in \mathfrak{X}_{s_1}$, we have $\mathbb{P}^1 = \mathbb{P}(V_2^{(1)})$ choices. To summarize: fiber of $\pi_{\underline{w}}$ over $\mathfrak{X}_{s_1}$ is isomorphic to $\mathbb{P}^1$, and over $\mathcal{B} \times \mathcal{B} \backslash \mathfrak{X}_{s_1}$ it is isomorphic to a point. Note that $\dim \mathfrak{X}_{s_1} = 4, \dim \mathcal{B} \times \mathcal{B} = 6$, so $\pi_{\underline{w}}$ is semismall in this case.

It will be convenient for us to introduce the following book-keeping device. Let $\mathcal{H}(W)$ be a free $\mathbb{Z}[v, v^{-1}]$-module with basis $T_w, w \in W$. Let $\mathcal{S}$ denote the stratification of $\mathcal{B} \times \mathcal{B}$ by orbits. For $\mathcal{F} \in D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$ write

$$h(\mathcal{F}) = \sum_{w \in W} \left(\sum_{i \in \mathbb{Z}} \dim H^{-i}(\mathcal{F}_w) v^i \right) T_w \in \mathcal{H}(W).$$

We have just computed

$$h(\pi_{s_1 s_2 *} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{s_1 s_2}}) = T_{s_1 s_2} + T_{s_1} + T_{s_2} + T_1,$$

$$h(\pi_{s_1 s_2 s_1 *} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{s_1 s_2 s_1}}[6]) = v^6(T_{s_1 s_2 s_1} + T_{s_2 s_1} + T_{s_1 s_2} + T_{s_2}) + (v^6 + v^4)(T_{s_1} + T_1).$$

Since we know that $\mathfrak{X}_{s_1 s_2 s_1}, \mathfrak{X}_{s_1}$ are smooth (first is the whole flag variety, second is isomorphic to a projective line) we have

$$h(\mathrm{IC}(\mathfrak{X}_{s_1})) = v^4(T_{s_1} + T_1),$$

$$h(\mathrm{IC}(\mathfrak{X}_{s_1 s_2 s_1})) = v^6(T_{s_1 s_2 s_1} + T_{s_2 s_1} + T_{s_1 s_2} + T_{s_2} + T_{s_1} + T_1).$$

So, by the decomposition theorem, we must have

$$\pi_{s_1 s_2 *} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{s_1 s_2 s_1}}[6] \simeq \mathrm{IC}(\mathfrak{X}_{s_1 s_2 s_1}) \oplus \mathrm{IC}(\mathfrak{X}_{s_1}).$$

We will now discuss the general case.

Note that, by definition of the IC-extension, we have, for all $w, y \in W$

$$\dim H^{-i}(\mathrm{IC}(\mathfrak{X}_w)_y) = 0 \text{ if } i < \dim O_y = \dim \mathcal{B} + l(y),$$

from perversity and

$$\dim H^{-i}(\mathrm{IC}(\mathfrak{X}_w)_y) = 0 \text{ if } i \leq \dim O_y = \dim \mathcal{B} + l(y), w \neq y,$$

$$\mathrm{IC}(\mathfrak{X}_w)_w = \mathbb{C}[\dim \mathcal{B} + l(w)],$$

from the IC property, which translates to

$$v^{-\dim \mathcal{B}} h(\mathrm{IC}(\mathfrak{X}_w)) \in v^{l(w)} T_w + \sum_{y < w} v^{l(y)+1} \mathbb{Z}[v] T_y,$$

Consider the variety $\mathcal{B}^3$ with projections

$$\pi_{12}, \pi_{13}, \pi_{23} : \mathcal{B}^3 \to \mathcal{B}^2,$$

π_{ij} being a projection to i^{th} and j^{th} factors. Consider the following operation, called convolution, on $D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$: for $\mathcal{A}, \mathcal{B} \in D^b_{\mathcal{S}}(\mathcal{B} \times \mathcal{B})$ write

$$\mathcal{A} * \mathcal{B} = \pi_{1,3*} \left(\pi_{1,2}^* \mathcal{A} \otimes \pi_{2,3}^* \mathcal{B} \right).$$

Lemma. *Fix $s \in S$, and let $\mathcal{F} \in D^b_S(\mathcal{B} \times \mathcal{B})$ be such that $\mathcal{H}^i(\mathcal{F}) = 0$ for all i even (resp. odd). Then $\underline{\mathbb{C}}_{\mathfrak{X}_s} * \mathcal{F}$ satisfies the same property and*

$$\dim H^{-i}((\underline{\mathbb{C}}_{\mathfrak{X}_s} * \mathcal{F})_w) = \begin{cases} \dim H^{-i}(\mathcal{F}_{sw}) + \dim H^{-i-2}(\mathcal{F}_w), & \textit{if } sw < w, \\ \dim H^{-i-2}(\mathcal{F}_{sw}) + \dim H^{-i}(\mathcal{F}_w), & \textit{if } sw > w. \end{cases}$$

