ON THE QUANTIZATION OF CONJUGACY CLASSES

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ABSTRACT. Let G be a compact, simple, simply connected Lie group. A theorem of Freed-Hopkins-Teleman identifies the level $k \geq 0$ fusion ring $R_k(G)$ of G with the twisted equivariant K-homology at level $k + \mathsf{h}^\vee$, where $\mathsf{h}^\vee$ is the dual Coxeter number of G. In this paper, we will review this result using the language of Dixmier-Douady bundles. We show that the additive generators of the group $R_k(G)$ are obtained as K-homology push-forwards of the fundamental classes of pre-quantized conjugacy classes in G.

1. INTRODUCTION

A classical result of Dixmier-Douady [10] states that the integral degree three cohomology group $H^3(X)$ of a space X classifies bundles of C^*-algebras $\mathcal{A} \to X$, with typical fiber the compact operators on a Hilbert space. For any such *Dixmier-Douady bundle* $\mathcal{A} \to X$, one defines the twisted K-homology and K-cohomology groups of X as the K-groups of the C^*-algebra of sections of $\mathcal{A}$, vanishing at infinity:

$$K_q(X, \mathcal{A}) := K^q(\Gamma_0(X, \mathcal{A})), \quad K^q(X, \mathcal{A}) := K_q(\Gamma_0(X, \mathcal{A})).$$

If a group G acts by automorphisms of $\mathcal{A}$, one has definitions of G-equivariant K-groups.

The twisted K-groups have attracted a lot of interest in recent years, mainly due to their applications in string theory. For the case of torsion twistings, they were pioneered by Donovan-Karoubi [11] in 1963, while the general case was developed by Rosenberg [36] in 1989. Rosenberg also gave an alternative characterization of $K^0(X, \mathcal{A})$ as homotopy classes of sections of a bundle of Fredholm operators; this viewpoint was further explored by Atiyah-Segal [4] (see [6, 43] for alternative approaches).

One of the most natural examples of an integral degree three cohomology class comes from Lie theory. Let G be a compact, simple, simply connected Lie group, acting on itself by conjugation. The generator of $H^3_G(G) = \mathbb{Z}$ is realized by a G-Dixmier-Douady bundle $\mathcal{A} \to G$. Let $\mathsf{h}^\vee$ be the dual Coxeter number of G, and $k \geq 0$ a non-negative integer (the *level*). A beautiful result of Freed-Hopkins-Teleman [13, 14, 15, 16, 17] asserts that the twisted equivariant K-homology at the shifted level $k + \mathsf{h}^\vee$ coincides with the level k fusion ring (Verlinde algebra) of G:

$$K_0^G(G, \mathcal{A}^{k+\mathsf{h}^\vee}) = R_k(G). \tag{1}$$

Here $R_k(G)$ may be defined as the ring of positive energy level k representations of the loop group LG, or equivalently as the quotient $R_k(G) = R(G)/I_k(G)$ of the usual representation ring by the level k *fusion ideal*. The quotient map $R(G) \to R_k(G)$ is realized on the K-homology side as push-forward under inclusion $\{e\} \hookrightarrow G$, while the product on $R_k(G)$ is given by push-forward under group multiplication.

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As a $\mathbb{Z}$-module, the fusion ring $R_k(G)$ is freely generated by the set Λ_k^* of *level k weights* of G. In this paper the isomorphism $R_k(G) = \mathbb{Z}[\Lambda_k^*]$ is realized as follows. Given $\mu \in \Lambda_k^* \subset \mathfrak{t}^*$, (where $\mathfrak{t}$ is the Lie algebra of a maximal torus), let $\mathcal{C}$ be the conjugacy class of the element $\exp(\mu/k) \in G$, where the basic inner product is used to identify $\mathfrak{t}^* \cong \mathfrak{t}$. We will show that there is a canonical stable isomorphism between the restriction $\mathcal{A}^{k+\mathsf{h}^\vee}|_{\mathcal{C}}$ and the Clifford algebra bundle $\mathrm{Cl}(T\mathcal{C})$. This then defines a push-forward map in twisted K-homology, and the image of the K-homology fundamental class $[\mathcal{C}] \in K_0^G(\mathcal{C}, \mathrm{Cl}(T\mathcal{C}))$ under the push-forward is exactly the generator of $R_k(G)$ labeled by μ. This is parallel to the fact that the generators of $R(G) = \mathbb{Z}[\Lambda_+^*]$ are obtained by geometric quantization of the coadjoint orbits through dominant weights. In fact, as shown in [15] the generators of $R_k(G)$ can also be obtained by geometric quantization of coadjoint orbits of the loop group of G. Hence, our modest observation is that this can also be carried out in finite-dimensional terms. In a forthcoming paper with A. Alekseev, we will discuss more generally the quantization of group-valued moment maps [1] along similar lines.

A second theme in this paper is the construction of a canonical resolution of $R_k(G)$ in the category of $R(G)$-modules,

$$0 \longrightarrow C_l \xrightarrow{\partial} C_{l-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} R_k(G) \longrightarrow 0 \tag{2}$$

where $l = \mathrm{rank}(G)$. In more detail, let $\{0, \ldots, l\}$ label the vertices of the extended Dynkin diagram of G. For each non-empty subset $I \subset \{0, \ldots, l\}$, let $G_I \subset G$ be the maximal rank subgroup whose Dynkin diagram is obtained by deleting the vertices labeled by I. These groups have canonical central extensions $1 \to \mathrm{U}(1) \to \widehat{G}_I \to G_I \to 1$ (described below). Let $R(\widehat{G}_I)_k$ denote the Grothendieck group of all $\widehat{G}_I$-representations for which the central circle acts with weight k. Define

$$C_p = \bigoplus_{|I|=p+1} R(\widehat{G}_I)_k. \tag{3}$$

The differentials ∂ in (2) are given by holomorphic induction maps relative to the inclusions $\widehat{G}_I \hookrightarrow \widehat{G}_J$ for $J \subset I$. As we will explain, the chain complex $(C_\bullet, \partial)$ arises as the E^1-term of a spectral sequence computing $K_\bullet^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$, and the exactness of (2) implies that the spectral sequence collapses at the E^2-term. Since $R_k(G)$ is free Abelian, there are no extension problems, and one recovers the equality $K_0^G(G, \mathcal{A}^{k+\mathsf{h}^\vee}) = R_k(G)$ as $R(G)$-modules, and hence also as rings.

This article does not make great claims of originality. In particular, I learned that a very similar computation of the twisted equivariant K-groups of a Lie group had appeared in the article *Thom Prospectra for loop group representations* by Kitchloo-Morava [25]. The argument itself may be viewed as a natural generalization of the Mayer-Vietoris calculation for $G = \mathrm{SU}(2)$, as explained by Dan Freed in [13]. Independently, the chain complex had been obtained by Christopher Douglas (unpublished), who used it to obtain information about the algebraic structure of the fusion ring $R_k(G)$.

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Contents

2. Review of twisted equivariant K-homology

Throughout this paper, all Hilbert spaces $\mathcal{H}$ will be taken to be separable, but not necessarily infinite-dimensional. All (topological) spaces X will be assumed to allow the structure of a countable CW-complex (respectively G-CW complex, in the equivariant case).

2.1. **Dixmier-Douady bundles.** [10, 35, 36] For any Hilbert space $\mathcal{H}$, we denote by $\mathrm{U}(\mathcal{H})$ the unitary group, with the strong operator topology. Let $\mathbb{K}(\mathcal{H})$ be the C^*-algebra of compact operators, that is, the norm closure of the finite rank operators. The conjugation action of the unitary group on $\mathbb{K}(\mathcal{H})$ descends to the projective unitary group, and provides an isomorphism, $\mathrm{Aut}(\mathbb{K}(\mathcal{H})) = \mathrm{PU}(\mathcal{H})$. A *Dixmier-Douady bundle* $\mathcal{A} \to X$ is a locally trivial bundle of C^*-algebras, with typical fiber $\mathbb{K}(\mathcal{H})$ and structure group $\mathrm{PU}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. That is,

$$\mathcal{A} = \mathcal{P} \times_{\mathrm{PU}(\mathcal{H})} \mathbb{K}(\mathcal{H}) \tag{4}$$

for a principal $\mathrm{PU}(\mathcal{H})$-bundle $\mathcal{P} \to X$. Dixmier-Douady bundles of finite rank are also known as Azumaya bundles [26, 27]. A *gauge transformation* of $\mathcal{A}$ is a bundle automorphism inducing the identity on X, and whose restriction to the fibers are C^*-algebra automorphisms. Equivalently, the group of gauge transformations consists of sections of the associated group bundle, $\mathrm{Aut}(\mathcal{A}) = \mathcal{P} \times_{\mathrm{PU}(\mathcal{H})} \mathrm{Aut}(\mathbb{K}(\mathcal{H}))$. This group bundle has a central extension

$$1 \to X \times \mathrm{U}(1) \to \widetilde{\mathrm{Aut}}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to 1, \tag{5}$$

where $\widetilde{\mathrm{Aut}}(\mathcal{A}) = \mathcal{P} \times_{\mathrm{PU}(\mathcal{H})} \mathrm{U}(\mathcal{H})$.

If $\mathcal{A}_1, \mathcal{A}_2$ are Dixmier-Douady bundles modeled on $\mathbb{K}(\mathcal{H}_1), \mathbb{K}(\mathcal{H}_2)$, then their (fiberwise) C^*-tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a Dixmier-Douady bundle modeled on $\mathbb{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Also, the (fiberwise) opposite $\mathcal{A}^{\mathrm{opp}}$ of a Dixmier-Douady bundle modeled on $\mathbb{K}(\mathcal{H})$ is a Dixmier-Douady bundle modeled on $\mathbb{K}(\mathcal{H}^{\mathrm{opp}})$. Here the Hilbert space $\mathcal{H}^{\mathrm{opp}}$ is equal to $\mathcal{H}$ as an additive group, but with the new scalar multiplication by $z \in \mathbb{C}$ equal to the old scalar multiplication by $\overline{z}$.

A *Morita isomorphism* between two Dixmier-Douady bundles $\mathcal{A}_1, \mathcal{A}_2 \to X$ is a lift of the structure group $\mathrm{PU}(\mathcal{H}_2) \times \mathrm{PU}(\mathcal{H}_1^{\mathrm{opp}})$ of $\mathcal{A}_2 \otimes \mathcal{A}_1^{\mathrm{opp}}$ to the group $\mathrm{P}(\mathrm{U}(\mathcal{H}_2) \times \mathrm{U}(\mathcal{H}_1^{\mathrm{opp}}))$. It is thus given by a bundle $\mathcal{E} \to X$ of $\mathcal{A}_2 - \mathcal{A}_1$-bimodules, modeled on the $\mathbb{K}(\mathcal{H}_2) - \mathbb{K}(\mathcal{H}_1)$-bimodule $\mathbb{K}(\mathcal{H}_1, \mathcal{H}_2)$. We will write $\mathcal{A}_1 \simeq_{\mathcal{E}} \mathcal{A}_2$ if $\mathcal{E}$ defines such a Morita isomorphism, and $\mathcal{A}_1 \simeq \mathcal{A}_2$ if $\mathcal{A}_1, \mathcal{A}_2$ are Morita isomorphic for *some* $\mathcal{E}$. Morita isomorphism is an equivalence relation: In particular, if $\mathcal{A}_1 \simeq_{\mathcal{E}} \mathcal{A}_2$ and $\mathcal{A}_2 \simeq_{\mathcal{F}} \mathcal{A}_3$, then the bundle $\mathcal{F} \otimes_{\mathcal{A}_2} \mathcal{E}$ (a completion of the algebraic tensor product over $\mathcal{A}_2$) defines a Morita isomorphism between $\mathcal{A}_1, \mathcal{A}_3$. The set of Morita isomorphism classes of Dixmier-Douady bundles over X is an Abelian group, with sum $[\mathcal{A}_1] + [\mathcal{A}_2] = [\mathcal{A}_1 \otimes \mathcal{A}_2]$, neutral element $0 = [\mathbb{C}]$, and inverse $-[\mathcal{A}] = [\mathcal{A}^{\mathrm{opp}}]$.

In particular, a *Morita trivialization* $\mathbb{C} \simeq_{\mathcal{E}} \mathcal{A}$ is a Hilbert space bundle $\mathcal{E}$ together with an isomorphism $\mathcal{A} \cong \mathbb{K}(\mathcal{E})$. The obstruction to the existence of a Morita trivialization is given by the Dixmier-Douady class[1] [10, 35]

$$\mathrm{DD}(\mathcal{A}) \in H^3(X).$$

The Dixmier-Douady class descends to a group isomorphism between Morita isomorphism classes of Dixmier-Douady bundles $\mathcal{A} \to X$ and $H^3(X)$.

Example 2.1. Let $V \to X$ be an oriented Euclidean vector bundle of rank k, and let $\mathrm{Cl}(V) \to X$ be the complex Clifford algebra bundle. If k is even, the bundle $\mathrm{Cl}(V)$ is a bundle of matrix algebras, and hence is a Dixmier-Douady bundle. A Morita trivialization

$$\mathbb{C} \simeq_{\mathsf{S}} \mathrm{Cl}(V)$$

is equivalent to the choice of a spinor module $\mathsf{S} \to X$, which in turn is equivalent to the choice of a Spin_c structure on V. For details, see Plymen [34]. The canonical anti-involution of $\mathrm{Cl}(V)$ defines an isomorphism $\mathrm{Cl}(V) \cong \mathrm{Cl}(V)^{\mathrm{opp}}$, thus

$$\mathrm{DD}(\mathrm{Cl}(V)) = \mathrm{DD}(\mathrm{Cl}(V)^{\mathrm{opp}}) = -\,\mathrm{DD}(\mathrm{Cl}(V))$$

showing that $\mathrm{DD}(\mathrm{Cl}(V))$ is 2-torsion. The Dixmier-Douady class $\mathrm{DD}(\mathrm{Cl}(V))$ is the third integral Stiefel-Whitney class $W^3(V) \in H^3(X)$ of the bundle. i.e. the image of $w_2(V) \in H^2(X, \mathbb{Z}_2)$

[1] We take all cohomology groups with integer coefficients, unless indicated otherwise.

under the Bockstein homomorphism. In the case of k odd, the even part $\mathrm{Cl}^+(V)$ is a Dixmier-Douady bundle, and a similar discussion applies.

If both $\mathcal{E},\mathcal{E}' \to X$ define Morita isomorphisms $\mathcal{A}_1 \simeq \mathcal{A}_2$, then the bundle of bi-module homomorphisms $L = \mathrm{Hom}_{\mathcal{A}_2-\mathcal{A}_1}(\mathcal{E},\mathcal{E}')$ is a Hermitian line bundle. We will call $\mathcal{E},\mathcal{E}'$ *equivalent* if this line bundle is isomorphic to the trivial line bundle. Conversely, if $\mathcal{E}$ is a Morita isomorphism then so is $\mathcal{E}' = \mathcal{E} \otimes L$, for any line bundle L. Thus, if $\mathcal{A}_1,\mathcal{A}_2$ have the same Dixmier-Douady class, then the equivalence classes of Morita isomorphisms $\mathcal{A}_1 \simeq_{\mathcal{E}} \mathcal{A}_2$ are a principal homogeneous space (torsor) over $H^2(X,\mathbb{Z})$. (In the example $\mathcal{A} = \mathrm{Cl}(V)$, this is the usual twist of Spin_c-structures by line bundles.)

Given a compact Lie group G acting on X, one may similarly define G-equivariant Dixmier-Douady bundles. All of the above extends to this equivariant setting: In particular, there is a G-equivariant Dixmier-Douady class $\mathrm{DD}_G(\mathcal{A}) \in H^3_G(X)$, which classifies G-Dixmier-Douady bundles up to G-equivariant Morita isomorphisms. The extension of the Dixmier-Douady theorem to the G-equivariant case was proved by Atiyah-Segal [4].

Still more generally, one can also consider $\mathbb{Z}_2$-graded G-Dixmier-Douady bundles $\mathcal{A} \to X$. Here, isomorphisms and tensor products are understood in the $\mathbb{Z}_2$-graded sense, and the bi-modules in the definition of Morita isomorphism are $\mathbb{Z}_2$-graded. We continue to denote by $\mathrm{DD}_G(\mathcal{A})$ the Dixmier-Douady class of $\mathcal{A}$ as an ungraded bundle. If $\mathrm{DD}_G(\mathcal{A}) = 0$, so that $\mathbb{C} \simeq_{\mathcal{E}} \mathcal{A}$, there is an obstruction in $H^1(X,\mathbb{Z}_2)$ for the existence of a compatible $\mathbb{Z}_2$-grading on $\mathcal{E}$. Hence, the map from Morita isomorphism classes of $\mathbb{Z}_2$-graded G-Dixmier-Douady bundles to those of ungraded G-Dixmier-Douady bundles is onto, with kernel $H^1(X,\mathbb{Z}_2)$. See Parker [32] and Atiyah-Segal [4] for details.

2.2. **Dixmier-Douady bundles related to central extensions.** We assume that G is compact and connected. Then $H^1_G(\mathrm{pt}) = 0$, while $H^2_G(\mathrm{pt})$ is the group of G-equivariant line bundles over a point, or equivalently $H^2_G(\mathrm{pt}) = \mathrm{Hom}(G,\mathrm{U}(1))$. The group $H^3_G(\mathrm{pt})$ is realized as the isomorphism classes of central extensions of G by $\mathrm{U}(1)$,

$$1 \to \mathrm{U}(1) \to \widehat{G} \to G \to 1. \tag{6}$$

For any such extension there is an associated G-equivariant line bundle $L = \widehat{G} \times_{\mathrm{U}(1)} \mathbb{C} \to G$ from which $\widehat{G}$ is recovered as the unit circle bundle. The group structure is encoded in an isomorphism

$$\mathrm{Mult}^* L \cong \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* L$$

where $\mathrm{Mult}\colon G\times G \to G$ is group multiplication, and pr_i are the two projections. For any $l \in \mathbb{Z}$, the l-th power $\widehat{G}^{(l)}$ of the extension is defined in terms of the l-th power of the corresponding line bundle. More generally one defines products of central extensions of G by $\mathrm{U}(1)$ in terms of the tensor products of the corresponding line bundles. The group of gauge transformations of a given central extension $\widehat{G}$ (i.e. group automorphisms covering the identity on G) is $H^2_G(\mathrm{pt}) = \mathrm{Hom}(G,\mathrm{U}(1))$.

From the interpretation via Dixmier-Douady bundles, the identification of $H^3_G(\mathrm{pt})$ with isomorphism classes of central extensions may be seen as follows: Given a G-equivariant Dixmier-Douady bundle $\mathcal{A} \to \mathrm{pt}$, the action of G defines a group homomorphism $G \to \mathrm{Aut}(\mathcal{A})$, and hence a central extension of G by pull-back of (5) (in the case $X = \mathrm{pt}$). Conversely, given a central extension $\widehat{G}$, choose a unitary representation $\widehat{G} \to \mathrm{U}(\mathcal{E})$ where the central circle $\mathrm{U}(1)$

acts by scalar multiplication. Then $\mathbb{K}(\mathcal{E}) \to \mathrm{pt}$ is a G-Dixmier-Douady bundle with the prescribed class in $H^3_G(\mathrm{pt})$. Note that we may take $\mathcal{E}$ to be of finite rank, reflecting that $H^3_G(\mathrm{pt})$ is torsion. (Recall that $H^p_G(\mathrm{pt}, \mathbb{R}) = H^p(BG, \mathbb{R}) = 0$ for p odd.)

Suppose X is a connected space, with $H^1(X)$ torsion-free, and with the trivial action of G. The Kuenneth theorem [38, Chapter 5.5] for $H^\bullet_G(X) = H^\bullet(X \times BG)$ gives a direct sum decomposition,

$$H^3_G(X) = H^3(X) \oplus (H^1(X) \otimes H^2_G(\mathrm{pt})) \oplus H^3_G(\mathrm{pt}).$$

For any G-Dixmier-Douady bundle $\mathcal{A} \to X$, we obtain a corresponding decomposition of $\mathrm{DD}_G(\mathcal{A})$. The first component is the non-equivariant class $\mathrm{DD}(\mathcal{A})$. The last summand is the class of the central extension of G, defined by the homomorphism $G \to \mathrm{Aut}(\mathcal{A}_{x_0})$ at any given base point $x_0 \in X$. To describe the middle summand, note that the family of actions $G \to \mathrm{Aut}(\mathcal{A}_x)$ defines a family of central extensions, by pull-back of (5),

$$1 \to \mathrm{U}(1) \to \widehat{G}_{(x)} \to G \to 1.$$

For any $x' \in X$, there exists an isomorphism $\widehat{G}_{(x)} \to \widehat{G}_{(x')}$ of central extensions, unique up to $\mathrm{Hom}(G, \mathrm{U}(1)) \cong H^2_G(\mathrm{pt})$. Since the latter group is discrete, it follows that the family $\widehat{G}_{(x)}$ carries a flat connection: Any path from a base point x_0 to x defines an isomorphism $\widehat{G} := \widehat{G}_{(x_0)} \to \widehat{G}_{(x)}$, depending only on the homotopy class of the path. We therefore obtain a holonomy homomorphism $\tau \colon \pi_1(X; x_0) \to H^2_G(\mathrm{pt})$, hence an element of $H^1(X) \otimes H^2_G(\mathrm{pt}) \subset H^3_G(X)$. This element is identified with the corresponding component of $\mathrm{DD}_G(\mathcal{A})$.

Remark 2.2. Any element of $H^1(X) \otimes H^2_G(\mathrm{pt})$ is realized in this way. Indeed, let $\mathcal{H} = L^2(G)$ with the left-regular representation of G. The homomorphism $\tau \colon \pi_1(X) \to H^2_G(\mathrm{pt}) = \mathrm{Hom}(G, \mathrm{U}(1))$ defines a unitary action of $\pi_1(X)$ on $\mathcal{H}$, where $\lambda \in \pi_1(X)$ acts as pointwise multiplication by the function $\tau(\lambda)$. The actions of G and $\pi_1(X)$ commute up to a scalar. The bundle $\mathcal{A} = \widetilde{X} \times_{\pi_1(X)} \mathbb{K}(\mathcal{H})$ associated to the universal covering $\widetilde{X} \to X$ is a G-equivariant Dixmier-Douady bundle, with $\mathrm{DD}_G(\mathcal{A})$ the prescribed class in $H^1(X) \otimes H^2_G(\mathrm{pt})$. Note that the component in $H^3(X)$ is zero, since non-equivariantly $\mathcal{A} = \mathbb{K}(\mathcal{E})$ for $\mathcal{E} = \widetilde{X} \times_{\pi_1(X)} \mathcal{H}$.

2.3. **Twisted K-homology.** The input for the twisted equivariant K-homology of a G-space X is a $\mathbb{Z}_2$-graded G-Dixmier-Douady bundle $\mathcal{A} \to X$. From now on, we will usually omit explicit mention of the $\mathbb{Z}_2$-grading (which may be trivial), with the understanding that all tensor products are in the $\mathbb{Z}_2$-graded sense, isomorphisms should preserve the $\mathbb{Z}_2$-grading, and so on.

Given $\mathcal{A} \to X$, the space $\mathsf{A} = \Gamma_0(X, \mathcal{A})$ of continuous sections of $\mathcal{A}$ vanishing at infinity is a ($\mathbb{Z}_2$-graded) $G - C^*$-algebra, with norm $||s|| = \sup_{x \in X} ||s_x||_{\mathcal{A}_x}$. Following J. Rosenberg [36], we define the twisted equivariant K-homology and K-cohomology groups as the equivariant C^*-algebra K-homology and K-cohomology groups of A:

$$K^G_q(X, \mathcal{A}) := K^q_G(\Gamma_0(X, \mathcal{A})), \quad K^q_G(X, \mathcal{A}) := K^G_q(\Gamma_0(X, \mathcal{A})).$$

In this paper, we will mostly work with the K-homology groups. See the appendix for a quick review of the K-homology of C^*-algebras, and some examples. We list some basic properties of the K-homology groups.

(i) **Morita isomorphisms.** Any Morita isomorphism $\mathcal{A}_1 \simeq_{\mathcal{E}} \mathcal{A}_2$ of G-Dixmier-Douady bundles over X induces an isomorphism in K-homology,

$$K_q^G(X, \mathcal{A}_1) \cong K_q^G(X, \mathcal{A}_2).$$

(ii) **Push-forwards**. The morphisms in the category of G-Dixmier-Douady bundles $(X, \mathcal{A})$ are the equivariant C^*-algebra bundle maps $\mathcal{A}_1 \to \mathcal{A}_2$ for which the induced map on the base $f\colon X_1 \to X_2$ is *proper*. Any such morphism induces a morphism of $G-C^*$-algebras $f^*\colon \Gamma_0(X_2, \mathcal{A}_2) \to \Gamma_0(X_1, \mathcal{A}_1)$, hence a push-forward in K-homology

$$K_q^G(f)\colon K_q^G(X_1, \mathcal{A}_1) \to K_q^G(X_2, \mathcal{A}_2).$$

In this way $K_\bullet^G$ becomes a covariant functor, invariant under proper G-homotopies.

(iii) **Excision**. For any closed, invariant subset $Y \subset X$, with complement $U = X \backslash Y$, there is a long exact sequence[2]

$$\cdots \to K_q^G(Y, \mathcal{A}|_Y) \to K_q^G(X, \mathcal{A}) \to K_q^G(U, \mathcal{A}|_U) \to K_{q-1}^G(Y, \mathcal{A}|_Y) \to \cdots$$

Here the restriction map $K_q^G(X, \mathcal{A}) \to K_q^G(U, \mathcal{A}|_U)$ is induced by the C^*-algebra morphism $\Gamma_0(U, \mathcal{A}|_U) \to \Gamma_0(X, \mathcal{A})$, given as extension by 0. More generally, one obtains a spectral sequence for any filtration of X by closed, invariant subspaces.

(iv) **Products**. Suppose $\mathcal{A} \to X$ and $\mathcal{B} \to Y$ are two G-Dixmier-Douady bundles. Then the exterior tensor product $\mathcal{A} \boxtimes \mathcal{B} \to X \times Y$ is again a G-Dixmier-Douady bundle. Its space of sections is the C^*-tensor product of the spaces of sections of $\mathcal{A}, \mathcal{B}$. As a special case of the Kasparov product in K-homology, one has a natural associative cross product,

$$K_\bullet^G(X, \mathcal{A}) \otimes K_\bullet^G(Y, \mathcal{B}) \to K_\bullet^G(X \times Y,\ \mathcal{A} \boxtimes \mathcal{B}).$$

(v) **Module structure**. The group $K_0^G(\mathrm{pt})$ is canonically identified with the representation ring $R(G)$. The ring structure on $K_0^G(\mathrm{pt})$ is defined by the cross product for $\mathbb{C} \boxtimes \mathbb{C} \to \mathrm{pt} \times \mathrm{pt}$. Similarly, if $\mathcal{A} \to X$ is a G-Dixmier-Douady bundle, the cross product for $\mathbb{C} \boxtimes \mathcal{A} \to \mathrm{pt} \times X$ makes $K_\bullet^G(X, \mathcal{A})$ into a module over $R(G)$. The maps $K_q^G(f)$ are $R(G)$-module homomorphisms.

If M is a manifold, one has the *Poincaré duality isomorphism* relating twisted K-homology and K-cohomology,

$$K_q^G(M, \mathcal{A}) \cong K_G^q(M, \mathcal{A}^{\mathrm{opp}} \otimes \mathrm{Cl}(TM)). \tag{7}$$

Here $\mathrm{Cl}(TM)$ is the Clifford algebra bundle for some choice of invariant metric. For $\mathcal{A} = \mathbb{C}$ the Poincaré duality was proved by Kasparov in [21, Section 8]; the result in the twisted case was obtained by J.-L. Tu [41, Theorem 3.1]. (See also [9, Section 2]). The image of $1 \in K_G^0(M)$ under this isomorphism is Kasparov's *K-homology fundamental class* [24]

$$[M] \in K_0^G(M, \mathrm{Cl}(TM)).$$

Remark 2.3. Note that $\mathrm{Cl}(TM)$ is a Dixmier-Douady bundle only if $\dim M$ is even. However, the definition of the twisted K-groups works for arbitrary bundles of C^*-algebras, and the isomorphism (7) holds in this sense (but with $\mathcal{A}$ a Dixmier-Douady bundle). Alternatively, one

[2] Note that K-homology is analogous to Borel-Moore homology (homology with non-compact supports), rather than ordinary homology.

may state the result in terms of Dixmier-Douady bundles, using $\mathrm{Cl}(TM) = \mathrm{Cl}^+(TM) \otimes \mathrm{Cl}(\mathbb{R})$ and the isomorphism $K^G_{q+1}(M, \mathcal{B}) = K^G_q(M, \mathcal{B} \otimes \mathrm{Cl}(\mathbb{R}))$.

The following basic computations in twisted equivariant K-homology may be deduced from their K-theory counterparts, using Poincaré duality.

(a) If $M = \mathrm{pt}$, the twisted K-homology is

$$K^G_0(\mathrm{pt}, \mathcal{A}) = R(\widehat{G})_{-1},$$

while $K^G_1(\mathrm{pt}, \mathcal{A}) = 0$. Here $\widehat{G}$ is the central extension defined by the action $G \to \mathrm{Aut}(\mathcal{A})$, and $R(\widehat{G})_{-1}$ is the Grothendieck group of $\widehat{G}$-representations where the central $\mathrm{U}(1)$ acts with weight -1.

(b) Suppose H is a closed subgroup of G. For any H-Dixmier-Douady bundle $\mathcal{B} \to Y$, there is a natural isomorphism

$$\mathbf{I}^G_H \colon K^H_q(Y, \mathcal{B} \otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{h})) \xrightarrow{\cong} K^G_q(G \times_H Y,\, G \times_H \mathcal{B}),$$

which is Poincaré dual to the isomorphism $K^q_G(G \times_H Y,\, G \times_H \mathcal{B}^{\mathrm{opp}}) \cong K^q_H(Y, \mathcal{B}^{\mathrm{opp}})$. If $Y = \mathrm{pt}$, the left hand side may be evaluated as in (a). If $H \subset H' \subset G$ are closed subgroups, we have

$$\mathbf{I}^G_H = \mathbf{I}^G_{H'} \circ \mathbf{I}^{H'}_H.$$

Here we are identifying $\mathrm{Cl}(\mathfrak{g}/\mathfrak{h}) \cong \mathrm{Cl}(\mathfrak{g}/\mathfrak{h}') \otimes \mathrm{Cl}(\mathfrak{h}'/\mathfrak{h})$, and we are using the canonical isomorphism $H' \times_H \mathrm{Cl}(\mathfrak{g}/\mathfrak{h}') \cong H'/H \times \mathrm{Cl}(\mathfrak{g}/\mathfrak{h}')$.

(c) Let $\mathcal{A} \to \mathrm{pt}$ be a G-Dixmier-Douady algebra as in (a), and let H be a closed subgroup of G. Then $G \times_H \mathcal{A}$ is canonically isomorphic to $\pi^*\mathcal{A}$, the pull-back under the map $\pi \colon G/H \to \mathrm{pt}$. By composing the map $\mathbf{I}^G_H$ with the push-forward $K^G_q(\pi)$, we obtain an *induction homomorphism*,

$$\mathrm{ind}^G_H \colon K^H_q(\mathrm{pt}, \mathcal{A} \otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{h})) \to K^G_q(\mathrm{pt}, \mathcal{A}).$$

An H-invariant complex structure on $\mathfrak{g}/\mathfrak{h}$ defines a spinor module S, hence a Morita trivialization $\mathbb{C} \simeq_{\mathsf{S}} \mathrm{Cl}(\mathfrak{g}/\mathfrak{h})$. In this case the induction map simplifies to a map

$$\mathrm{ind}^G_H \colon K^H_0(\mathrm{pt}, \mathcal{A}) = R(\widehat{H})_{-1} \to K^G_0(\mathrm{pt}, \mathcal{A}) = R(\widehat{G})_{-1}$$

known as *holomorphic induction*.

For other examples of calculations of twisted K-groups, see [6, Section 8].

3. The Dixmier-Douady bundle over G

For the rest of this paper, G will denote a compact, simple, simply connected Lie group, acting on itself by conjugation. Then $H^3_G(G)$ is canonically isomorphic to $\mathbb{Z}$. Hence there exists a G-Dixmier-Douady bundle $\mathcal{A} \to G$, unique up to Morita isomorphism, such that $\mathrm{DD}_G(G, \mathcal{A})$ corresponds to the generator $1 \in \mathbb{Z}$. Any two bundles $\mathcal{A}, \mathcal{A}' \to G$ representing the generator are related by a G-equivariant Morita isomorphism, unique up to equivalence (since $H^2_G(G) = 0$). The quickest construction of $\mathcal{A}$ is as an associated bundle

$$\mathcal{A} = P_eG \times_{L_eG} \mathbb{K}(\mathcal{H}),$$

where P_eG is the space of based paths in G, $L_eG = LG \cap P_eG$ the based loop group, and $\mathcal{H}$ a representation of the standard central extension $\widehat{LG}$ of LG where the central circle acts with weight -1. The construction given in this Section is essentially just a slow-paced version of this model for $\mathcal{A}$, avoiding some infinite-dimensional technicalities. Our strategy is to first give a direct construction of the family of central extensions of the centralizers $G_g \subset G$, corresponding to their action on $\mathcal{A}$.