Proof. We will prove the first equality and live the second one as an exercise. Pick $(x, z) \in O_w$. By the proper base change, we need to compute the cohomology of a constructible sheaf $\mathcal{F}'$ on

$$\mathcal{C} = \{y \in \mathcal{B} : (x, y) \in \mathfrak{X}_s\} \simeq \mathbb{P}^1$$

stratified as $\mathrm{pt} \sqcup \mathbb{A}^1$, where pt is the unique point $y_0 \in \mathcal{C}$ with $(y_0, z) \in O_{sw}$. Stalk of $\mathcal{F}'$ at pt is isomorphic to $\mathcal{F}_{sw}$, and stalk over $\mathbb{A}^1$ is isomorphic to $\mathcal{F}_w$. Writing

$$i : \mathrm{pt} \hookrightarrow \mathbb{P}^1 \hookleftarrow \mathbb{A}^1 : j,$$

consider the long exact sequence of cohomology associated to the triangle

$$j_! j^* \mathcal{F}' \to \mathcal{F}' \to i_* i^* \mathcal{F}' \to .$$

It gives

$$\to H_c^{-i}(j^* \mathcal{F}') \to H^{-i}(\mathcal{F}') \to H^{-i}(\mathcal{F}'_{\mathrm{pt}}) \to$$

or

$$\to H^{-i-2}(\mathcal{F}_w) \to H^{-i}(\mathcal{F}') \to H^{-i}(\mathcal{F}_{sw}) \to$$

where we used the fact that $j^* \mathcal{F}'$ is the sum of shifted constant sheaves and that $H_c^\bullet(\mathbb{A}^1) = \mathbb{C}[-2]$. Now parity assumption on $\mathcal{F}$ gives that this long exact sequence splits into short exact sequences, and we are done. □

Assume that we have computed the stalks of IC complexes for all $y < w$. Pick an expression

$$w = s_{i_1} s_{i_2} \dots s_{i_k}$$

and write $\underline{w}$ for the corresponding word.

Repeatedly applying the proper base change theorem, it is easy to see that

$$\underline{\mathbb{C}}_{\mathfrak{X}_{s_{i_1}}} * \underline{\mathbb{C}}_{\mathfrak{X}_{s_{i_2}}} * \dots * \underline{\mathbb{C}}_{\mathfrak{X}_{s_{i_k}}} \simeq \pi_{\underline{w}*} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{\underline{w}}}.$$

So $\pi_{\underline{w}*} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{\underline{w}}}$ satisfies the parity condition of the Lemma. Assume from now on that $\underline{w}$ is reduced. Applying the decomposition theorem, we get

$$\pi_{\underline{w}*} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{\underline{w}}}[\dim \mathcal{B} + l(w)] \simeq \mathrm{IC}(\mathfrak{X}_w) \oplus \bigoplus_{y<w} V_y \otimes \mathrm{IC}(\mathfrak{X}_y),$$

where V_y are graded vector spaces counting multiplicities. We get that $\mathrm{IC}(\mathfrak{X}_w)$ must also satisfy the parity condition of the Lemma, as a direct summand of $\pi_{\underline{w}*} \underline{\mathbb{C}}_{\tilde{\mathfrak{X}}_{\underline{w}}}[\dim \mathcal{B} + l(w)]$.

Choose s such that $sw < w$. Applying the results of [Zho] and the decomposition theorem, one can show that $\underline{\mathbb{C}}_{\mathfrak{X}_s} * \mathrm{IC}(\mathfrak{X}_{sw})[1]$ is semisimple. Applying the Lemma, on the other hand, we get that

$$h(\underline{\mathbb{C}}_{\mathfrak{X}_s} * \mathrm{IC}(\mathfrak{X}_{sw})[1]) \in v^{l(w)} T_w + \sum_{y<w} v^{l(y)} \mathbb{Z}[v] T_y.$$

From this one may deduce that the multiplicity of $\mathrm{IC}(\mathfrak{X}_y), y < w$, in $\underline{\mathbb{C}}_{\mathfrak{X}_s} * \mathrm{IC}(\mathfrak{X}_{sw})[1]$ is the coefficient of $v^{l(y)} T_y$, and, by induction, compute $h(\mathrm{IC}(\mathfrak{X}_w))$.

Remark. It is easy to see that the restriction of h to semisimple complexes is surjective onto $\mathcal{H}(W)$. One can show that the convolution operation on such complexes defines a $\mathbb{Z}[v, v^{-1}]$-algebra structure on $\mathcal{H}(W)$. The corresponding algebra is called Hecke algebra. It has a simple algebraic definition, for any coxeter group W, independent of geometry. $v^{-\dim \mathcal{B}} h(\mathrm{IC}(\mathfrak{X}))$ gives a basis of this algebra, called the Kazdhan-Lusztig basis. It was first defined, purely algebraically, in [KL79]. Kazhdan and Lusztig then proved, in [KL], that this algebraic definition coincides with the geometric one given above. Coefficients of $h(\mathrm{IC}(\mathfrak{X}))$ (after a multiplication by a suitable power of v) are called Kazhdan-Luzstig polynomials.

1.2 Springer and Grothendieck-Springer resolutions

The exposition of this subsection is from Chapter 8.1 of the book [Ach].

Let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{g}$, $\mathfrak{b}$ be the Lie algebra of B, $\mathfrak{t}$ be the Lie algebra of T. Let N be a unipotent radical of B, and let $\mathfrak{n}$ be the Lie algebra of N. We have an identification $\mathfrak{t} \simeq \mathfrak{b}/\mathfrak{n}$.

Consider a variety

$$\tilde{\mathfrak{g}} = G \times^B \mathfrak{b} = \{(xB, a) \in \mathcal{B} \times \mathfrak{g} : a \in \mathrm{Ad}(x)(\mathfrak{b})\}.$$

$\tilde{\mathfrak{g}}$ is a smooth variety (being a fiber bundle over $\mathcal{B}$), and we have a proper projection map

$$\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}.$$

This variety is called the Grothendieck-Springer simultaneous resolution.

There is also a map

$$\theta : \tilde{\mathfrak{g}} \to \mathfrak{t}, (xB, a) \mapsto \mathrm{Ad}(x^{-1})(a) \bmod \mathfrak{n}.$$

Let

$$\tilde{\mathcal{N}} = \theta^{-1}(0).$$

Restriction $\mu_{\tilde{\mathcal{N}}}$ of μ to $\tilde{\mathcal{N}}$ lands into the variety $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$. $\tilde{\mathcal{N}}$ is again smooth and in fact is isomorphic to the cotangent bundle $T^*\mathcal{B}$. It is called the Springer resolution.

Note that the subvariety $\mathfrak{g}_{rs}$ of regular semisimple elements in $\mathfrak{g}$ is open and dense. Let $\tilde{\mathfrak{g}}_{rs} = \mu^{-1}(\mathfrak{g}_{rs})$. Then $\mu_{rs} := \mu|_{\tilde{\mathfrak{g}}_{rs}}$ is a finite $|W|$-to-1 map.

Similarly, note that the subvariety $\mathcal{N}_r$ of regular nilpotent elements is open dense in $\mathcal{N}$, and all regular nilpotent elements are conjugate, so $\dim \mathcal{N} = \dim G - \dim \mathfrak{t}$.

Example. Let $G = SL(n)$. Then $\tilde{\mathfrak{g}}$ consists of pairs $(x, V_\bullet)$, $x \in \mathfrak{sl}_n$, $V_\bullet$ – flag of vector subspaces, preserved by x. $\tilde{\mathcal{N}}$ is given by the condition that x is nilpotent. Each regular nilpotent element preserves a unique flag, so $\tilde{\mathcal{N}}$ is indeed a resolution of singularities. Consider the adjoint action of G on $\mathcal{N}$. Its orbits are numbered by partition of n, corresponding to Jordan normal forms of nilpotent matrices. Thus, regular orbit corresponds to a partition (n). For a partition λ, let $\mathcal{O}_\lambda$ denote the corresponding orbit. Write

$$\mu \leq \lambda \iff \mathcal{O}_\mu \subset \overline{\mathcal{O}}_\lambda.$$

Decomposition

$$\mathcal{N} = \bigsqcup_\lambda \mathcal{O}_\lambda$$

gives an algebraic stratification of $\mathcal{N}$, which we will denote $\mathcal{S}$. G also acts on $\tilde{\mathcal{N}}$ by the formula

$$g(x, V_\bullet) = (\mathrm{Ad}(g)(x), gV_\bullet),$$

and $\mu_{\tilde{\mathcal{N}}}$ is a G-equivariant map, so $\mu_{\tilde{\mathcal{N}}*}\underline{\mathbb{C}}_{\tilde{\mathcal{N}}} \in D^b_{\mathcal{S}}(\mathcal{N})$. Let $\mathcal{A}$ be a free $\mathbb{Z}[v, v^{-1}]$-module with basis T_λ, where λ runs through the set of all partitions of n. Define, as in the previous subsection, the map $h : D^b_{\mathcal{S}}(\mathcal{N}) \to \mathcal{A}$, via

$$h(\mathcal{F}) = \sum_\lambda \left(\sum_{i \in \mathbb{Z}} \dim H^{-i}(\mathcal{F}_\lambda) v^i \right) T_\lambda.$$

Again, IC-property translates to

$$h(\mathrm{IC}(\overline{\mathcal{O}}_\lambda)) \in v^{\dim \mathcal{O}_\lambda} T_\lambda + \sum_{\mu < \lambda} v^{\dim(\mathcal{O}_\mu)+1} \mathbb{Z}[v] T_\mu.$$

Remark. Varieties $\mathcal{O}_\lambda$ are not, in general, simply-connected (even when $G = SL(2)$). However, in case of $G = SL(n)$, we have an action of $GL(n)$ on both $\tilde{\mathcal{N}}, \mathcal{N}$, and $\mu_{\tilde{\mathcal{N}}}$ is $GL(n)$-equivariant. It can be deduced from this and the fact that stabilizers of $GL(n)$-action on $\mathcal{N}$ are connected, that the local systems appearing in cohomology of $\mu_{\tilde{\mathcal{N}}*}\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}$ restricted to $\mathcal{O}_\lambda$ are trivial in this case.

Example. Let $G = SL(2)$. Then $\mathcal{N} = \{x \in \mathrm{End}(\mathbb{C}^2), x^2 = 0\}$ – quadratic cone, $\tilde{\mathcal{N}} = \{(x, l) \in \mathcal{N} \times \mathbb{P}^1 : xl = \{0\}\}$. Fiber over any $x \neq 0$ consists of a single line, and fiber over $x = 0$ is the whole $\mathbb{P}^1$. We also see that $\dim \mathcal{N} = 2$, so the map $\mu_{\tilde{\mathcal{N}}}$ is semismall. We have two G-orbits in $\mathcal{N}$, $\mathcal{O}_{(1,1)} = \{0\}, \mathcal{O}_{(2)}$ – regular orbit. So

$$h(\mu_{\tilde{\mathcal{N}}*}\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[2]) = v^2 T_{(2)} + (1 + v^2) T_{(1,1)}.$$

We get $\mu_{\tilde{\mathcal{N}}*}\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[2] \simeq \mathrm{IC}(\overline{\mathcal{O}}_{(2)}) \oplus \mathrm{IC}(\overline{\mathcal{O}}_{(1,1)})$.