3.1. **Pull-back to the maximal torus.** Let $T \subset G$ be a maximal torus of G, with Lie algebra $\mathfrak{t}$. Consider the map

$$H^3_G(G) \to H^3_T(T) \tag{8}$$

obtained by first restricting the action to T and then pulling back to T. We will compute the image of the generator of $H^3_G(G)$ under this map. Denote by $\Lambda \subset \mathfrak{t}$ the integral lattice (i.e. the kernel of $\exp\colon \mathfrak{t} \to T$). Recall that the *basic inner product* B on the Lie algebra $\mathfrak{g}$ is the unique invariant inner product, with the property that the smallest length of a non-zero element $\lambda \in \Lambda$ equals $\sqrt{2}$. One of the key properties of B is that it restricts to an integer-valued bilinear form on Λ. That is, $B|_{\mathfrak{t}} \in \Lambda^* \otimes \Lambda^*$ where $\Lambda^* = \mathrm{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$ is the (real) weight lattice.

Proposition 3.1. *The map* (8) *is injective, and takes the generator of $H^3_G(G)$ to the element*

$$-B|_{\mathfrak{t}} \in \Lambda^* \otimes \Lambda^* \cong H^2_T(\mathrm{pt}) \otimes H^1(T) \subset H^3_T(T) \tag{9}$$

given by minus the basic inner product.

Proof. Since $H_G(G)$ and $H_T(T)$ have no torsion in degree ≤ 3, we may pass to real coefficients, and hence work with Cartan's equivariant de Rham model $\Omega^p_G(M) = \bigoplus_{2i+j=p}(S^i\mathfrak{g}^* \otimes \Omega^j(M))^G$ for the equivariant cohomology $H_G(M, \mathbb{R})$ of a G-manifold, with differential $(\mathrm{d}_G\alpha)(\xi) = \mathrm{d}\alpha(\xi) - \iota(\xi_M)\alpha(\xi)$ where ξ_M is the vector field defined by $\xi \in \mathfrak{g}$. Note $H^3_T(T, \mathbb{R}) = \mathfrak{t}^* \otimes H^1(T) \oplus H^3(T, \mathbb{R})$ since the T-action on T is trivial. Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left-, right- invariant Maurer Cartan forms. The generator of $H^3_G(G)$ is represented by an equivariant de Rham form,

$$\eta_G(\xi) = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) - \frac{1}{2}B(\theta^L + \theta^R, \xi). \tag{10}$$

Its pull-back to T is $\iota_T^*\eta_G(\xi) = -B(\theta_T, \xi)$, where $\theta_T \in \Omega^1(T, \mathfrak{t})$ the Maurer-Cartan form for T. Thus

$$\iota_T^*[\eta_G] = [B^\flat(\theta_T)] \in \mathfrak{t}^* \otimes H^1(T, \mathbb{R}) \subset H^3_T(T, \mathbb{R}).$$

The identification $H^1(T, \mathbb{R}) \cong \mathfrak{t}^*$ takes $[B^\flat(\theta_T)]$ to $B|_{\mathfrak{t}} \in \mathfrak{t}^* \otimes \mathfrak{t}^*$. ∎

3.2. **The family of central extensions $\widehat{T}_{(t)}$.** As discussed in Section 2.2, any element of $H^2_T(\mathrm{pt}) \otimes H^1(T)$ is realized as the holonomy of a family of central extensions. For any $\mu \in \Lambda^*$ let $T \to \mathrm{U}(1),\ t \mapsto t^\mu$ be the corresponding homomorphism. Let the lattice Λ act on $\widehat{T} = T \times \mathrm{U}(1)$ as

$$\Lambda \times \widehat{T} \to \widehat{T}, \quad \lambda.(h, z) = (h, h^{-B^\flat(\lambda)}z).$$

Then the holonomy of the family

$$\mathfrak{t} \times_\Lambda \widehat{T} \to \mathfrak{t}/\Lambda = T. \tag{11}$$

is the element $B|_{\mathfrak{t}}$. The action of the Weyl group $W = N(T)/T$ on T lifts to an action on this family, by

$$w.[(\xi; h, z)] = [(w\xi; wh, z)]. \tag{12}$$

Let $\widehat{T}_{(t)}$ be the fiber of (11) over $t \in T$. The choice of ξ with $\exp\xi = t$ defines a trivialization

$$T \to \widehat{T}_{(t)}, \quad h \mapsto [(\xi; h, 1)] \in \mathfrak{t} \times_\Lambda \widehat{T}. \tag{13}$$

Shifting ξ by $\lambda \in \Lambda$ changes the trivialization by the homomorphism $T \to \mathrm{U}(1),\ h \mapsto h^{-B^\flat(\lambda)}$.

3.3. **Simplicial description.** It will be useful to have the following equivalent description of the bundle (11). Let $\mathfrak{t}_+ \subset \mathfrak{t}$ be the choice of a closed Weyl chamber, and let $\Delta \subset \mathfrak{t}_+$ be the corresponding closed Weyl alcove. Recall that Δ labels the W-orbits in T, in the sense that every orbit contains a unique point in $\exp(\Delta)$. Label the vertices of Δ by $0, \ldots, l = \mathrm{rank}(G)$, in such a way that the label 0 corresponds to the origin. For every non-empty subset $I \subset \{0, \ldots, l\}$ let Δ_I denote the closed simplex spanned by the vertices in I, and let $W_I \subset W$ denote the subgroup fixing $\exp(\Delta_I) \subset T$. Then the maps $W/W_I \times \Delta_I \to T,\ (wW_I, \xi) \mapsto w \exp\xi$ define an isomorphism

$$T \cong \coprod_I W/W_I \times \Delta_I \Big/ \sim \tag{14}$$

using the identifications,

$$(x, \iota_J^I(\xi)) \sim (\phi_I^J(x), \xi), \quad J \subset I. \tag{15}$$

Here $\iota_J^I\colon \Delta_J \hookrightarrow \Delta_I$ is the natural inclusion, giving rise to an inclusion $W_I \hookrightarrow W_J$ of Lie groups and hence to projection $\phi_I^J\colon W/W_I \to W/W_J$.

Let $\lambda_I\colon W_I \to \Lambda$ be defined by $w\Delta_I = \Delta_I - \lambda_I(w)$. It is a group cocycle, $\lambda_I(uv) = \lambda_I(u) + u \cdot \lambda_I(v)$, and $\lambda_J|_{W_I} = \lambda_I$ for $J \subset I$. We thus obtains compatible actions of w_I on $\widehat{T} = T \times \mathrm{U}(1)$:

$$w.(h, z) = (wh,\ h^{-B^\flat(\lambda_I(w^{-1}))} z). \tag{16}$$

Lemma 3.2. *The isomorphism* (14) *extends to an isomorphism of the family* (11) *of central extensions,*

$$\bigcup_{t \in T} \widehat{T}_t = \mathfrak{t} \times_\Lambda \widehat{T} \cong \coprod_I (W \times_{W_I} \widehat{T}) \times \Delta_I \Big/ \sim . \tag{17}$$

Proof. The maps $\widehat{T} \times \Delta_I \to \mathfrak{t} \times_\Lambda \widehat{T},\ (h, z; \xi) \mapsto [(\xi, h, z)]$ are W_I-equivariant, by the calculation (for $\xi \in \Delta_I, \quad w \in W_I$)

$$\begin{aligned} w.[(\xi; h, z)] &= [(w\xi; wh, z)] = [(\xi - \lambda_I(w); wh, z)] \\ &= [(\xi; wh, (wh)^{B^\flat(\lambda_I(w))} z)] = [(\xi; wh,\ h^{-B^\flat(\lambda_I(w^{-1}))} z)]. \end{aligned}$$

They hence extend to W-equivariant maps $(W \times_{W_I} \widehat{T}) \times \Delta_I \to \mathfrak{t} \times_\Lambda \widehat{T}$, which glue to the desired isomorphism. $\square$

3.4. **The centralizers G_I and their central extensions.** For any $g \in G$, we denote by G_g its centralizer. For any given I, the centralizer $G_{\exp\xi}$ for ξ in the interior of Δ_I is independent of the choice of ξ, and will be denote G_I. Equivalently, G_I is the closed subgroup of G fixing

$\exp \Delta_I$. Each G_I is a connected subgroup containing T, and we have $W_I = N_{G_I}(T)/T$. For $J \subset I$ we have $G_I \subset G_J$. The description (14) of the maximal torus extends to the group G:

$$G \cong \coprod_I G/G_I \times \Delta_I \Big/ \sim \tag{18}$$

using the equivalence relations (15) for the natural maps $\phi_I^J \colon G/G_I \to G/G_J$ for $J \subset I$. In this Section, we generalize (17) to define a G-equivariant collection of central extensions,

$$\bigcup_{g \in G} \widehat{G}_g \cong \coprod_I (G \times_{G_I} \widehat{G}_I) \times \Delta_I \Big/ \sim .$$

(Of course, this is no longer a fiber bundle.) Our construction of $\mathcal{A} \to G$ will realize $\widehat{G}_g$ as the opposite of the central extension, defined by action of G_g on the fiber $\mathcal{A}_g$.

Lemma 3.3. *There are distinguished central extensions*

$$1 \to \mathrm{U}(1) \to \widehat{G}_I \to G_I \to 1,$$

together with lifts $\widehat{i}_I^J \colon \widehat{G}_I \hookrightarrow \widehat{G}_J$ of the inclusions $i_I^J \colon G_I \hookrightarrow G_J$ for $J \subset I$, such that

(a) $\widehat{G}_{\{0,\ldots,l\}} = \widehat{T}$,
(b) *the lifted inclusions satisfy the coherence condition $\widehat{i}_I^K = \widehat{i}_J^K \circ \widehat{i}_I^J$ for $K \subset J \subset I$,*
(c) *the W_I-action on $\widehat{T} \subset \widehat{G}_I$ (cf. (16)) is induced by the conjugation action of $N_{G_I}(T)$.*

Proof. Recall $\pi_1(G_I) = \Lambda/\Lambda_I$, where Λ_I is the co-root lattice of G_I [8, Theorem (7.1)]. But

$$\lambda \in \Lambda_I, \quad t \in \exp(\Delta_I) \Rightarrow t^{B^\flat(\lambda)} = 1$$

(see [28, Proposition 5.4]). Hence, for any given $t \in \exp(\Delta_I)$, there is a homomorphism

$$\varrho_{t,I} \colon \pi_1(G_I) = \Lambda/\Lambda_I \to \mathrm{U}(1), \quad \lambda + \Lambda_I \mapsto t^{-B^\flat(\lambda)}.$$

We therefore obtain a family of central extensions $\widehat{G}_{I,(t)} = \widetilde{G_I} \times_{\pi_1(G_I)} \mathrm{U}(1)$ parametrized by the points of $\exp(\Delta_I)$. Since $\exp(\Delta_I) \cong \Delta_I$ is contractible, we may use the flat connection on the family of central extensions (cf. Section 2.2) to identify all $\widehat{G}_{I,(t)}$. The resulting $\widehat{G}_I$ has the desired properties. In particular, if $J \subset I$ and $t \in \exp(\Delta_J) \subset \exp(\Delta_I)$, the homomorphism $\varrho_{t,I}$ is given by the inclusion $\pi_1(G_I) \to \pi_1(G_J)$ followed by $\varrho_{t,J}$. This defines an inclusion $\widehat{G}_{I,(t)} \hookrightarrow \widehat{G}_{J,(t)}$, compatible with the flat connection and (hence) satisfying the coherence condition. Fix $\xi \in \Delta$ with $\exp_T \xi = t$. The inclusion of $\widehat{T} = T \times \mathrm{U}(1)$ into $\widehat{G}_I \cong \widehat{G}_{I,(t)}$ is explicitly given as

$$i^I \colon (\exp_T \zeta, z) \mapsto [(\exp_{\widetilde{G}_I} \zeta,\, e^{-2\pi\sqrt{-1}B(\xi,\zeta)} z)], \tag{19}$$

for $\zeta \in \mathfrak{t},\ z \in \mathrm{U}(1)$. If $g \in N_{G_I}(T)$ lifts $w \in W_I$, we have

$$\begin{aligned} g.[(\exp_{\widetilde{G}_I} \zeta,\, e^{-2\pi\sqrt{-1}B(\xi,\zeta)} z)] &= [(\exp_{\widetilde{G}_I}(w.\zeta),\, e^{-2\pi\sqrt{-1}B(\xi,\zeta)} z)] \\ &= i^I(\exp_T(w.\zeta), e^{-2\pi\sqrt{-1}B(\xi,\zeta - w.\zeta)} z)) \\ &= i^I(w.(\exp_T \zeta,\ z)) \end{aligned}$$

proving that i^I is equivariant for the actions of W_I and $N_{G_I}(T)$. $\square$

Remarks 3.4. (i) The central extension $\widehat{G}_I$ admits a trivialization if and only if the affine span of $B^\flat(\Delta_I) \subset \mathfrak{t}^*$ contains a point in the weight lattice, Λ^*. In particular, this is the case whenever $0 \in I$. If G is of type A_n or C_n, then all $\widehat{G}_I$ are isomorphic to trivial extensions. (ii) The choice of any $t \in \exp(\Delta_I)$ gives a trivialization $\widehat{\mathfrak{g}}_I \cong \widehat{\mathfrak{g}}_{I,(t)} = \mathfrak{g}_I \times \mathbb{R}$, by the definition of $\widehat{G}_{I,(t)}$ as a quotient of $\widetilde{G}_I \times \mathrm{U}(1)$.

3.5. **Construction of the Dixmier-Douady bundle $\mathcal{A} \to G$.** Our construction of the Dixmier-Douady bundle $\mathcal{A} \to G$ involves a suitable Hilbert space $\mathcal{H}$.

Lemma 3.5. *There exists a Hilbert space $\mathcal{H}$, equipped with unitary representations of the central extensions $\widehat{G}_I$ such that (i) the central* $\mathrm{U}(1)$ *acts with weight -1, and (ii) for $J \subset I$ the action of $\widehat{G}_J$ restricts to the action of $\widehat{G}_I$.*

One may construct such an $\mathcal{H}$ using the theory of affine Lie algebras. Let $\mathcal{L}(\mathfrak{g}) = \mathfrak{g}^{\mathbb{C}} \otimes \mathbb{C}[z, z^{-1}]$ be the loop algebra associated to $\mathfrak{g}$. For all roots α of G, let $e_\alpha \in \mathfrak{g}^{\mathbb{C}}$ be the corresponding root vector. Then $\mathfrak{g}_I^{\mathbb{C}}$ is spanned by $\mathfrak{t}^{\mathbb{C}}$ together with the root vectors e_α such that $\langle \alpha, \xi \rangle \in \mathbb{Z}$ for $\xi \in \Delta_I$. The map $j_I \colon \mathfrak{g}_I^{\mathbb{C}} \to \mathcal{L}(\mathfrak{g})$ given by $\zeta \mapsto \zeta \otimes 1$ for $\zeta \in \mathfrak{t}^{\mathbb{C}}$ and

$$e_\alpha \mapsto e_\alpha \otimes z^{\langle \alpha, \xi \rangle},$$

for $\langle \alpha, \xi \rangle \in \mathbb{Z}$ is an injective Lie algebra homomorphism (independent of ξ). Consider the standard central extension $\widehat{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}\mathfrak{c}$, with bracket

$$[\zeta_1 \otimes f_1 + s_1\mathfrak{c},\ \zeta_2 \otimes f_2 + s_2\mathfrak{c}] = ([\zeta_1, \zeta_2] \otimes f_1 f_2) + B(\zeta_1, \zeta_2)\,\mathrm{Res}(f_1 \mathrm{d} f_2)\mathfrak{c}.$$

Its restriction to constant loops is canonically trivial, thus $\widehat{\mathfrak{t}}^C$ is embedded in $\mathcal{L}(\mathfrak{g}^{\mathbb{C}})$ by the map $(\zeta, s) \mapsto \zeta + s\mathfrak{c}$. The inclusions j_I lift to inclusions $\widehat{j}_I \colon \widehat{\mathfrak{g}}_I \hookrightarrow \widehat{\mathcal{L}}(\mathfrak{g})$ extending the given inclusion of $\widehat{\mathfrak{t}}^{\mathbb{C}}$. To see this, take $\xi \in \Delta_I$ (defining a trivialization $\mathfrak{g}_I \cong \mathfrak{g}_{I,(\exp\xi)} = \mathfrak{g}_I \times \mathbb{R}$). Then the desired lift reads,

$$\widehat{j}_{I,\xi} \colon \widehat{\mathfrak{g}}^{\mathbb{C}}_{I,(\exp\xi)} \to \widehat{\mathcal{L}}(\mathfrak{g}), \quad \widehat{j}_{I,\xi}(\zeta, s) = j_I(\zeta) + (s + B(\xi, \zeta))\mathfrak{c}.$$

By the theory of affine Lie algebras [20], there exists a unitarizable $\widehat{L\mathfrak{g}}$-module where the central element $\mathfrak{c}$ acts as -1. Unitarizibility means in particular that the $\widehat{\mathfrak{t}}$-action exponentiates to a unitary $\widehat{T}$-action, and hence all $\widehat{\mathfrak{g}}_I$-actions exponentiate to unitary $\widehat{G}_I$-actions.