Example. Let $G = SL(3)$. We have three G-orbits, $\mathcal{O}_{(3)}, \mathcal{O}_{(2,1)}, \mathcal{O}_{(1,1,1)}$ with $\dim \mathcal{O}_{(3)} = \dim \mathcal{N} = 6, \dim \mathcal{O}_{(2,1)} = 4, \dim \mathcal{O}_{(1,1,1)} = 0$. Lets study $\mathcal{O}_{(2,1)}$ more carefully. This is the orbit consisting of nilpotent matrices of rank 1. Let X be such a matrix, and consider the variety of flags $\{V_\bullet\}$ that are preserved by X. Let the line ℓ be the image of X, and let the plane K be the kernel of X. Then we must have $V_2 \supset \ell, V_1 \subset K$. Now if $V_1 = \ell$, for any $V_2 \supset \ell, X$ preserves $V_\bullet$, and if $V_2 = K$ for any $V_1 \subset K, X$ preserves $V_\bullet$. We get that the fiber of $\mu_{\tilde{\mathcal{N}}}$ over X is a union of two projective lines intersection at the flag $V = (\ell \subset K)$. In particular,

$$2 \dim \mu^{-1}_{\tilde{\mathcal{N}}}(X) = 2 \leq \mathrm{codim}\, \mathcal{O}_{(2,1)} = 2.$$

We also have

$$2 \dim \mu^{-1}_{\tilde{\mathcal{N}}}(\{0\}) = 2 \dim \mathcal{B} = 6 \leq \mathrm{codim}\, \mathcal{O}_{(1,1,1)} = \dim \mathcal{N} = 6,$$

so $\mu_{\tilde{\mathcal{N}}}$ is semismall with respect to $\mathcal{O}_{(3)}$.

Consider the variety $\tilde{\mathcal{N}}' = \{(x, \ell) \in \mathrm{End}(\mathbb{C}^3) \times \mathbb{P}^2 : x^3 = 0, \mathrm{rk}\, x \leq 1, \mathrm{Im}\, x \subset \ell\}$.

Exercise. $\tilde{\mathcal{N}}'$ is a semismall resolution of $\overline{\mathcal{O}}_{(2,1)}$. Deduce that

$$h(\mathrm{IC}(\overline{\mathcal{O}}_{(2,1)})) = v^4 T_{(2,1)} + (v^4 + v^2) T_{(1,1,1)}.$$

To summarize, we have

$$h(\mu_{\tilde{\mathcal{N}}*}\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[6]) = v^6 T_{(3)} + (v^6 + 2v^4) T_{(2,1)} + (v^6 + 2v^4 + 2v^2 + 1) T_{(1,1,1)},$$

and we deduce

$$\mu_{\tilde{\mathcal{N}}*}\underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[6] \simeq \mathrm{IC}(\overline{\mathcal{O}}_{(3)}) \oplus \mathrm{IC}(\overline{\mathcal{O}}_{(2,1)})^{\oplus 2} \oplus \mathrm{IC}(\overline{\mathcal{O}}_{(1,1,1)}).$$

The main goal of this subsection is to prove the following

Proposition 1. *μ is a small map and $\mu_{\tilde{\mathcal{N}}}$ is a semismall resolution of singularities.*

Proof. We first prove that $\mu_{\tilde{\mathcal{N}}}$ is semismall. Denote $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Z is called the Steinberg variety. Semismallness of $\mu_{\tilde{\mathcal{N}}}$ is equivalent, as was shown in the previous talk [Zho], to

$$\dim Z \leq \dim \mathcal{N}.$$

We have a projection map $\pi : Z \to \mathcal{B} \times \mathcal{B}$. Let $Z_w = \pi^{-1}(O_w), w \in W$. We will show that

$$\dim Z_w \leq \dim \mathcal{N}$$

for all $w \in W$. Choose a representative $\dot{w}$ of w. There is an isomorphism of varieties

$$G \times^{B \cap \dot{w} B \dot{w}^{-1}} (\mathfrak{n} \cap \mathrm{Ad}(\dot{w})\mathfrak{n}) \to Z_w,$$

(exercise). It follows that

$$\dim Z_w = \dim G - \dim(B \cap \dot{w} B \dot{w}^{-1}) + \dim(\mathfrak{n} \cap \mathrm{Ad}(\dot{w})\mathfrak{n}).$$

But the Lie algebra of $B \cap \dot{w} B \dot{w}^{-1}$ is isomorphic, as a vector space, to $\mathfrak{t} \oplus (\mathfrak{n} \cap \mathrm{Ad}(\dot{w})\mathfrak{n})$, so

$$\dim Z_w = \dim G - \dim \mathfrak{t} = \dim \mathcal{N}.$$

Proof of the smallness of μ is similar. Let $Z' = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}, \pi : Z' \to \mathcal{B} \times \mathcal{B}$ – projection, $Z'_w = \pi^{-1}(O_w)$. Similarly, we have

$$G \times^{B \cap \dot{w} B \dot{w}^{-1}} (\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b}) \xrightarrow{\sim} Z'_w,$$

so $\dim Z'_w = \dim \mathfrak{g}$ for all w.

We already know that over $\mathfrak{g}_{rs}$ μ is a finite map. Let

$$Z'' = \{(xB, yB, a) \in Z' : a \notin \mathfrak{g}_{rs}\}, Z''_w = Z'' \cap Z'_w.$$

We have

$$G \times^{B \cap \dot{w} B \dot{w}^{-1}} (\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b} \cap (\mathfrak{g} \backslash \mathfrak{g}_{rs})) \xrightarrow{\sim} Z''_w.$$