With $\mathcal{H}$ as in the Lemma, put $\mathcal{A}_I = G \times_{G_I} \mathbb{K}(\mathcal{H})$. For $J \subset I$, the map $\phi_I^J \colon G/G_I \to G/G_J$ is covered by a homomorphism of Dixmier-Douady bundles, $\mathcal{A}_I \to \mathcal{A}_J$. Hence we may define a G-Dixmier-Douady bundle,

$$\mathcal{A} = \coprod_I (\mathcal{A}_I \times \Delta_I)/\sim \tag{20}$$

with identifications similar to those in (18). By construction, the central extension of G_I defined by the restriction $\mathcal{A}|_{\exp(\Delta_I)}$ coincides with the opposite of $\widehat{G}_I$. Hence, the family of central extensions defined by the action of T on $\mathcal{A}|_T$ is the opposite of the family $\widehat{T}_{(t)}$. We had seen that the class in $H^2_T(\mathrm{pt}) \otimes H^1(T) \subset H^3_T(T)$ is the class defined by $-B_{\mathfrak{t}}$, and hence coincides with the image of the generator of $H^3_G(G) \cong \mathbb{Z}$. It follows that $\mathrm{DD}_G(\mathcal{A})$ is a generator of $H^3_G(G)$.

4. Conjugacy classes

As is well-known, coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$ carry a distinguished invariant complex structure, hence a Spin_c-structure. If $\mathcal{O}$ admits a pre-quantum line bundle $L \to \mathcal{O}$ (i.e. a line bundle with curvature equal to the symplectic form), one may twist the original Spin_c-structure by this line bundle. The resulting equivariant index is the irreducible representation parametrized by $\mathcal{O}$. In this Section, we will describe a similar picture for conjugacy classes $\mathcal{C} \subset G$.

4.1. **Pull-back to conjugacy classes.** Given $\xi \in \Delta$, define a G-equivariant map $\Psi\colon G/T \to G,\ gT \mapsto \mathrm{Ad}_g(\exp\xi)$. The pull-back $\Psi^*\mathcal{A}$ admits a canonical Morita trivialization, defined by the Hilbert space bundle $G \times_T \mathcal{H}$. More generally, for any $l \in \mathbb{Z}$ and any weight $\mu \in \Lambda^*$ there is a Morita trivialization,

$$\mathbb{C} \simeq_{\mathcal{E}} \Psi^*\mathcal{A}^l, \quad \mathcal{E} = G \times_T (\mathcal{H}^l \otimes \mathbb{C}_\mu) \tag{21}$$

where $\mathbb{C}_\mu$ is the 1-dimensional 1-dimensional T-representation of weight μ. Equivariant Dixmier-Douady bundles over G, together with Morita trivializations of their pull-backs by Ψ, are classified by the relative cohomology group $H^3_G(\Psi)$. (See Appendix A.) The map $\Psi =: \Psi_1$ is equivariantly homotopic to the constant map $\Psi_0\colon gT \mapsto e$, by the homotopy $\Psi_t(gT) = \exp(t\,\mathrm{Ad}_g(\xi))$. Hence $H^3_G(\Psi) = H^3_G(\Psi_0) = H^2_G(G/T) \oplus H^3_G(G)$. Identifying $H^2_G(G/T) = H^2_T(\mathrm{pt}) = \Lambda^*$ and $H^3_G(G) = \mathbb{Z}$, we obtain an isomorphism

$$H^3_G(\Psi) = \Lambda^* \oplus \mathbb{Z},$$

The element $(\mu, l) \in H^3_G(\Psi)$ is realized by the Morita trivialization (21).

Now let $\mathcal{C}$ be the conjugacy class of $\exp(\xi)$, and $\Phi\colon \mathcal{C} \to G$ the inclusion. Let $\pi\colon G/T \to \mathcal{C}$ be the G-invariant projection such that $\Psi = \Phi \circ \pi$. We obtain a map of long exact sequences in relative cohomology,

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & H^2_G(\mathcal{C}) & \longrightarrow & H^3_G(\Phi) & \longrightarrow & H^3_G(G) & \longrightarrow & H^3_G(\mathcal{C}) & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle =} & & \big\downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & H^2_G(G/T) & \longrightarrow & H^3_G(\Psi) & \longrightarrow & H^3_G(G) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

From the identifications $H^2_G(\mathcal{C}) = \mathrm{Hom}(G_{\exp\xi}, \mathrm{U}(1))$ and $H^2_G(G/T) = \mathrm{Hom}(T, \mathrm{U}(1))$, it is evident that the second vertical map is injective. Hence the 5-Lemma implies that the map $H^3_G(\Phi) \to H^3_G(\Psi)$ is injective. Hence we obtain an injective map,

$$H^3_G(\Phi) \to H^3_G(\Psi) = \Lambda^* \oplus \mathbb{Z}.$$

By a parallel discussion with real coefficients, there is an injective map $H^3_G(\Phi, \mathbb{R}) \to H^3_G(\Psi, \mathbb{R}) = \mathfrak{t}^* \oplus \mathbb{R}$.

4.2. **Pre-quantization of conjugacy classes.** We return to Cartan's de Rham model for $H^\bullet_G(M, \mathbb{R})$ (cf. the proof of Proposition 3.1) with $\eta_G \in \Omega^3_G(G)$ representing the generator of $H^3_G(G)$. The conjugacy class $\mathcal{C}$ carries a unique invariant 2-form $\omega \in \Omega^2(\mathcal{C})^G \subset \Omega^2_G(\mathcal{C})$ with the property [1, 18],

$$\mathrm{d}_G\omega = \Phi^*\eta_G. \tag{22}$$

The triple $(\mathcal{C}, \omega, \Phi)$ is an example of a quasi-Hamiltonian G-space in the terminology of [1]. Equation (22) together with $\mathrm{d}_G \eta_G = 0$ say that $(\omega, \eta_G) \in \Omega^3_G(\Phi)$ is a relative equivariant cocycle. Let $[(\omega, \eta_G)]$ be its class in $H^3_G(\Phi, \mathbb{R})$.

Lemma 4.1. *The inclusion* $H^3_G(\Phi, \mathbb{R}) \to \mathfrak{t}^* \oplus \mathbb{R}$ *takes the class* $[(\omega, \eta_G)]$ *to the element* $(B^\flat(\xi), 1)$.

Proof. Let $h_t \colon \Omega^\bullet_G(G) \to \Omega^{\bullet-1}_G(G/T)$ be the homotopy operator defined by homotopy Ψ_t. Thus $\mathrm{d} \circ h_t + h_t \circ \mathrm{d} = \Psi_t^* - \Psi_0^*$. Then

$$\Omega^\bullet_G(\Psi_t) \to \Omega^\bullet_G(\Psi_0), \quad (\alpha, \beta) \mapsto (\alpha - h_t(\beta), \beta)$$

is an isomorphism of chain complexes, inducing the isomorphism $H^\bullet_G(\Psi_t, \mathbb{R}) \to H^\bullet_G(\Psi_0, \mathbb{R})$. In particular, the isomorphism $H^3_G(\Psi_1, \mathbb{R}) \to H^\bullet_G(\Psi_0, \mathbb{R})$ takes $[(\omega, \eta_G)]$ to $[(\omega - h_1^* \eta_G, \eta_G)]$.

The family of maps Ψ_t is a composition of the map $f \colon G/T \to \mathfrak{g},\ gT \mapsto \mathrm{Ad}_g(\xi)$ with the family of maps $\mathfrak{g} \to G,\ \zeta \mapsto \exp(t\zeta)$. Let $j_t \colon \Omega^\bullet_G(G) \to \Omega^{\bullet-1}(\mathfrak{g})$ be the homotopy operator for the second family of maps. Then $h_t = f^* \circ j_t$. By [28], we have $j_1 \eta_G = \varpi_G$, where $\varpi_G \in \Omega^2_G(\mathfrak{g})$ is of the form $\varpi_G(\zeta)|_\xi = \varpi|_\xi - B(\xi, \cdot)$. It follows that the image of $[(\omega, \eta_G)]$ under the map to $\mathfrak{t}^* \oplus \mathbb{R}$ is $(B^\flat(\xi), 1)$. □

As a special case of pre-quantization of group-valued moment maps [2], we define:

Definition 4.2. A *level* $k \in \mathbb{Z}$ *pre-quantization* of a conjugacy class $\mathcal{C}$ is a lift of the class $k\,[(\omega, \eta_G)] \in H^3_G(\Phi, \mathbb{R})$ to an integral class.

By the long exact sequence in relative cohomology, if $\mathcal{C}$ admits a level k pre-quantization, then the latter is unique (since $H^2_G(\mathcal{C})$ has no torsion).

Proposition 4.3. *The conjugacy class* $\mathcal{C}$ *of the element* $\exp \xi$ *with* $\xi \in \Delta$ *admits a pre-quantization at level* k *if and only if* $(B^\flat(k\xi), k) \in \Lambda^* \times \mathbb{Z}$.

Proof. According to the Lemma, $k\,[(\omega, \eta_G)]$ maps to $(B^\flat(k\xi), k) \in \mathfrak{t}^* \times \mathbb{R}$. Since all maps in the commutative diagram

$$\begin{array}{ccc} H^3_G(\Phi) & \longrightarrow & \Lambda^* \oplus \mathbb{Z} \\ \big\downarrow & & \big\downarrow \\ H^3_G(\Phi, \mathbb{R}) & \longrightarrow & \mathfrak{t}^* \oplus \mathbb{R} \end{array}$$

are injective, it follows that $k\,[(\omega, \eta_G)]$ is integral if and only if $(B^\flat(k\xi), k) \in \Lambda^* \times \mathbb{Z}$. □

Geometrically, a level k pre-quantization is given by a G-equivariant Morita trivialization of $\Phi^* \mathcal{A}^k$. This can be seen explicitly, as follows.

Lemma 4.4. *Let* $\xi \in \Delta_I$, *and suppose that* $B^\flat(k\xi) \in \Lambda^*$. *Then the* k*-th power of the central extension of* G_I *admits a unique trivialization* $G_I \to \widehat{G}_I^{(k)}$ *extending the map*

$$T \to \widehat{T}^{(k)} = T \times \mathrm{U}(1),\ h \mapsto (h, h^{B^\flat(k\xi)}). \tag{23}$$

Proof. By G_I-equivariance, a trivialization $G_I \to \widehat{G}_I^{(k)}$ is uniquely determined by its restriction to T. For existence, recall that $t = \exp \xi$ determines an identification $\widehat{G}_I \cong \widehat{G}_{I,(t)} = \widetilde{G}_I \times_{\pi_1(G_I)} \mathrm{U}(1)$, using the homomorphism $\varrho_{t,I} \colon \pi_1(G_I) = \Lambda/\Lambda_I \to \mathrm{U}(1),\ \lambda + \Lambda_I \mapsto t^{-B^\flat(\lambda)}$. The powers

$\widehat{G}^{(l)}$ are obtained similarly, using the l-th powers of the homomorphism $\varrho_{t,I}$. Since $B^\flat(k\xi)$ is a weight, we have

$$(\varrho_{t,I})^k(\lambda + \Lambda_I) = e^{-2\pi\sqrt{-1}B(k\xi,\lambda)} = 1.$$

This defines a trivialization,

$$\widehat{G}_I^{(k)} \cong \widehat{G}_{I,(t)}^{(k)} = G_I \times \mathrm{U}(1).$$

By (19), this trivialization intertwines the standard inclusion $\widehat{T}^{(k)} \to \widehat{G}_I^{(k)}$ with the map

$$\widehat{T} = T \times \mathrm{U}(1) \to G_I \times \mathrm{U}(1),\ (h, z) \mapsto (h, h^{-B^\flat(k\xi)} z).$$

The composition of this map with (23) is $h \mapsto (h, 1)$, as required. $\square$

Let $\Phi\colon \mathcal{C} \hookrightarrow G$ be the conjugacy class of $t = \exp\xi$, and let I be the unique index set such that ξ lies in the relative interior of Δ_I. If $\mathcal{C}$ is pre-quantizable at level k, so that $B^\flat(k\xi) \in \Lambda^*$, the Lemma defines a trivialization of $G_I^{(k)}$. Hence, its action on $\mathcal{H}^k$ descends to an action of G_I, and the Hilbert bundle $\mathcal{E} = G \times_{G_I} \mathcal{H}^k$ defines a Morita trivialization of $\Phi^*\mathcal{A}^k$.

Proposition 4.5. *The relative Dixmier-Douady class* $\mathrm{DD}_G(\mathcal{A}^k, \mathcal{E}) \in H^3_G(\Phi)$ *(cf. Appendix A) is an integral lift of the class* $k[(\omega, \eta_G)] \in H^3_G(G, \mathcal{C}, \mathbb{R})$.

Proof. We have to show that the image of $\mathrm{DD}_G(\mathcal{A}, \mathcal{E})$ in $H^3_G(\Psi) = \Lambda^* \oplus \mathbb{Z}$ is $(B^\flat(k\xi), k)$. But this follows from the discussion in the last Section, since the pull-back of $\mathcal{E}$ under the map $\pi\colon G/T \to \mathcal{C}$ is

$$\pi^*\mathcal{E} = G \times_T (\mathcal{H}^k \otimes \mathbb{C}_{B^\flat(k\xi)}).$$

$\square$

4.3. **The $\mathsf{h}^\vee$-th power of the Dixmier-Douady bundle.** For any coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$, the compatible complex structure defines a G-invariant Spin_c-structure, i.e. Morita trivialization of $\mathrm{Cl}(T\mathcal{O})$. We show show that similarly, for all conjugacy classes $\mathcal{C} \subset G$, there is a distinguished Morita isomorphism between $\mathrm{Cl}(T\mathcal{C})$ and $\mathcal{A}^{\mathsf{h}^\vee}|_{\mathcal{C}}$, where $\mathsf{h}^\vee$ is the dual Coxeter number. That is, conjugacy classes carry a canonical 'twisted Spin_c-structure'. There are examples of conjugacy classes that do not admit Spin_c-structures let alone almost complex structures.

Example 4.6. The simplest example of a conjugacy class not admitting an almost complex structure is the conjugacy class $\mathcal{C} \cong \mathrm{Spin}(5)/\mathrm{Spin}(4) \cong S^4$ of the group $\mathrm{Spin}(5)$. (Its image in $\mathrm{SO}(5)$ is the conjugacy class of the matrix with entries $(-1, -1, -1, -1, 1)$ down the diagonal.) Similarly, the group $G = \mathrm{Spin}(9)$ has a conjugacy class G/H with $H = (\mathrm{SU}(2) \times \mathrm{Spin}(6))/\mathbb{Z}_2$ that does not admit a Spin_c-structure. Indeed, if such a Spin_c-structure existed it could be made G-equivariant (since G is simply connected), hence it would give an H-invariant Spin_c-structure on $\mathfrak{g}/\mathfrak{h}$. Since H is semi-simple, this is equivalent to the condition that the half-sum of positive roots of H, is a weight of H. But by explicit calculation, one checks that this is not the case. I thank Reyer Sjamaar for discussion of these and similar examples.

We will need some additional notation. Let $\mathfrak{S}_0 = \{\alpha_1, \ldots, \alpha_l\}$, $l = \mathrm{rank}(G)$, be a set of simple roots for $\mathfrak{g}$, relative to our choice of fundamental Weyl chamber. We denote by $\alpha_0 = -\alpha_{\max}$ minus the highest root, and let

$$\mathfrak{S} = \mathfrak{S}_0 \cup \{\alpha_0\} = \{\alpha_0, \ldots, \alpha_l\}.$$

Thus $\Delta \subset \mathfrak{t}_+$ is the l-simplex cut out by the inequalities $\langle \alpha_i, \cdot \rangle + \delta_{i,0} \geq 0$ for $i = 0, \ldots, l$, and $\mathfrak{t}_+$ is cut out by the inequalities with $i > 0$. The roots of G_I are those roots α of G for which $\langle \alpha, \xi \rangle \in \mathbb{Z}$ for $\xi \in \Delta_I$, and a set of simple roots is

$$\mathfrak{S}_I = \{\alpha_i \in \mathfrak{S} |\ i \notin I\}$$

That is, the Dynkin diagram of G_I is obtained from the extended Dynkin diagram of G by removing the vertices labeled by $i \in I$. Let ρ be the half-sum of positive roots of G, let $\rho^\sharp = B^\sharp(\rho)$ with $B^\sharp = (B^\flat)^{-1}$, and let

$$\mathsf{h}^\vee = 1 + \langle \alpha_{\max}, \rho^\sharp \rangle$$

be the dual Coxeter number.

Theorem 4.7. *For any conjugacy class $\Phi \colon \mathcal{C} \hookrightarrow G$, there is a distinguished G-equivariant Morita isomorphism* $\mathrm{Cl}(T\mathcal{C}) \simeq \Phi^* \mathcal{A}^{\mathsf{h}^\vee}$.

Proof. Let $\xi \in \Delta$ be the unique point of the alcove corresponding with $\exp \xi \in \mathcal{C}$, and I the index set such that $\xi \in \mathrm{int}(\Delta_I)$. Thus $\mathcal{C} = G/G_I$ and $\mathrm{Cl}(T\mathcal{C}) = G \times_{G_I} \mathrm{Cl}(\mathfrak{g}_I^\perp)$, where $\mathfrak{g}_I^\perp$ is the orthogonal complement of $\mathfrak{g}_I$ in $\mathfrak{g}$. By construction, $\Phi^* \mathcal{A}^{\mathsf{h}^\vee} = G \times_{G_I} \mathbb{K}(\mathcal{H}^{\mathsf{h}^\vee})$. Hence it is our task to construct a G_I-equivariant Morita isomorphism

$$\mathrm{Cl}(\mathfrak{g}_I^\perp) \simeq \mathbb{K}(\mathcal{H}^{\mathsf{h}^\vee}).$$

Let $\widehat{G}'_I$ be the central extension of G_I defined by its action on $\mathrm{Cl}(\mathfrak{g}_I^\perp)$. It fits into a pull-back diagram,

$$\begin{array}{ccc} \widehat{G}'_I & \longrightarrow & \mathrm{Spin}_c(\mathfrak{g}_I^\perp) \\ \downarrow & & \downarrow \\ G_I & \longrightarrow & \mathrm{SO}(\mathfrak{g}_I^\perp). \end{array}$$

Equivalently, $\widehat{G}'_I = \widetilde{G}_I \times_{\pi_1(G_I)} \mathrm{U}(1)$ where $\widetilde{G}_I$ is the universal covering group, and the homomorphism $\pi_1(G_I) \to \mathrm{U}(1)$ is defined by the commutative diagram,

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(G_I) & \longrightarrow & \widetilde{G}_I & \longrightarrow & G_I & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{Spin}_c(\mathfrak{g}_I^\perp) & \longrightarrow & \mathrm{SO}(\mathfrak{g}_I^\perp) & \longrightarrow & 1 \end{array}$$

Let Λ_I be the co-root lattice of G_I, so that $\pi_1(G_I) = \Lambda/\Lambda_I$. By a direct calculation (cf. Sternberg [40, Section 9.2]), the homomorphism $\pi_1(G_I) \to \mathrm{U}(1)$ is

$$\pi_1(G_I) = \Lambda/\Lambda_I \to \mathrm{U}(1),\ \lambda \mapsto e^{2\pi\sqrt{-1}\langle \rho - \rho_I, \lambda \rangle} = \pm 1 \tag{24}$$

where ρ is the half-sum of positive roots of G, and ρ_I is the half-sum of positive roots of G_I, relative to the given system $\mathfrak{S}_I$ of simple roots. Let

$$\nu_I = \frac{1}{\mathsf{h}^\vee}(\rho - \rho_I), \quad \nu_I^\sharp = B^\sharp(\nu_I). \tag{25}$$

The element $\nu_I^\sharp$ is contained in the the interior of the face Δ_I (see e.g. [30]). Hence, the homomorphism (24) is just the $-\mathsf{h}^\vee$-th power of the homomorphism $\varrho_{t,I}$, $t = \exp \nu_I^\sharp$ in the definition of $\widehat{G}_{I,(t)} \cong \widehat{G}_I$. That is, we have identified

$$\widehat{G}'_I = \widehat{G}_I^{(-\mathsf{h}^\vee)}.$$

Recall that $\widehat{G}_I$ acts with weight $-\mathsf{h}^\vee$ on $\mathcal{H}^{\mathsf{h}^\vee}$, or equivalently $\widehat{G}_I^{-\mathsf{h}^\vee}$ acts with weight 1. Hence, if S_I is any spinor module over $\mathrm{Cl}(\mathfrak{g}_I^\perp)$, the $\mathrm{Cl}(\mathfrak{g}_I^\perp) - \mathbb{K}(\mathcal{H}^{\mathsf{h}^\vee})$-bimodule

$$\mathrm{Hom}(\mathcal{H}^{\mathsf{h}^\vee}, \mathsf{S}_I)$$

is G_I-equivariant, and gives the desired Morita isomorphism $\mathrm{Cl}(\mathfrak{g}_I^\perp) \simeq \mathbb{K}(\mathcal{H}^{\mathsf{h}^\vee})$. An *explicit* spinor module S_I for $\mathrm{Cl}(\mathfrak{g}_I^\perp)$, constructed as follows. Let $\mathfrak{n}_+ \subset \mathfrak{g}^\mathbb{C}$ and $\mathfrak{n}_{I,+} \subset \mathfrak{g}_I^\mathbb{C}$ be the sum of root spaces for positive roots of G and G_I, respectively. (Here positivity is defined by the respective sets $\mathfrak{S}_0, \mathfrak{S}_I$ of simple roots.) Then $\mathsf{S} = \wedge \mathfrak{n}_+$ is a spinor module for $\mathrm{Cl}(\mathfrak{t}^\perp)$, and $\mathsf{S}^I = \wedge \mathfrak{n}_{I,+}$ is a spinor module for $\mathrm{Cl}(\mathfrak{g}_I \cap \mathfrak{t}^\perp)$. (Cf. [40, Section 9.2].) We define

$$\mathsf{S}_I = \mathrm{Hom}_{\mathrm{Cl}(\mathfrak{g}_I \cap \mathfrak{t}^\perp)}(\mathsf{S}^I, \mathsf{S}). \tag{26}$$

□

The spinor modules S, S^I are T-equivariant, since they are constructed using T-invariant complex structures on $\mathfrak{t}^\perp, \mathfrak{g}_I \cap \mathfrak{t}^\perp$. Hence S_I is T-equivariant as well.

Proposition 4.8. *Let $\mathcal{C}$ be the conjugacy class of* $\exp\xi$, $\xi \in \Delta$. *The pull-back of* $\mathrm{Cl}(T\mathcal{C})$ *under the projection map*

$$\pi\colon G/T \to \mathcal{C},\ \ gT \mapsto \mathrm{Ad}_g(\exp(\xi)).$$

admits a canonical G-equivariant Morita trivialization

$$\mathbb{C} \simeq \pi^* \mathrm{Cl}(T\mathcal{C}). \tag{27}$$

Proof. Let I be the index set such that G_I is the stabilizer of $\exp\xi$. We have $\pi^* \mathrm{Cl}(T\mathcal{C}) = \mathrm{Cl}(\pi^* T\mathcal{C}) = G \times_T \mathrm{Cl}(\mathfrak{g}_I^\perp)$. Hence we need a T-equivariant Morita trivialization of $\mathrm{Cl}(\mathfrak{g}_I^\perp)$, and this is provided by S_I. □

If the conjugacy class $\mathcal{C}$ is pre-quantized at level k, the Morita equivalences $\mathrm{Cl}(T\mathcal{C}) \simeq \Phi^*\mathcal{A}^{\mathsf{h}^\vee}$ and $\mathbb{C} \simeq \Phi^*\mathcal{A}^k$, combine to a Morita isomorphism

$$\mathrm{Cl}(T\mathcal{C}) \simeq \Phi^*\mathcal{A}^{k+\mathsf{h}^\vee} \tag{28}$$

Recall $\Psi = \Phi \circ \pi\colon G/T \to G$. The composition of the Morita isomorphisms (27) and $\mathrm{Cl}(T\mathcal{C}) \simeq \Phi^*\mathcal{A}^{\mathsf{h}^\vee}$ is the Morita trivialization $\mathbb{C} \simeq \Psi^*\mathcal{A}^{\mathsf{h}^\vee}$ defined by the bundle $G \times_T \mathcal{H}^{\mathsf{h}^\vee}$. It is thus labeled by $(0, \mathsf{h}^\vee) \in \Lambda^* \oplus \mathbb{Z}$. Hence, in the pre-quantized case, the composition of (27) and (28) is the Morita trivialization of $\Psi^*\mathcal{A}^{k+\mathsf{h}^\vee}$ parametrized by $(B^\flat(k\xi), k + \mathsf{h}^\vee) \in \Lambda^* \oplus \mathbb{Z}$.

4.4. **Freed-Hopkins-Teleman.** The twisted equivariant K-homology group

$$K_\bullet^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$$

carries a ring structure, with product given by the cross-product for $G \times G$, followed by push-forward under group multiplication $\mathrm{Mult}\colon G \times G \to G$. Indeed, since $\mathrm{Mult}^* x = \mathrm{pr}_1^* x + \mathrm{pr}_2^* x$ for all $x \in H^3_G(G, \mathbb{Z})$, there is a Morita isomorphism,

$$\mathrm{pr}_1^* \mathcal{A}^{k+\mathsf{h}^\vee} \otimes \mathrm{pr}_2^* \mathcal{A}^{k+\mathsf{h}^\vee} \simeq \mathrm{Mult}^* \mathcal{A}^{k+\mathsf{h}^\vee}.$$

The Morita bimodule is unique up to equivalence since $H^2_G(G\times G)=0$. It defines a product structure

$$K^G_\bullet(\mathrm{Mult})\colon K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})\otimes K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})\to K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee}),$$

given by the cross product $K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})\otimes K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})\to K^G_\bullet(G\times G,\mathrm{pr}_1^*\mathcal{A}^{k+\mathsf{h}^\vee}\otimes \mathrm{pr}_2^*\mathcal{A}^{k+\mathsf{h}^\vee})$ followed by $K^G_\bullet(\mathrm{Mult})$. The product is commutative and associative, again since the relevant Morita bimodules are unique up to equivalence. (For non-simply connected groups G, the existence of a ring structures on the twisted K-homology is a much more subtle matter [42].)

The inclusion $\iota\colon\{e\}\hookrightarrow G$ of the group unit induces a ring homomorphism

$$K^G_\bullet(\iota)\colon R(G)=K^G_\bullet(\mathrm{pt})\to K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee}). \tag{29}$$

Theorem 4.9 (Freed-Hopkins-Teleman). *For all non-negative integers $k\ge 0$ the ring homomorphism* (29) *is onto, with kernel the level k fusion ideal $I_k(G)\subset R(G)$. That is, $K^G_1(G,\mathcal{A}^{k+\mathsf{h}^\vee})=0$, while $K^G_0(G,\mathcal{A}^{k+\mathsf{h}^\vee})$ is canonically isomorphic to the level k fusion ring, $R_k(G)=R(G)/I_k(G)$.*

We will explain a proof of this Theorem in Section 5. The ring $R_k(G)$ may be defined as the ring of level k projective representations of the loop group LG, or in finite-dimensional terms (cf. [3]): Let

$$\Lambda^*_k=\Lambda^*\cap B^\flat(k\Delta)$$

be the set of *level k weights*. Identify $R(G)$ with ring of characters of G. Then $R_k(G)=R(G)/I_k(G)$, where $I_k(G)$ is the vanishing ideal of the set of elements $\{t_\nu\in T,\ \nu\in\Lambda^*_k\}$ where

$$t_\nu=\exp(B^\sharp(\tfrac{\nu+\rho}{k+\mathsf{h}^\vee})).$$

It turns out that as an additive group, $R_k(G)$ is freely generated by the images of irreducible characters χ_μ for $\mu\in\Lambda^*_k$. Thus $R_k(G)=\mathbb{Z}[\Lambda^*_k]$ additively.

Remark 4.10. If G has type ADE (so that all roots have equal length), the lattice $B^\sharp(\Lambda^*)\subset\mathfrak{t}$ is identified with the set of elements $\xi\in\mathfrak{t}$ with $\exp\xi\in Z(G)$, the center of G. Hence the ideal $I_k(G)$ may be characterized, in this case, as the vanishing ideal of the set of all $g\in G^{\mathrm{reg}}$ such that $g^{k+\mathsf{h}^\vee}\in Z(G)$.

Remark 4.11. Freed-Hopkins-Teleman compute twisted K-homology groups of G for arbitrary compact groups, not necessarily simply connected. The case of simple, simply connected groups considered here is considerably easier than the general case.

Remark 4.12. It is also very interesting to consider the non-equivariant twisted K-homology rings $K_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})$. These are studied are in the work of V. Braun [7] and C. Douglas [12].

4.5. **Quantization of conjugacy classes.** Suppose $\Phi\colon\mathcal{C}\hookrightarrow G$ is the conjugacy class of $\exp\xi,\ \xi\in\Delta$, pre-quantized at level $k\ge 0$. Thus $\mu:=B^\flat(k\xi)$ is a weight. The Morita isomorphism (28) defines a push-forward map in K-homology,

$$K^G_0(\Phi)\colon K^G_0(\mathcal{C},\mathrm{Cl}(T\mathcal{C}))\to K^G_0(G,\mathcal{A}^{k+\mathsf{h}^\vee}) \tag{30}$$

where $\Phi\colon\mathcal{C}\hookrightarrow G$ is the inclusion.

Theorem 4.13. *The push-forward map* (30) *takes the fundamental class $[\mathcal{C}]\in K^G_0(\mathcal{C},\mathrm{Cl}(T\mathcal{C}))$ to the equivalence class of the character χ_μ in $R_k(G)=R(G)/I_k(G)$.*

Proof. Let $\pi\colon G/T \to \mathcal{C}$ and $\Psi = \Phi \circ \pi\colon G/T \to G$ be as in Section 4.1. The Morita trivializations

$$\mathbb{C} \simeq \mathrm{Cl}(T(G/T)), \quad \mathbb{C} \simeq \pi^* \mathrm{Cl}(T\mathcal{C})$$

defined by $G \times_T \mathsf{S}$ resp. $G \times_T \mathsf{S}_I$ (cf. Proposition 4.8) define a push-forward map

$$K_0^G(\pi)\colon K_0^G\big(G/T, \mathrm{Cl}(T(G/T))\big) \cong K_0^G(G/T) \to K_0^G(\mathcal{C}, \mathrm{Cl}(T\mathcal{C}))$$

with $K_0^G(\pi)([G/T]) = [\mathcal{C}]$. Hence

$$K_0^G(\Phi)([\mathcal{C}]) = K_0^G(\Psi)([G/T]).$$

Recall now that $\Psi = \Psi_1$ is equivariantly homotopic to the constant map Ψ_0 onto $e \in G$. That is, the diagram

$$\begin{tikzcd} G/T \arrow[r, "\pi"'] \arrow[d, "p"] & \mathcal{C} \arrow[d, "\Phi"] \\ \mathrm{pt} \arrow[r, "\iota"'] & G, \end{tikzcd}$$

commutes up to a G-equivariant homotopy. As discussed at the end of Section 4.3, the composition of the Morita isomorphisms $\mathbb{C} \simeq \pi^* \mathrm{Cl}(T\mathcal{C})$ and $\mathrm{Cl}(T\mathcal{C}) \simeq \Phi^* \mathcal{A}^{k+\mathsf{h}^\vee}$ (see Equations (27) and (28)) is the Morita trivialization,

$$\Psi^* \mathcal{A}^{k+\mathsf{h}^\vee} \cong \mathbb{K}(G \times_T (\mathbb{C}_\mu \otimes \mathcal{H}^{k+\mathsf{h}^\vee})).$$

On the other hand, $\iota^* \mathcal{A}^{k+\mathsf{h}^\vee} = \mathbb{K}(\mathcal{H}^{k+\mathsf{h}^\vee})$ by construction of $\mathcal{A}$, hence

$$\Psi_0^* \mathcal{A}^{k+\mathsf{h}^\vee} \cong p^* \mathbb{K}(\mathcal{H}^{k+\mathsf{h}^\vee}) = \mathbb{K}(G \times_T \mathcal{H}^{k+\mathsf{h}^\vee}).$$