Now $(\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b} \cap \mathfrak{g}_{rs})$ is a non-empty open subset in a vector space $\mathfrak{b} \cap \mathrm{Ad}(\dot{w})\mathfrak{b}$, so its complement has a strictly positive codimension. We deduce that

$$\dim Z''_w < \dim Z'_w = \dim \mathfrak{g},$$

so $\dim Z'' < \dim \mathfrak{g}$. Choose any locally-closed subvariety $Y \subset \mathfrak{g} \backslash \mathfrak{g}_{rs}$ such that fibers of μ have the same dimension over all points $x \in Y$. Let $\mu' : Z' \to \mathfrak{g}$ be the projection, let

$$Z'_Y = \mu'^{-1}(Y).$$

$Z'_Y \subset Z''$, so $\dim Z'_Y < \dim \mathfrak{g}$. Pick a point $x \in Y$. We have

$$\mu'^{-1}(x) = \mu^{-1}(x) \times \mu^{-1}(x).$$

So

$$\dim Z'_Y = 2 \dim \mu^{-1}(x) + \dim Y < \dim \mathfrak{g},$$

and we are done. □

Corollary 1.1. *Let $i_{\mathcal{N}} : \mathcal{N} \hookrightarrow \mathfrak{g}$ be the closed embedding.*

$$\mu_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}}[\dim \mathfrak{g}] = \mathrm{IC}(\mu_{rs*} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}}[\dim \mathfrak{g}]),$$

$$\mu_{\tilde{\mathcal{N}}*} \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}] = i_{\mathcal{N}}^* \mathrm{IC}(\mu_{rs*} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}}[\dim \mathfrak{g}])[-\dim \mathfrak{t}] \in \mathrm{Perv}(\mathcal{N}).$$

Moreover, by the decomposition theorem, we know that the latter sheaf is semisimple.

We will now construct the W-action on these sheaves.

Proposition 2. *μ_{rs} is a Galois covering with the Galois group W.*

Proof. Let $\mathfrak{b}_{rs} = \mathfrak{b} \cap \mathfrak{g}_{rs}, \mathfrak{t}_{rs} = \mathfrak{g}_{rs} \cap \mathfrak{t}$. We have $\tilde{\mathfrak{g}}_{rs} = G \times^B \mathfrak{b}_{rs}$. We have the following map, indcued by the inclusion $\mathfrak{t}_{rs} \hookrightarrow \mathfrak{b}_{rs}$:

$$G \times^T \mathfrak{t}_{rs} \to G \times^B \mathfrak{b}_{rs}.$$

This is an isomorphism (exercise). We get an action of W on $\tilde{\mathfrak{g}}_{rs}$ induced by the free action of W on $\mathfrak{t}_{rs}$. Using the fact that each Cartan subalgebra is contained in excatly $|W|$ Borel subalgebras, one gets an isomorphism

$$\tilde{\mathfrak{g}}_{rs}/W \simeq \mathfrak{g}_{rs},$$

as desired. □

Using the fact from the talk [Dyk] that IC is a fully faithful functor $\mathrm{Perv}(\mathfrak{g}_{rs}) \to \mathrm{Perv}(\mathfrak{g})$, we get

Corollary 1.2.
$$\mathrm{End}(\mathrm{IC}(\mu_{rs*} \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}_{rs}}[\dim \mathfrak{g}])) \simeq \mathbb{C}[W].$$

Corollary 1.3. *There is an action of W on the sheaf $\mu_{\tilde{\mathcal{N}}*} \underline{\mathbb{C}}_{\tilde{\mathcal{N}}}[\dim \mathcal{N}]$, and, in particular, on the cohomology of its fibers.*

This action is originally due to Springer, and the construction given here is due to Lusztig.

2 Beilinson's theorem

Theorem 3. *Let X be a variety. Then there is a triangulated equivalence of categories*

$$\mathrm{real} : D^b(\mathrm{Perv}(X)) \to D^b_c(X),$$

such that $\mathrm{real}|_{\mathrm{Perv}(X)}$ is the tautological embedding.

We indicate some ideas behind the proof. Start we the following simple lemma:

Lemma 2.1. *Let $f : \mathcal{T} \to \mathcal{T}'$ be a triangulated functor between two triangulated categories. Assume that there is a subset $\mathcal{C} \subset \mathrm{Ob}(\mathcal{T})$ generating $\mathcal{T}$ as a triangulated category, such that $f(\mathcal{C})$ generates $\mathcal{T}'$ and*

$$\mathrm{Hom}(X, Y[k]) = \mathrm{Hom}(f(X), f(Y)[k])$$

for all $X, Y \in \mathcal{C}$. Then f is an equivalence of categories.

This lemma, assuming that we can construct *some* functor $D^b(\mathrm{Perv}(X)) \to D^b_c(X)$, reduces Beilinson's theorem to the comparison of Ext-groups in $\mathrm{Perv}(X)$ computed in $D^b(\mathrm{Perv}(X))$ and $D^b_c(X)$.

The following notion is used repeatedly throughout the proof. Let $\mathcal{T}$ be a triangulated category with a t-structure, let $\mathcal{C}$ be its heart. For $X, Y \in \mathcal{C}$, a morphism $f \in \mathrm{Hom}(X, Y[k])$ is called ***effaceable***, if there exist morphisms $p : X' \to X$, $i : Y \to Y'$, p being surjective and i injective, such that $i[k] \circ f \circ p : X' \to Y'[k]$ is 0.

Lemma. *If $\mathcal{A}$ is abelian, $\mathcal{T} = D^b(\mathcal{A})$ (so that $\mathcal{A}$ is the heart of the natural t-structure), then for all $k > 0, X, Y \in \mathcal{A}$, any morphism $f \in \mathrm{Hom}(X, Y[k])$ is effaceable. Moreover, there always exist a surjective map $p : X' \to X$ such that $f \circ p = 0$ and an injective map $i : Y \to Y'$ such that $i[n] \circ f = 0$.*

E.g., if $k = 1$, rotate the triangle $X \to Y[1] \to Cone(f) \to$ to the exact sequence $0 \to Y \to Cone(f)[-1] \to X \to 0$ and take $Y' = Cone(f)[-1]$ (or $X' = Cone(f)[-1]$). Proof for the general k is similar and is left as an exercise.