The two Morita isomorphisms are thus related by a twist by the line bundle $G \times_T \mathbb{C}_\mu$. It follows that $K_0^G(\Psi)$ is the automorphism of $K_0(G/T)$ defined by the class of the line bundle $G \times_T \mathbb{C}_\mu$, followed by $K_0^G(\Psi_0) = K_0^G(\iota) \circ K_0^G(p)$. But $K_0^G(p)$ is just the equivariant index map for G/T. As is well-known, it takes $[G/T]$, twisted by $G \times_T \mathbb{C}_\mu$, to the class $[V_\mu] \in K_0^G(\mathrm{pt})$ of the irreducible G-representation labeled by μ. We conclude that

$$K_0^G(\Psi)([G/T]) = K_0^G(\iota)([V_\mu]).$$

The identification $K_0^G(\mathrm{pt}) \cong R(G)$ takes $[V_\mu]$ to the character χ_μ. $\square$

4.6. **Twisted K-homology of the conjugacy classes.** Suppose $\Phi\colon \mathcal{C} \hookrightarrow G$ is an arbitrary conjugacy class (not necessarily pre-quantized) corresponding to $\xi \in \Delta$. Let I be the index set such that $\xi \in \mathrm{int}(\Delta_I)$, thus $\mathcal{C} = G/G_I$. Write $\mathcal{B} = \mathbb{K}(\mathcal{H})$ so that $\mathcal{A}_I = G \times_{G_I} \mathcal{B}$. In 4.3 we had constructed a G_I-equivariant Morita isomorphism $\mathrm{Cl}(\mathfrak{g}_I^\perp) \simeq \mathcal{B}^{\mathsf{h}^\vee}$, or equivalently (since $\mathrm{Cl}(\mathfrak{g}_I^\perp) \cong \mathrm{Cl}(\mathfrak{g}_I^\perp)^{\mathrm{opp}}$) $\mathbb{C} \simeq \mathcal{B}^{\mathsf{h}^\vee} \otimes \mathrm{Cl}(\mathfrak{g}_I^\perp)$. We have, by 2.3(a)-(c),

$$\begin{aligned} K_q^G(\mathcal{C}, \Phi^* \mathcal{A}^{k+\mathsf{h}^\vee}) &= K_q^G(G/G_I,\ G \times_{G_I} \mathcal{B}^{k+\mathsf{h}^\vee}) \\ &= K_q^{G_I}(\mathrm{pt},\ \mathcal{B}^{k+\mathsf{h}^\vee} \otimes \mathrm{Cl}(\mathfrak{g}_I^\perp)) \\ &= K_q^{G_I}(\mathrm{pt},\ \mathcal{B}^k). \end{aligned}$$

This vanishes for $q=1$, and is equal to $R(\widehat{G}_I^{(-k)})_{-1}$ for $q=0$. But a representation of $\widehat{G}_I^{(-k)})$, where the central circle acts with weight -1, is the same as a representation of $\widehat{G}_I$ where the central circle acts with weight k. Thus

$$K_0^G(\mathcal{C},\Phi^*\mathcal{A}^{k+\mathsf{h}^\vee}) = K_0^G(G/G_I,\ G\times_{G_I}\mathcal{B}^{k+\mathsf{h}^\vee}) \cong R(\widehat{G}_I)_k \tag{31}$$

as $R(G)$-modules. (The module structure is given by the restriction homomorphism $R(G)\to R(G_I)=R(\widehat{G}_I)_0$, which acts on $R(\widehat{G}_I)$ by multiplication.) If $J\subset I$, we have a natural map $\phi_I^J\colon G/G_I\to G/G_J$ covered by a map of Dixmier-Douady bundles $G\times_{G_I}\mathcal{B}\to G\times_{G_J}\mathcal{B}$. Hence we obtain a push-forward map,

$$K_0^G(\phi_I^J)\colon K_0^G(G/G_I,\ G\times_{G_I}\mathcal{B}^{k+\mathsf{h}^\vee}) \to K_0^G(G/G_J, G\times_{G_J}\mathcal{B}^{k+\mathsf{h}^\vee}) \tag{32}$$

The naturality of the maps $\mathbf{I}_H^G$ (cf. 2.3(b)) and the definition of $\mathrm{ind}_I^J\equiv \mathrm{ind}_{G_I}^{G_J}$ (cf. 2.3(c)) gives a commutative diagram,

$$\begin{tikzcd}
K_0^{G_I}(\mathrm{pt},\ \mathcal{B}^{k+\mathsf{h}^\vee}\otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I)) \arrow[r, "\mathrm{ind}_I^J"] \arrow[d, "\mathbf{I}_{G_I}^{G_J}"] & K_0^{G_J}(\mathrm{pt},\ \mathcal{B}^{k+\mathsf{h}^\vee}\otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_J)) \arrow[d, "="] \\
K_0^{G_J}(G_J/G_I,\ \big(G_J\times_{G_I}\mathcal{B}^{k+\mathsf{h}^\vee}\big)\otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_J)) \arrow[r] \arrow[d, "\mathbf{I}_{G_J}^G"] & K_0^{G_J}(\mathrm{pt},\ \mathcal{B}^{k+\mathsf{h}^\vee}\otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_J)) \arrow[d, "\mathbf{I}_{G_J}^G"] \\
K_0^G(G/G_I, G\times_{G_I}\mathcal{B}^{k+\mathsf{h}^\vee}) \arrow[r, "K_0^G(\phi_I^J)"] & K_0^G(G/G_J, G\times_{G_J}\mathcal{B}^{k+\mathsf{h}^\vee})
\end{tikzcd}$$

That is, $K_0^G(\phi_I^J)\circ \mathbf{I}_{G_I}^G = \mathbf{I}_{G_J}^G\circ \mathrm{ind}_I^J$. The entries on the top row are identified with $R(\widehat{G}_I)_k$ and $R(\widehat{G}_J)_k$, and (cf. 2.3(c)) the map ind_I^J is the holomorphic induction map

$$\mathrm{ind}_I^J\colon R(\widehat{G}_I)_k\to R(\widehat{G}_J)_k, \tag{33}$$

relative to the complex structure on $G_J/G_I=\widehat{G}_J/\widehat{G}_I$ defined by the collections of simple roots $\mathfrak{S}_J\subset \mathfrak{S}_I$. To summarize,

Proposition 4.14. *The identifications $K_0^G(G/G_I, G\times_{G_I}\mathcal{B}^{k+\mathsf{h}^\vee})\cong R(\widehat{G}_I)_k$ intertwine the push-forward maps $K_0^G(\phi_I^J)$ with the holomorphic induction maps ind_I^J.*

5. Computation of $K_\bullet^G(G,\mathcal{A}^{k+\mathsf{h}^\vee})$

The Dixmier-Douady bundle $\mathcal{A}\to G$, as described in (20), may be viewed as the geometric realization of a co-simplicial Dixmier-Douady bundle, with non-degenerate p-simplices the bundle $\coprod_{|I|=p+1}\mathcal{A}_I$ over $\coprod_{|I|=p+1}G/G_I$. This defines a spectral sequence computing the K-homology group $K_\bullet^G(G,\mathcal{A}^{k+\mathsf{h}^\vee})$, in terms of the known K-homology groups $K_\bullet^G(G/G_I,\mathcal{A}_I^{k+\mathsf{h}^\vee})=R(\widehat{G}_I)_k$ and the holomorphic induction maps between these groups. As it turns out, the spectral sequence collapses at the E_2-stage, and computes the level k fusion ring.

5.1. **The spectral sequence for $K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})$.** The construction (20) of $\mathcal{A}\to G$ as a quotient of $\coprod_I \mathcal{A}^I\times\Delta_I \to \coprod_I G/G_I\times\Delta_I$ may be thought of as the geometric realization of a 'co-simplicial Dixmier-Douady bundle'. See [37] and [31] for background on co-simplicial (semi-simplicial) techniques. Here the G-Dixmier-Douady bundles

$$\coprod_{|I|=p+1}\mathcal{A}_I \to \coprod_{|I|=p+1} G/G_I$$

are the *non-degenerate* p-simplices; the full set of p-simplices is a union $\coprod_f \mathcal{A}_{f([p])}\to\coprod_f G/G_{f([p])}$ over all non-decreasing maps $f\colon [p]=\{0,\ldots,p\}\to\{0,\ldots,l\}$. By the theory of co-simplicial spaces (see [37, Section 5]), one obtains a spectral sequence $E^1_{p,q}\Rightarrow K^G_{p+q}(G,\mathcal{A}^{k+\mathsf{h}^\vee})$ where

$$E^1_{p,q}=\bigoplus_{|I|=p+1} K^G_q(G/G_I,\mathcal{A}_I^{k+\mathsf{h}^\vee}). \tag{34}$$

The differential $\mathrm{d}^1\colon E^1_{p,q}\to E^1_{p-1,q}$ is given on $K^G_q(G/G_I,\mathcal{A}_I^{k+\mathsf{h}^\vee})$ as an alternating sum,

$$\mathrm{d}^1=\sum_{r=0}^p(-1)^r K^G_q(\phi_I^{\delta_r I}).$$

Here $\delta_r I$ is obtained from I by omitting the r-th entry: $\delta_r I=\{i_0,\ldots,\widehat{i_r},\ldots,i_p\}$ for $I=\{i_0,\ldots,i_p\}$ with $i_0<\cdots<i_p$. Recall that $\phi_I^J\colon G/G_I\to G/G_J$ are the natural maps for $J\subset I$.

By mod 2 periodicity of the K-homology, we have $E^1_{p,q}=E^1_{p,q+2}$. Since the groups G_I are connected, and since $\dim G/G_I$ is even, one has $K^G_1(G/G_I,\mathcal{A}_I^{k+\mathsf{h}^\vee})=0$, thus $E^1_{\bullet,1}=0$. Hence, the E^1-term is described by a single chain complex $(C_\bullet,\partial)$, where

$$C_p=E^1_{p,0},\quad \partial=\mathrm{d}^1.$$

The map $R(G)\to K^G_\bullet(G,\mathcal{A}^{k+\mathsf{h}^\vee})$ defined by the inclusion $\iota\colon e\hookrightarrow G$ may be also be described by the spectral sequence. Think of ι as the geometric realization of a map of co-simplicial manifolds, given as the inclusion of $\{e\}=G/G_{\{0\}}$ into $\coprod_{i=0}^l G/G_{\{i\}}$. The co-simplicial map gives rises to a morphism of spectral sequences, $\tilde{E}^\bullet\to E^\bullet$, where

$$\tilde{E}^1_{p,q}=\begin{cases}K^G_q(\mathrm{pt},\mathbb{C}) & \text{if } p=0\\ 0 & \text{otherwise}\end{cases}$$

At the E^1-stage, this boils down to a chain map

$$R(G)\to C_\bullet \tag{35}$$

where $R(G)=\tilde{E}^1_{0,0}$ carries the zero differential. Our goal is to show that the homology of $(C_\bullet,\partial)$ vanishes in positive degrees, while the induced map in homology $R(G)\to H_0(C,\partial)$ is onto, with kernel $I_k(G)$.

5.2. **The induction maps in terms of weights.** To get started, we express the chain complex in terms of weights of representations. Recall that $R(T)$ is isomorphic to the group ring $\mathbb{Z}[\Lambda^*]$. The restriction map $R(G)\to R(T)$ is injective, and identifies

$$R(G)\cong\mathbb{Z}[\Lambda^*]^W.$$

Let us next describe $R(\widehat{G}_I)_k$ in terms of weights. Each $\widehat{G}_I$ has maximal torus $\widehat{T} = T \times \mathrm{U}(1)$, hence the weight lattice is

$$\widehat{\Lambda}^* = \Lambda^* \times \mathbb{Z} \subset \widehat{\mathfrak{t}}^* = \mathfrak{t}^* \times \mathbb{R}.$$

The simple roots for $\widehat{G}_I$ are $(\alpha_i, 0)$ with $\alpha_i \in \mathfrak{S}_I$, the corresponding co-roots are

$$(\alpha_i^\vee, \delta_{i,0}) \in \widehat{\mathfrak{t}} = \mathfrak{t} \times \mathbb{R}, \quad \alpha_i \in \mathfrak{S}_I. \tag{36}$$

These define a fundamental Weyl chamber

$$\widehat{\mathfrak{t}}^*_{I,+} = \{(\nu, s)|\ \langle \nu, \alpha_i^\vee \rangle + s\delta_{i,0} \geq 0,\ \alpha_i \in \mathfrak{S}_I\} \tag{37}$$

The elements ν_I satisfy $\langle \nu_I, \alpha_i^\vee \rangle + \delta_{i,0} = 0$. Hence, $(\nu, s) \in \widehat{\mathfrak{t}}^*_{I,+}$ is and only if $\nu - s\nu_I \in \mathfrak{t}^*_{I,+}$. Let $\Lambda^*_{I,k} \subset \Lambda^*$ be the intersection of (37) with $\Lambda^* \times \{k\} \cong \Lambda^*$. Thus

$$\Lambda^*_{I,k} = \{\nu \in \Lambda^*|\ \langle \nu, \alpha_i^\vee \rangle + k\delta_{i,0} \geq 0,\ i \notin I\}$$

labels the irreducible $\widehat{G}_I$-representations for which the central circle acts with weight k. The Weyl group W_I of G_I is also the Weyl group of $\widehat{G}_I$. Its action on $\widehat{\Lambda^*}$ preserves the levels $\Lambda^* \times \{k\}$, hence it takes the form $w.(\nu, k) = (w \bullet_k \nu, k)$ for a *level k-action* $\nu \mapsto w \bullet_k \nu$ on Λ^*. Explicitly,

$$w \bullet_k \nu = w(\nu - k\nu_I) + k\nu_I. \tag{38}$$

Fix k, and denote by $\mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}}$ the anti-invariant part for the W_I-action $\nu \mapsto w \bullet_{k+\mathsf{h}^\vee} \nu$ at the shifted level $k + \mathsf{h}^\vee$. Observe that this space is invariant under the action of $\mathbb{Z}[\Lambda^*]^W$. Let

$$\mathrm{Sk}^I \colon \mathbb{Z}[\Lambda^*] \to \mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}},\ \nu \mapsto \sum_{w \in W_I} (-1)^{\mathrm{length}(w)} w \bullet_{k+\mathsf{h}^\vee} \nu$$

denote skew-symmetrization relative to the action at level $k + \mathsf{h}^\vee$. For $\mu \in \Lambda^*_k$, let $\chi^I_\mu \in R(\widehat{G}_I)_k$ be the character of the irreducible $\widehat{G}_I$-representation of weight (μ, k).

Lemma 5.1. *The map $\chi^I_\mu \mapsto \mathrm{Sk}^I(\mu + \rho)$ extends to an isomorphism*

$$R(\widehat{G}_I)_k \to \mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}}. \tag{39}$$

Under this isomorphism, the $R(G) \cong \mathbb{Z}[\Lambda^*]^W$-module structure is given by multiplication in the group ring. Furthermore, the identification (39) *intertwines the holomorphic induction maps $\mathrm{ind}_I^J \colon R(\widehat{G}_I)_k \to R(\widehat{G}_J)_k$ for $J \subset I$ with skew-symmetrizations*

$$\mathrm{Sk}_I^J = \frac{1}{|W_I|} \mathrm{Sk}_J \colon \mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}} \to \mathbb{Z}[\Lambda^*]^{W_J-\mathrm{as}}.$$

Note that the statement involves a shift by ρ, rather than ρ_I. Thus, even in the case $I = \{0, \ldots, l\}$ where $G_I = T$ and $W_I = \{1\}$, $\rho_I = 0$, the identification $R(\widehat{T})_k \to \mathbb{Z}[\Lambda^*]$ involves a ρ-shift.