We have the following:

Proposition 3. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{C}_0$ be a heart of t-structure on $\mathcal{T}$, $\mathcal{C} \subset \mathcal{C}_0$ – Serre subcategory. Suppose we have a triangulated functor*

$$\rho : D^b(\mathcal{C}) \to \mathcal{T}$$

such that $\rho|\mathcal{C}$ is isomorphic to the inclusion functor. For $X, Y \in \mathcal{C}$, consider the map

$$\mathrm{Ext}^k_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(X, Y[k])$$

induced by ρ. We have

- *This map is an isomorphism for $k = 0, 1$.*
- *If this map is an isomorphism for $k = 0, 1, \ldots, n-1$, it is injective for $k = n$ and its image consists of effaceable morphisms.*

Corollary 2.1. *ρ is an equivalence of categories if and only if, for $k > 0, X, Y \in \mathcal{C}$, all morphisms in $\mathrm{Hom}_{\mathcal{T}}(X, Y[k])$ are effaceable.*

This can already be used to reduce Beilinson's theorem to the case when X is affine. Indeed, assume that it is known for all affine varieties U, and let X be any variety. Choose $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(X), f \in \mathrm{Hom}(\mathcal{F}, \mathcal{G}[k])$. Choose an affine open cover $\{j_i : U_i \to X\}$ of X. By the lemma above, we can choose surjective maps $p_i : \mathcal{P}_i \to j_i^* \mathcal{F}$ with $j_i^* f \circ p_i = 0$. By adjunction, we have maps $\overline{p}_i : j_{i!}\mathcal{P}_i \to \mathcal{F}$ with $f \circ \overline{p}_i = 0$. But $j_i^* \overline{p}_i$ are surjective, so

$$\overline{p} = \oplus_i \overline{p}_i : \oplus j_{i!}\mathcal{P}_i \to \mathcal{F}$$

is surjective (because perverse sheaves can be glued on open covers) and satisfies $f \circ \overline{p} = 0$, so f is effaceable.

Note that, if we replace the categories in Beilinson's theorem with categories of sheaves constructible with respect to a fixed stratification, the theorem is false: take, for example, the trivial stratification of $X = \mathbb{P}^1$. In this stratification, the category of perverse sheaves is semisimple, but $\mathrm{Hom}_{D^b_c(X)}(\underline{\mathbb{C}}_X, \underline{\mathbb{C}}_X[2]) \neq 0$, as was discussed in [Zho]. However if we take X to be an open subset of $\mathbb{A}^1$ (with trivial stratification), the theorem holds:

Proposition 4. *Let $X \subset \mathbb{A}^1$ be an affine open subset. Then we have an equivalence of categories*

$$D^b(\mathrm{Loc}(X)) \simeq D^b_{loc}(X),$$

where D^b_{loc} stands for the derived category of sheaves with locally-constant cohomologies.

Proof. Let $\mathcal{F}, \mathcal{G}$ be two local systems. Since X is affine, $\mathrm{Ext}^k_{D^b_{loc}(X)}(\mathcal{F}, \mathcal{G})$ vanishes for $k \geq 2$. $\mathrm{Ext}^k_{\mathrm{Loc}(X)}(\mathcal{F}, \mathcal{G})$ also vanishes for $k \geq 2$, since $\pi_1(X)$ is free and so has homological dimension 1. And since local systems are closed under extensions, we have

$$\mathrm{Ext}^1_{\mathrm{Loc}(X)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^1_{D^b_{loc}(X)}(\mathcal{F}, \mathcal{G}).$$

□

By induction on dimension, one then proves the following theorem:

Theorem 4. *For any variety X there exists a smooth, open, connected affine subvariety U such that*

$$D^b(\mathrm{Loc}(U)) \simeq D^b_{loc}(U).$$

For local systems, effaceability can be checked using the following

Proposition 5. *For a connected affine variety U the following conditions are equivalent:*

- *for all $k > 0, \mathcal{F}, \mathcal{G} \in \mathrm{Loc}(U)$ all morphisms in $\mathrm{Hom}_{D^b_{loc}(U)}(\mathcal{F}, \mathcal{G}[k])$ are effaceable.*
- *for all $k > 0, \mathcal{G} \in \mathrm{Loc}(U)$ there is an injective morphism $\mathcal{G} \to \mathcal{G}'$ in $\mathrm{Loc}(U)$ such that the induced morphism $\mathbb{H}^k(\mathcal{G}) \to \mathbb{H}^k(\mathcal{G}')$ is 0.*

Induction is based on the following two lemmas:

Lemma. *If $U \to V$ is a locally-trivial fibration with fibers of $\dim \leq 1$, and the statement of the theorem is true for V, then it is true for U.*

This Lemma, along with the Noether's normalization, reduces the theorem to the case when X is an open subset of $\mathbb{A}^n$.

Lemma. *Let X be an open subset of $\mathbb{A}^n$. Then, there is an open subset $U \subset X$ such that U is a locally-trivial fibration over some $V \subset \mathbb{A}^{n-1}$ with fibers isomorphic to an affine subset of $\mathbb{A}^1$.*

Then we need the following two propositions.

Proposition 6. *Let $Z \subset X$ be a closed subset of the form $f^{-1}(0)$ for some regular function $f : X \to \mathbb{C}$, $i : Z \to X$ be the corresponding closed embedding. Let $D^b_Z(\mathrm{Perv}(X))$ be the full subcategory of $D^b(\mathrm{Perv}(X))$ consisting of complexes with cohomology supported on Z. Then i_* defines the equivalence*

$$i_* : D^b(\mathrm{Perv}(Z)) \to D^b_Z(\mathrm{Perv}(X))$$

Remark. This is a derived version of the proposition that appeared in [Dyk].

We will prove this proposition using the following fact: there exist exact functors $\Phi^{un}_f : \mathrm{Perv}(X) \to \mathrm{Perv}(Z), \Upsilon_f : \mathrm{Perv}(X) \to \mathrm{Perv}(X)$, with canonical morphisms

$$\mathcal{F} \to \Upsilon_f(\mathcal{F}) \leftarrow i_*\Phi^{un}_f(\mathcal{F})$$

that become isomorphisms when $\mathcal{F} \in \mathrm{Perv}_Z(X)$.

Now the fact that i_* admits the right inverse Φ_f^{un}, shows that the map

$$\mathrm{Ext}^k_{\mathrm{Perv}(Z)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^k_{\mathrm{Perv}(X)}(i_*\mathcal{F}, i_*\mathcal{G})$$

is injective for all k. To show surjectivity, recall the Yoneda's definition of Ext. Elements of $\mathrm{Ext}^k(\mathcal{F}, \mathcal{G}), k \geq 1$, are given by equivalence classes of exact sequences

$$0 \to \mathcal{G} \to C_1 \to \cdots \to C_k \to \mathcal{F} \to 0,$$

with two sequences considered equivalent, if they can be connected by a chain of morphisms of sequences that are isomorphisms on $\mathcal{F}$ and $\mathcal{G}$. Take

$$(0 \to i_*\mathcal{G} \to C_1 \to \cdots \to C_k \to i_*\mathcal{F} \to 0) \in \mathrm{Ext}^k_{\mathrm{Perv}(X)}(i_*\mathcal{F}, i_*\mathcal{G}).$$

This sequence is equivalent to the sequence

$$(0 \to i_*\mathcal{G} \to i_*\Phi_f^{un} C_1 \to \cdots \to i_*\Phi_f^{un} C_k \to i_*\mathcal{F} \to 0) \in i_* \mathrm{Ext}^k_{\mathrm{Perv}(Z)}(\mathcal{F}, \mathcal{G}),$$

via

$$(0 \to i_*\mathcal{G} \to \Upsilon_f C_1 \to \cdots \to \Upsilon_f C_k \to i_*\mathcal{F} \to 0) \in \mathrm{Ext}^k_{\mathrm{Perv}(X)}(i_*\mathcal{F}, i_*\mathcal{G}),$$

and so i_* is surjective onto $\mathrm{Ext}^k_{\mathrm{Perv}(X)}(i_*\mathcal{F}, i_*\mathcal{G})$.

Exercise. Prove a similar statement where $D^b(\mathrm{Perv}(Z)), D^b_Z(\mathrm{Perv}(X))$ are replaced by $D^b(C(Z)), D^b_Z(C(X))$, where $C(Z), C(X)$ are abelian categories of constructible sheaves.

Proposition 7. *Let $U \subset X$ be an* affine *open subset, $j : U \to X$ be the corresponding open embedding. Then, for any $\mathcal{F} \in \mathrm{Perv}(X), \mathcal{G} \in \mathrm{Perv}(U)$ and for any k, we have*

$$\mathrm{Ext}^k_{\mathrm{Perv}(U)}(j^*\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^k_{\mathrm{Perv}(X)}(\mathcal{F}, j_*\mathcal{G}),$$

$$\mathrm{Ext}^k_{\mathrm{Perv}(X)}(j_!\mathcal{G}, \mathcal{F}) = \mathrm{Ext}^k_{\mathrm{Perv}(U)}(\mathcal{G}, j^*\mathcal{F}).$$

Note that, since U is assumed to be affine, $j_!, j_*$ are exact functors on perverse sheaves, by Artin's theorem, so the proposition above makes sense.

Proof. We have $j^! = j^* : \mathrm{Perv}(X) \to \mathrm{Perv}(U)$ exact and right (resp. left) adjoint to $j_!$ (resp. j_*). Lets prove the first adjunction of the proposition, the second is similar.

We have natural transformations

$$\mathrm{id}_{\mathrm{Perv}(X)} \to j_*j^*, j^*j_* \to \mathrm{id}_{\mathrm{Perv}(U)}$$

such that compositions

$$j_* \to j_*j^*j_* \to j_*, j^* \to j^*j_*j^* \to j^*$$

are identity transformations. Since the functors j_*, j^* are exact, they induce maps on Yoneda Ext-groups:

$$\mathrm{Ext}^k(j^*\mathcal{F}, \mathcal{G}) \to \mathrm{Ext}^k(j_*j^*\mathcal{F}, j_*\mathcal{G}) \to \mathrm{Ext}^k(\mathcal{F}, j_*\mathcal{G}),$$

$$\mathrm{Ext}^k(\mathcal{F}, j_*\mathcal{G}) \to \mathrm{Ext}^k(j^*\mathcal{F}, j^*j_*\mathcal{G}) \to \mathrm{Ext}^k(j^*\mathcal{F}, \mathcal{G}).$$

These maps are mutually inverse isomorphism: proof is the same as the proof that unit-counit adjunction induces an isomorphism on Hom-sets. $\square$

Knowing the above theorem and propositions, one then deduces Beilinson's theorem by induction in dimension, when X is affine: if $\mathcal{F}, \mathcal{G} \in \mathrm{Perv}(X)$ are supported on some closed subvariety, we use Proposition 6 and inductive assumption. If $\mathcal{F}$ has full support and $\mathcal{G}$ has closed support, one chooses an open affine set U disjoint from the support of $\mathcal{G}$ and deduces the theorem from the long sequences of Ext's coming from the exact triangle $j_! \mathcal{F}_U \to \mathcal{F} \to \mathcal{K} \to$ and five lemma.

If $\mathcal{F}, \mathcal{G}$ have full support, we choose U such that the restriction of $\mathcal{F}, \mathcal{G}$ to U are shifted local systems, and apply Theorem 4 along with the Propositions 6, 7 to the corresponding long exact sequences. We leave out the (somewhat complicated) details.