Proof. Let $\Lambda^{*,\mathrm{reg}}_{I,k+\mathsf{h}^\vee}$ be the intersection of $\Lambda^* \times \{k+\mathsf{h}^\vee\}$ with $\mathrm{int}(\widehat{\mathfrak{t}}^*_{I,+})$. Since obviously $R(\widehat{G}_I)_k = \mathbb{Z}[\Lambda^*_{I,k}]$, the first part of the Lemma amounts to the assertion that

$$\mu \in \Lambda^*_{I,k} \Leftrightarrow \mu + \rho \in \Lambda^{*,\mathrm{reg}}_{I,k+\mathsf{h}^\vee}.$$

We have $\mu \in \Lambda^*_{I,k}$ if and only if $\langle \mu, \alpha_i^\vee \rangle + k\delta_{i,0} \geq 0$ for $i \notin I$. Since $\langle \rho, \alpha_i^\vee \rangle + \mathsf{h}^\vee \delta_{i,0} = 1$ this is equivalent to $\langle \mu + \rho, \alpha_i^\vee \rangle + (k + \mathsf{h}^\vee)\delta_{i,0} \geq 1,\ i \notin I$, i.e. $\mu + \rho \in \Lambda^{*,\mathrm{reg}}_{I,k+\mathsf{h}^\vee}$ as claimed. The assertion about the $R(G)$-module structure is obvious. Finally, for $J \subset I$ the holomorphic induction map ind_I^J is given by

$$\mathrm{ind}_I^J(\chi_\mu^I) = (-1)^{\mathrm{length}(w)} \chi^J_{w \bullet_k (\mu+\rho_J)-\rho_J}$$

if there exists $w \in W_J$ with $w \bullet_k (\mu + \rho_J) - \rho_J \in \Lambda^*_{J,k}$, while $\mathrm{ind}_I^J(\chi_\mu^I) = 0$ if there is no such w. Using (38) together with $\rho_I - k\nu_I = \rho - (k + \mathsf{h}^\vee)\nu_I$ (by the definition of ν_I), this may be re-written in terms of the action at level $k + \mathsf{h}^\vee$:

$$w \bullet_k (\mu + \rho_J) - \rho_J = w \bullet_{k+\mathsf{h}^\vee} (\mu + \rho) - \rho.$$

□

By combining this discussion with Proposition 4.14, we have established a commutative diagram

$$\begin{array}{ccccc} K_0^G(G/G_J, \mathcal{A}_J^{k+\mathsf{h}^\vee}) & \xrightarrow[\cong]{} & R(\widehat{G}_J)_k & \xrightarrow[\cong]{} & \mathbb{Z}[\Lambda^*]^{W_J-\mathrm{as}} \\ \uparrow{\scriptstyle K_0(\phi_I^J)} & & \uparrow{\scriptstyle \mathrm{ind}_I^J} & & \uparrow{\scriptstyle \mathrm{Sk}_I^J} \\ K_0^G(G/G_I, \mathcal{A}_I^{k+\mathsf{h}^\vee}) & \xrightarrow[\cong]{} & R(\widehat{G}_I)_k & \xrightarrow[\cong]{} & \mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}} \end{array} \tag{40}$$

We can thus re-express the chain complex $(C_\bullet, \partial)$ in terms of weights:

$$C_p = \bigoplus_{|I|=p+1} \mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}}, \quad \partial\phi^I = \sum_{r=0}^{p} (-1)^r \,\mathrm{Sk}_I^{\delta^r I}(\phi^I), \tag{41}$$

for $\phi^I \in \mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}}$. The map $R(G) \to C_0 \subset C_\bullet$ given by (35) is expressed as the inclusion of $\mathbb{Z}[\Lambda^*]^{W-\mathrm{as}}$, as the summand corresponding to $I = \{0\}$. By construction, $C_\bullet$ is a complex of $R(G)$-modules, and the map (35) is an $R(G)$-module homomorphism.

5.3. **Fusion ring.** Let us also describe the fusion ring in terms of weights. The subset $B^\flat(k\Delta) \subset \mathfrak{t}^*$ defining the set $\Lambda_k^* = \Lambda^* \cap B^\flat(k\Delta)$ of level k weights is cut out by the inequalities

$$\langle \nu, \alpha_i^\vee \rangle + k\delta_{i,0} \geq 0.$$

It is a fundamental domain for the level k action $\nu \mapsto w \bullet_k \nu$ of the affine Weyl group, generated by the simple affine reflections

$$\nu \mapsto \nu - (\langle \nu, \alpha_i^\vee \rangle + s\delta_{i,0})\alpha_i, \quad i = 0, \ldots, l.$$

This is consistent with our earlier notation: The level k action of W_{aff} restricts to the level k action of the subgroup W_I, generated by the affine reflections with $i \notin I$.

Let $\mathbb{Z}[[\Lambda^*]]$ be the $\mathbb{Z}[\Lambda^*]$-module consisting of all functions $\Lambda^* \to \mathbb{Z}$, not necessarily of finite support. Let

$$\mathrm{Sk}_{\mathrm{aff}} \colon \mathbb{Z}[\Lambda^*] \to \mathbb{Z}[[\Lambda^*]]^{W_{\mathrm{aff}}-\mathrm{as}}, \ \nu \mapsto \sum_{w \in W_{\mathrm{aff}}} (-1)^{\mathrm{length}(w)}\, w \bullet_{k+\mathsf{h}^\vee} \nu$$

be skew-symmetrization, using the action at the shifted level $k+\mathsf{h}^\vee$. The map $\mu \mapsto \mathrm{Sk}_{\mathrm{aff}}(\mu+\rho)$ extends to an isomorphism, $\mathbb{Z}[\Lambda_k^*] \to \mathbb{Z}[[\Lambda^*]]^{W_{\mathrm{aff}}-\mathrm{as}}$. This identifies

$$R_k(G) \cong \mathbb{Z}[[\Lambda^*]]^{W_{\mathrm{aff}}-\mathrm{as}} \tag{42}$$

as an Abelian group. For any I we have $R(G)=\mathbb{Z}[\Lambda^*]^W$-module homomorphisms $R(\widehat{G}_I)_k \to R_k(G)$,

$$\mathbb{Z}[\Lambda^*]^{W_I-\mathrm{as}} \to \mathbb{Z}[[\Lambda^*]]^{W_{\mathrm{aff}}-\mathrm{as}}, \quad \phi_I \mapsto \frac{1}{|W_I|}\,\mathrm{Sk}_{\mathrm{aff}}\,\phi_I. \tag{43}$$

For $I=\{0\}$ we may use the obvious trivialization $\widehat{G}=G\times \mathrm{U}(1)$ to identify $R(G)=R(\widehat{G}_0)_k$. The following is clear from the description of the quotient map $R(G)\to R_k(G)$ (see e.g. [3]):

Lemma 5.2. *The identifications $R(G)=\mathbb{Z}[\Lambda^*]^{W-\mathrm{as}}$ and (42) intertwine the quotient map $R(G)\to R_k(G)$ with the skew-symmetrization map,*

$$\frac{1}{|W|}\,\mathrm{Sk}_{\mathrm{aff}}\colon \mathbb{Z}[\Lambda]^{W-\mathrm{as}} \to \mathbb{Z}[[\Lambda^*]]^{W_{\mathrm{aff}}-\mathrm{as}}. \tag{44}$$

In particular, (42) is an isomorphism of $R(G)\cong \mathbb{Z}[\Lambda^*]^W$-modules.

In fact, we could *define* the ideal $I_k(G)\subset R(G)$ as the kernel of the map (44). Let $\epsilon\colon C_0\to R_k(G)$ be the direct sum of the morphisms (43) for $|I|=1$.

5.4. **A resolution of the $R(G)$-module $R_k(G)$.**

Theorem 5.3. *For all $k\ge 0$ the chain complex $(C_\bullet,\partial)$ defines a resolution*

$$0\to C_l \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} R_k(G)\to 0$$

of $R_k(G)$ as an $R(G)$-module.

The proof will be given below. As mentioned in the introduction, Theorem 5.3 is implicit in the work of Kitchloo-Morava [25].

Remark 5.4. It turns out that the twisted representations $R(\widehat{G}_I)_k$ are projective modules over $R(G)$, hence (by the Quillen-Suslin theorem) free modules over $R(G)$. That is, $(C_\bullet,\partial)$ is a *free resolution* of the $R(G)$-module $R_k(G)$. If $\widehat{G}_I^{(k)}\cong G_I\times \mathrm{U}(1)$, the $R(G)$ module $R(\widehat{G}_I)_k$ is isomorphic to $R(G_I)$, and the claim follows from the Pittie-Steinberg theorem [33, 39]. The general case requires a mild generalization of the Pittie-Steinberg theorem [29].

Remark 5.5. Theorem 5.3 implies the Freed-Hopkins-Teleman theorem (1): By acyclicity of the chain complex $C_\bullet$ the spectral sequence E^r collapses at the E^2-term, with

$$E^2_{p,q}=E^\infty_{p,q}=\begin{cases} R_k(G) & \text{if } p=0 \text{ and } q \text{ even}\\ 0 & \text{otherwise}\end{cases}$$

Since $R_k(G)$ is free Abelian as a $\mathbb{Z}$-module, there are no extension problems and we conclude $K_1^G(G,\mathcal{A}^{k+\mathsf{h}^\vee})=0$, while

$$K_0^G(G,\mathcal{A}^{k+\mathsf{h}^\vee})=R_k(G) \tag{45}$$

as modules over $R(G)$. This isomorphism takes the ring homomorphism $R(G)\to K_0^G(G,\mathcal{A}^{k+\mathsf{h}^\vee})$ to the quotient map $R(G)\to R_k(G)$, hence (45) is an isomorphism of rings.

The statement of Theorem 5.3 can be simplified. Indeed, the chain complex $C_\bullet$ breaks up as a direct sum of sub-complexes $C_\bullet(\mu),\ \mu \in \Lambda_k^*$, given as

$$C_p(\mu) = \bigoplus_{|I|=p+1} \mathbb{Z}[W_{\mathrm{aff}} \bullet_{k+\mathsf{h}^\vee} \mu]^{W_I-\mathrm{as}}.$$

Similarly the map $\epsilon\colon C_0 \to R_k(G)$ splits into a direct sum of maps

$$\epsilon\colon C_0(\mu) \to \mathbb{Z}[W_{\mathrm{aff}} \bullet_{k+\mathsf{h}^\vee} \mu]^{W_{\mathrm{aff}}-\mathrm{as}} = \begin{cases} \mathbb{Z} & \text{for } \mu \in \Lambda^{*,\mathrm{reg}}_{k+\mathsf{h}^\vee} \\ 0 & \text{otherwise.} \end{cases}$$

Finally the chain map $R(G) \hookrightarrow C_\bullet$ splits into inclusions of $\mathbb{Z}[W_{\mathrm{aff}} \bullet_{k+\mathsf{h}^\vee} \mu]^{W-\mathrm{as}}$ as the term corresponding to $I = \{0\}$. Clearly, $(C_\bullet(\mu), \partial)$ depends only on the open face $B^\flat((k+\mathsf{h}^\vee)\Delta_J)$ of $B^\flat((k+\mathsf{h}^\vee)\Delta)$ containing μ. Indeed, since $\mathbb{Z}[W_{\mathrm{aff}} \bullet_{k+\mathsf{h}^\vee} \mu] = \mathbb{Z}[W_{\mathrm{aff}}/W_J]$ we have

$$C_p(J) = \bigoplus_{|I|=p+1} \mathbb{Z}[W_{\mathrm{aff}}/W_J]^{W_I-\mathrm{as}}.$$

The differential ∂ is again given by anti-symmetrization as in (41), but with ϕ^I now an element of $\mathbb{Z}[W_{\mathrm{aff}}/W_J]^{W_I-\mathrm{as}}$. The map $\epsilon\colon C_0 \to R_k(G)$ translates into the zero map $C_0(J) \to 0$ unless $J = \{0\ldots,l\}$, in which case it becomes a map $\epsilon\colon C_0(J) \to \mathbb{Z}$, given as the direct sum for $i = 0,\ldots,l$ of the maps,

$$\mathbb{Z}[W_{\mathrm{aff}}]^{W_i-\mathrm{as}} \to \mathbb{Z}, \quad \sum_w n_w w \mapsto \sum_W n_w (-1)^{\mathrm{length}(w)}.$$

The map $R(G) \to C_\bullet$ is again the inclusion of the summand of $C_0(J)$ corresponding to $I = \{0\}$.

Theorem 5.3 is now reduced to the following simpler statement:

Theorem 5.6. *The homology $H_\bullet(J)$ of the chain complex $C_\bullet(J)$ vanishes in degree $p > 0$, while*

$$H_0(J) = \begin{cases} 0 & \text{if } J \neq \{0,\ldots,l\} \\ \mathbb{Z} & \text{if } J = \{0,\ldots,l\} \end{cases}$$

In the second case, the isomorphism is induced by the augmentation map $\epsilon\colon C_0(J) \to \mathbb{Z}$.

5.5. **Proof of Theorem 5.6.** Throughout this Section, we consider a given face Δ_J of the alcove. We may think of W_{aff}/W_J as the W_{aff}-orbit of a point in the interior of the face Δ_J, under the standard action of W_{aff} on $\mathfrak{t}$. To be concrete, let us take the point $\nu_J^\sharp$. Denote its orbit by

$$V = W_{\mathrm{aff}}.\nu_J^\sharp \subset \mathfrak{t}.$$

We introduce a length function length: $V \to \mathbb{Z}$, defined in terms of the function on W_{aff} as

$$\mathrm{length}(x) = \min\{\mathrm{length}(w) | w \in W_{\mathrm{aff}},\ x = w.\nu_J^\sharp\}, \quad x \in V.$$

Geometrically, length(x) is the number of affine root hyperplanes in the Stiefel diagram, crossed by a line segment from the a point in the interior of Δ to the point x.

For any I let $\mathfrak{t}_{I,+}$ be defined by the inequalities $\langle \alpha_i, \cdot \rangle + \delta_{i,0} \geq 0$ for $\alpha_i \in \mathfrak{S}_I$. (Equivalently, it is the affine cone over Δ at $\nu_I^\sharp$.) Then $\mathfrak{t}_{I,+}$ is a fundamental domain for the W_I-action. Let

$V^I \subset \overline{V}^I \subset V$ be the subsets,

$$V^I = V \cap \operatorname{int}(\mathfrak{t}_{I,+}), \quad \overline{V}^I = V \cap \mathfrak{t}_{I,+}.$$

Every $W_I \subset W_{\mathrm{aff}}$-orbit contains a unique point in $\overline{V}^I$. Thus, if $x \in V$, we may choose $u \in W_I$ with $u.x \in \overline{V}^I$. Then

$$\operatorname{length}(u.x) \leq \operatorname{length}(x),$$

with equality if and only if $x \in \overline{V}^I$ and hence $u.x = x$.

The elements

$$\beta_I(x) = \operatorname{Sk}^I(x), \quad x \in V^I \tag{46}$$

form a basis of the $\mathbb{Z}$-module $\mathbb{Z}[V]^{W_I-\mathrm{as}}$. (Note that if $x \in \overline{V}^I \backslash V^I$ then $\operatorname{Sk}^I(x) = 0$.) Let us describe the differential in terms of this basis. For $|I| = p+1$ and $x \in V^I$ we have,

$$\partial \beta_I(x) = \sum_{r=0}^{p} (-1)^r \operatorname{Sk}^{\delta_r I}(x).$$

In general, the terms $\operatorname{Sk}^{\delta_r I}(x)$ are not standard basis elements, since x need not lie in $V^{\delta_r I}$. Letting $u_r \in W_{\delta^r I}$ be the unique element such that $u_r x \in V^{\delta^r I}$, we have

$$\partial \beta_I(x) = \sum_{r=0}^{n} (-1)^{r+\operatorname{length}(u_r)} \beta_{\delta^r I}(u_r x). \tag{47}$$

5.5.1. *Computation of $H_0(J)$.* Consider $C_0(J) = \bigoplus_{i=0}^{p} \mathbb{Z}[V]^{W_i-\mathrm{as}}$. For all i, j and all x, the elements $\operatorname{Sk}^j(x), \operatorname{Sk}^i(x)$ are homologous since they differ by the boundary of $\operatorname{Sk}^{ij}(x) \in C_1(J)$. Together with $\operatorname{Sk}^j(x) = (-1)^{\operatorname{length}(w)} \operatorname{Sk}^j(wx)$ for $w \in W_j$, this implies

$$\operatorname{Sk}^i(x) \sim (-1)^{\operatorname{length}(w)} \operatorname{Sk}^i(wx)$$

for $w \in W_j$. Since the subgroups W_j generate W_{aff}, this holds in fact for all $w \in W_{\mathrm{aff}}$. Thus

$$\operatorname{Sk}^j(w.\nu_J^\sharp) \sim \operatorname{Sk}^i(w.\nu_J^\sharp) \sim (-1)^{\operatorname{length}(w)} \operatorname{Sk}^i(\nu_J^\sharp)$$

for all i, j, and all $w \in W_{\mathrm{aff}}$. If $J \neq \{0, \ldots, l\}$, the choice of any $i \notin J$ gives $\operatorname{Sk}^i(\nu_J^\sharp) = 0$. This proves $H_0(J) = 0$. Suppose now $J = \{0, \ldots, l\}$. The augmentation map $C_0(J) \to \mathbb{Z}$ is described in terms of the basis by $\beta_i(x) \mapsto (-1)^{\operatorname{length}(x)}$. It has a right inverse $\mathbb{Z} \to C_0(J)$, $1 \mapsto \beta_0(\nu_0^\sharp)$. Hence the induced map in homology $\mathbb{Z} \to H_0(J)$ is injective, but also surjective since $\operatorname{Sk}^i(x) \sim (-1)^{\operatorname{length}(x)} \beta_0(\nu_0^\sharp)$. Thus $H_0(J) = \mathbb{Z}$ in this case.