Exercise. Do it for $\dim X = 1$.

2.1 Realization functor

We will now describe how to construct the functor real from the theorem. We follow Section 1.10 of the book [Ach].

Let $\mathcal{A}$ be an abelian category. Denote by $F\mathcal{A}$ an additive category of filtered objects in $\mathcal{A}$, namely objects equipped with an increasing bounded filtration indexed by $\mathbb{Z}$:

$$\{0\} \subset \cdots \subset F_i M \subset F_{i+1} M \subset \cdots \subset M,$$

where the word bounded means that $F_i M = 0$ for $i \ll 0$, and $F_i M = M$ for $i \gg 0$. Morphisms in $F\mathcal{A}$ are morphisms in $\mathcal{A}$ preserving the filtration. We can form categories $\mathrm{Ch}^b(F\mathcal{A})$ and $\mathrm{Ho}^b(F\mathcal{A})$, a bounded category of chain complexes and a bounded homotopy category of $F\mathcal{A}$.

Category $\mathrm{Ho}^b(F\mathcal{A})$ comes equipped with the following structures:

1. the collection of strictly full triangulated subcategories $\mathcal{F}_{\leq n}, \mathcal{F}_{\geq n}$. $\mathcal{F}_{\leq n}$ is a subcategory consisting of complexes M with filtration F satisfying $F_i M = M$ for $i \geq n$, and $\mathcal{F}_{\geq n}$ is a subcategory consisting of complexes M with filtration F satisfying $F_i M = 0$ for $i < n$;

2. triangulated functors

$$\sigma_{\leq n} : \mathrm{Ho}^b(F\mathcal{A}) \to \mathcal{F}_{\leq n}, \sigma_{\geq n} : \mathrm{Ho}^b(F\mathcal{A}) \to \mathcal{F}_{\geq n},$$

truncating the filtration:

$$F_k(\sigma_{\leq n} M) = \begin{cases} F_n M & \text{if } k > n, \\ F_k M & \text{if } k \leq n; \end{cases}$$

$$F_k(\sigma_{\geq n} M) = \begin{cases} F_k M / F_{n-1} M & \text{if } k \geq n, \\ 0 & \text{if } k < n; \end{cases}$$

3. There is a unique natural transformation $\delta : \sigma_{\geq n+1} \to \sigma_{\leq n}[1]$ such that for any $X \in \mathrm{Ho}^b(F\mathcal{A})$, the diagram

$$\sigma_{\leq_n} X \to X \to \sigma_{\geq n+1} X \xrightarrow{\delta} \sigma_{\leq n} X[1]$$

is a distinguished triangle.

4. associated graded functors $\mathrm{gr}_n = \sigma_{\geq n}\sigma_{\leq n}$;
5. functors $i : \mathrm{Ho}^b(\mathcal{A}) \to \mathrm{Ho}^b(F\mathcal{A})$, given by

$$F_k i(M) = \begin{cases} M & \text{if } k \geq 0, \\ 0 & \text{if } k < 0; \end{cases}$$

and $\omega : \mathrm{Ho}^b(F\mathcal{A}) \to \mathrm{Ho}^b(\mathcal{A})$, forgetting the filtration. i is an equivalence $\mathrm{Ho}^b(\mathcal{A}) \to \mathcal{F}_{\leq 0} \cap \mathcal{F}_{\geq 0}$.

We say that a morphism $f : X \to Y$ in $\mathrm{Ho}^b(F\mathcal{A})$ is a filtered quasi-isomorphism if $\omega\,\mathrm{gr}_n\, f$ is a quasi-isomorphism in $\mathrm{Ho}^b(\mathcal{A})$. It can be shown that filtered quasi-isomorphisms satisfy the localization conditions. The localized category $DF^b(\mathcal{A})$ is called the filtered derived category of $\mathcal{A}$. It inherits all structures indicated above from $\mathrm{Ho}^b(F\mathcal{A})$.

Theorem 5. *Let $\mathcal{T}$ be a full triangulated subcategory of $D^b(\mathcal{A})$. Suppose that it is equipped with a bounded t-structure, and let $\mathcal{C}$ be its heart. Then there is a t-exact, triangulated functor*

$$\mathrm{real} : D^b(\mathcal{C}) \to \mathcal{T}$$

whose restriction to $\mathcal{C} \subset D^b(\mathcal{C})$ is the inclusion functor $\mathcal{C} \to \mathcal{T}$.

We only indicate how the functor is constructed. Let $\tilde{\mathcal{C}}$ be the full additive subcategory of $DF^b(\mathcal{A})$ given by

$$\tilde{\mathcal{C}} = \{X \in \mathcal{F} | \omega\,\mathrm{gr}_n\, X \in \mathcal{C}[n]\}.$$

There is a functor $\beta : \tilde{\mathcal{C}} \to \mathrm{Ch}^b(\mathcal{C})$ constructed as follows. For $X \in \tilde{\mathcal{C}}, i \in \mathbb{Z}$, consider the canonical exact triangle for $\sigma_{\geq -i-1}\sigma_{\leq -i}X$:

$$\mathrm{gr}_{-i-1} X \to \sigma_{\geq -i-1}\sigma_{\leq -i}X \to \mathrm{gr}_{-i} X \xrightarrow{\delta} \mathrm{gr}_{-i-1} X[1].$$

Define $\beta(X)$ to be the chain complex $Y^\bullet$ with $Y^i = \omega\,\mathrm{gr}_{-i}\, X[i]$ and with differential given by $-\omega\delta[i]$. One then needs to check that this is indeed a complex, functorial in X, that $\omega\beta^{-1}$ factors through $\mathrm{Ho}^b(C)$ and sends quasi-isomorphisms to isomorphisms, all of which we skip.

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