5.5.2. *Computation of $H_l(J)$.* Suppose $\phi \in C_l(J) = \mathbb{Z}[V]$. Then $\partial\phi = 0$ if and only if $\operatorname{Sk}^{0 \cdots \hat{i} \cdots l} \phi = 0$ for all i. That is, ϕ is invariant under every reflection $\sigma_i \in W_{\mathrm{aff}}$, hence under the full affine Weyl group W_{aff}. But since ϕ has finite length this is impossible unless $\phi = 0$. This shows $H_l(J) = 0$.

5.5.3. *Computation of $H_p(J)$, $0 < p < l$.* To simplify notation, we will write $C_\bullet$ instead of $C_\bullet(J)$. (This should of course not be confused with the chain complex $C_\bullet$ considered in previous sections.) Introduce a $\mathbb{Z}$-filtration

$$0 = F_{-1}C_\bullet \subset F_0C_\bullet \subset F_1C_\bullet \subset \cdots$$

where F_NC_p is spanned by basis elements (46) with $|I| = p+1$ and $\mathrm{length}(x) \le N$. Formula (47) shows that for any basis element $\beta_I(x) \in F_NC_p$,

$$\partial\beta_I(x) = {\sum_r}' (-1)^r \beta_{\delta_r I}(x) \mod F_{N-1}C_{p-1} \tag{48}$$

where the sum is only over those r for which $x \in V^{\delta^r I} \subset V^I$, i.e. $u_r = 1$ (other terms lower the filtration degree since $\mathrm{length}(u_r x) < \mathrm{length}(x)$ unless $x = u_r x$). In particular, ∂ preserves the filtration. Define operators $h_i : C_p \to C_{p+1}$ on basis elements, as follows:

$$h_i\beta_I(x) = \begin{cases} (-1)^r\beta_{I\cup\{i\}}(x) & \text{if } \ i_{r-1} < i < i_r, \\ 0 & \text{if } i = i_r, \text{ some } r. \end{cases}$$

Note that h_i preserves the filtration: $h_i(F_NC_p) \subset F_NC_{p+1}$. Let

$$A_i = \mathrm{id} - h_i\partial - \partial h_i.$$

Then A_i is a chain map, which is homotopic to the identity map.

Lemma 5.7. *Let $p > 0$. For any basis element $\beta_I(x) \in F_NC_p$ we have $A_i\,\beta_I(x) \in F_{N-1}C_p$* unless *$i \in I$ and $x \notin V^{I-\{i\}}$. In the latter case,*

$$A_i\beta_I(x) = \beta_I(x) \mod F_{N-1}C_p.$$

Proof. Write $I = \{i_0, \ldots, i_p\}$ where $i_0 < \cdots < i_p$. Using (48) we obtain

$$h_i\partial\beta_I(x) = {\sum_r}' (-1)^r h_i\beta_{\delta^r I}(x) \mod F_{N-1}C_p, \tag{49}$$

summing over indices with $x \in V^{\delta^r I} \subset V^I$. The calculation of $A_i\beta_I(x)$ divides into two cases:
Case 1: $i \in I$. Thus $i = i_s$ for some index s, and $(-1)^r h_i\beta_{\delta^r I}(x) = 0$ unless $r = s$, in which case one obtains $\beta_I(x)$. Hence all terms in the sum (49) vanish, except possibly for the term $r = s$ which appears if and only if $x \in V^{\delta^s I} = V^{I-\{i\}}$. That is,

$$h_i\partial\beta_I(x) = \begin{cases} \beta_I(x) \mod F_{N-1}C_p & \text{if } x \in V^{I-\{i\}} \\ 0 \mod F_{N-1}C_p & \text{if } x \notin V^{I-\{i\}} \end{cases}$$

(using the assumption $p > 0$). Since $h_i\beta_I(x) = 0$ this shows $A_i\beta_I(x) \in F_{N-1}C_p$ *unless* $x \notin V^{I-\{i\}}$, in which case $A_i\beta_I(x) = \beta_I(x) \mod F_{N-1}C_p$.
Case 2: $i \notin I$. Exactly one of the terms in $\partial h_i\beta_I(x)$ reproduces $\beta_I(x)$. The remaining terms are organized in a sum similar to (47):

$$\partial h_i\beta_I(x) = \beta_I(x) - {\sum_r}'' (-1)^r h_i\beta_{\delta^r I}(x) \mod F_{N-1}C_p,$$

where the sum is over all r such that $x \in V^{I\cup\{i\}-\{i_r\}}$. But $x \in V^{\delta^r I} \Leftrightarrow x \in V^{I\cup\{i\}-\{i_r\}}$, since

$$V^{\delta^r I} = V^{I\cup\{i\}-\{i_r\}} \cap V^I.$$

Hence the sum $\sum_r'$ and $\sum_r''$ are just the same. This proves $A_i\,\beta_I(x) \in F_{N-1}C_p$. $\square$

Consider now the product $A := A_0 \cdots A_l$. By iterated application of the Lemma, we find that if $0 < p < l$, then $A\beta_I(x) \in F_{N-1}C_p$ (because at least one index i is not in I). Thus

$$A\colon F_N C_p \to F_{N-1}C_p$$

for $0 < p < l$. The chain map A is chain homotopic to the identity, since each of its factors are. Thus, if $\phi \in F_N C_p$ is a cycle,

$$\phi \sim A\phi \sim \cdots A^{N+1}\phi = 0.$$

This proves $H_p(J) = 0$ for $0 < p < l$, and concludes the proof of Theorem 5.6.

Remark 5.8. N. Kitchloo pointed out a more elegant proof of Theorem 5.6, along the lines of Kitchloo-Morava [25]. His argument produces an inclusion of $C_\bullet(J)$ as a direct summand of $S_\bullet \otimes_{\mathbb{Z}[W_J]} \mathbb{Z}$, where $S_\bullet$ is the simplicial complex with respect to the Stiefel diagram, and $\mathbb{Z}[W_J]$ acts on $\mathbb{Z}$ by the sign representation. The acyclicity of $C_\bullet(J)$ then follows from the W_J-equivariant acyclicity of $S_\bullet$.

Appendix A. Relative Dixmier-Douady bundles

For any map $f\colon Y \to X$, and $\operatorname{cone}(f)$ its mapping cone, obtained by gluing $\operatorname{cone}(Y) = Y \times I/Y \times \{0\}$ with X by the identification $(y,1) \sim f(y)$. Let $H^\bullet(f) = H^\bullet(\operatorname{cone}(f))$ denote the relative cohomology of f. Equivalently $H^\bullet(f)$ is the cohomology of the algebraic mapping cone $C^\bullet(f)$ of the cochain map $C^\bullet(Y) \to C^\bullet(X)$, i.e. $C^p(f) = C^{p-1}(Y) \oplus C^p(X)$ with differential $\mathrm{d}(a,b) = (\mathrm{d}\,a - f^*b,\, \mathrm{d}c)$. If f is a smooth map of manifolds, the cohomology $H^\bullet(f,\mathbb{R})$ may be computed using differential forms, replacing the singular cochains in the above.

The group $H^2(f)$ has a geometric interpretation as isomorphism classes of relative line bundles, i.e. pairs (L,ψ_Y), where L is a Hermitian line bundle over X, and $\psi_Y\colon Y \times \mathbb{C} \to f^*L$ is a unitary trivialization of its pull-back to Y. The class of a relative line bundle is the Chern class of the line bundle $\tilde{L} \to \operatorname{cone}(f)$, obtained by gluing $\operatorname{cone}(Y) \times \mathbb{C}$ with L via ψ_Y.

Similarly, $H^3(f)$ is interpreted in terms of relative Dixmier-Douady bundles, i.e. pairs $(\mathcal{A},\mathcal{E}_Y)$, where $\mathcal{A} \to X$ is a Dixmier-Douady bundle, and $\mathcal{E}_Y \to Y$ is a Morita trivialization of the pull-back $f^*\mathcal{A}$.

Given such a triple, one may construct a Dixmier-Douady bundle $\tilde{\mathcal{A}} \to \operatorname{cone}(f)$. First stabilize: Let $\mathbb{H}$ be a fixed infinite-dimensional Hilbert space, and $\mathbb{K} = \mathbb{K}(\mathbb{H}) =$ the compact operators. Then $\mathcal{E}_Y^{\mathrm{st}} = \mathcal{E}_Y \otimes \mathbb{H}$ defines a Morita trivialization of the pull-back of $\mathcal{A}^{\mathrm{st}} = \mathcal{A} \otimes \mathbb{K}$. Since the Hilbert space bundle $\mathcal{E}_Y^{\mathrm{st}}$ is stable, it is equivariantly isomorphic to the trivial bundle $Y \times \mathbb{H}$. Define $\tilde{\mathcal{A}}$ by gluing the trivial bundle $\operatorname{cone}(Y) \times \mathbb{K}$ with $f^*\mathcal{A}^{\mathrm{st}}$, using this identification. We define the relative Dixmier-Douady class $\mathrm{DD}(\mathcal{A},\mathcal{E}_Y) := \mathrm{DD}(\tilde{\mathcal{A}}) \in H^3(f)$.

Tensor products and opposites of relative Dixmier-Douady bundles are defines in the obvious way. A Morita trivialization $(\mathcal{A},\mathcal{E}_Y)$ is a Morita trivialization $\mathbb{C} \simeq_{\mathcal{E}_X} \mathcal{A}$ together with an isomorphism $\mathcal{E}_Y \cong f^*\mathcal{E}_X$ intertwining the module structures. From the usual Dixmier-Douady theorem, one deduces that $\mathrm{DD}(\mathcal{A},\mathcal{E}_Y)$ is the obstruction to the existence of a relative Morita trivialization.

More generally, one may define relative equivariant Dixmier-Douady bundles; these are classified by an equivariant class $\mathrm{DD}_G(\mathcal{A},\mathcal{E}_Y,\psi_Y) \in H^3_G(f) := H^3(f_G)$, where $f_G\colon Y_G \to X_G$ is the induced map of Borel constructions. (For the stabilization procedure, one replaces $\mathbf{H}$ with the stable G-Hilbert space $\mathbf{H}_G$ containing all G-representations with infinite multiplicity.)

Appendix B. Review of Kasparov K-homology

In this Section we review Kasparov's definition of K-homology [23, 22] for C^*-algebras. Excellent references for this material are the books by Higson-Roe [19] and Blackadar [5]. Suppose A is a $\mathbb{Z}_2$-graded C^*-algebra, equipped with an action of a compact Lie group G by automorphisms. An equivariant Fredholm module over A is a triple $x = (\mathcal{H}, \varrho, F)$, where $\mathcal{H}$ is a G-equivariant $\mathbb{Z}_2$-graded Hilbert space, $\varrho\colon \mathsf{A} \to L(\mathcal{H})$ is a morphism of $\mathbb{Z}_2$-graded G-C^*-algebras, and $F \in L(\mathcal{H})$ is a G-invariant odd operator such that for all $a \in \mathsf{A}$,

$$(F^2 - I)\varrho(a) \sim 0,\ (F^* - F)\varrho(a) \sim 0,\ [F, \varrho(a)] \sim 0.$$

Here $\sim$ denotes equality modulo compact operators. There is an obvious notion of direct sum of Fredholm modules over A. One defines a semi-group $K_0^G(\mathsf{A})$, with generators $[x]$ for each Fredholm module over A, and equivalence relations

$$[x] + [x'] = [x \oplus x'],$$

and

$$[x_0] = [x_1]$$

provided x_0, x_1 are related by an 'operator homotopy' $x_t = (\mathcal{H}, \varrho, F_t)$ (cf. [5, 19]). One then proves that every element in this semi-group has an additive inverse, so that $K_0^G(\mathsf{A})$ is actually a group. More generally, for $q \leq 0$ one defines $K_q^G(\mathsf{A}) = K_0^G(\mathsf{A} \otimes \mathrm{Cl}(\mathbb{R}^q))$. This has the mod 2 periodicity property $K_G^{q+2}(\mathsf{A}) = K_G^q(\mathsf{A})$, which is then used to extend the definition to all $q \in \mathbb{Z}$. The assignment $\mathsf{A} \to K_G^q(\mathsf{A})$ is homotopy invariant, contravariant functor, depending only on the Morita isomorphism class of A. It has the *stability property*, $K_G^q(\mathsf{A} \otimes \mathbb{K}_G) = K_G^q(\mathsf{A})$, where $\mathbb{K}_G$ are the compact operators on a G-Hilbert-space $\mathcal{H}_G$ containing all G-representations with infinite multiplicity. With this definition, let us now review some basic examples of twisted K-homology groups $K_q^G(X, \mathcal{A}) = K_G^q(\Gamma_0(X, \mathcal{A}))$ for Dixmier-Douady bundles $\mathcal{A} \to X$.

Example B.1. Let $\mathcal{A} \to \mathrm{pt}$ be a G-equivariant Dixmier-Douady bundle over a point. Disregarding the G-action, we have $\mathcal{A} \cong \mathbb{K}(\mathcal{E})$ for some Hilbert space $\mathcal{E}$. As in Section 2.2 the action $G \to \mathrm{Aut}(\mathcal{A})$ defines a central extension $\widehat{G}$ of G by U(1). The group $\widehat{G}$ acts on $\mathcal{E}$, in such a way that the central circle acts with weight 1. Let V be a $\widehat{G}$-module where the central circle acts with weight -1. Then the Hilbert space $\mathcal{H} = V \otimes \mathcal{E}$ is a G-module. Letting $\rho\colon \mathbb{C} \to L(\mathcal{H})$ be the action by scalar multiplication, the triple $(\mathcal{H}, \varrho, 0)$ is a G-equivariant Fredholm module over $C(\mathrm{pt}) = \mathbb{C}$. This construction realizes the isomorphism $R(\widehat{G})_{-1} \to K_0^G(\mathrm{pt}, \mathcal{A})$.

Example B.2. Let M be a compact Riemannian G-manifold, and D an invariant first order elliptic operator acting on a G-equivariant $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Suppose also that a finite rank $\mathbb{Z}_2$-graded G-Dixmier-Douady bundle $\mathcal{A} \to M$ acts on $\mathcal{E}$, where the action is equivariant and compatible with the grading. Let $\mathcal{H}$ be the space of L^2-sections of $\mathcal{E}$, with the natural representation ϱ of $\Gamma(M, \mathcal{A})$, and $F = D(1 + D^2)^{-1/2} \in L(\mathcal{H})$. The commutator of F with elements $\varrho(a)$ for $a \in \Gamma(M, \mathcal{A})$ are pseudo-differential operators of degree -1, hence are compact. Thus $(\mathcal{H}, \varrho, F)$ is an equivariant Fredholm module over $\Gamma(M, \mathcal{A})$, defining a class in $K_0^G(M, \mathcal{A})$.

Example B.3. [24, page 114] Let M be a compact Riemannian G-manifold, and $\mathcal{A} = \mathrm{Cl}(TM)$ its Clifford bundle. Take $\mathcal{E} = \wedge T^*M$, $\mathcal{H}$ its space of L^2-sections, and ϱ the usual action of sections of $\Gamma(M, \mathrm{Cl}(TM))$. Let $D = \mathrm{d} + \mathrm{d}^*$ be the de-Rham Dirac operator. By B.2 above, we obtain a Fredholm module $(\mathcal{H}, \varrho, F)$ over $\Gamma(M, \mathrm{Cl}(TM))$, defining a class $[M] \in K_0^G(M, \mathrm{Cl}(TM))$. This is the *Kasparov fundamental class* of M. (Actually, $\mathrm{Cl}(TM)$ is a Dixmier-Douady bundle only if $\dim M$ is even. If $\dim M$ is odd, one can use the isomorphism $K_0^G(M, \mathrm{Cl}(TM)) = K_1^G(M, \mathrm{Cl}^+(TM))$ if needed.)

Example B.4. Let H be a closed subgroup of G, and $\mathcal{B} \to \mathrm{pt}$ an H-Dixmier-Douady bundle of finite rank. As explained in B.1, any class in $K_0^H(\mathrm{pt}, \mathrm{Cl}(\mathfrak{g}/\mathfrak{h}) \otimes \mathcal{B})$ is realized by a Fredholm module of the form $(\mathcal{E}, \varrho, 0)$ where $\mathcal{E}$ if a Hilbert space of finite dimension. Let $\widehat{\mathcal{E}} = G \times_H \mathcal{E}$. The action of $\mathrm{Cl}(T(G/H))$ defines a Dirac operator, which together with the action of $\mathbf{I}_H^G(\mathcal{B})$ yields a Fredholm module and hence an element of $K_0^G(G/H, \mathbf{I}_H^G(\mathcal{B}))$. This construction realizes the isomorphism $K_0^H(\mathrm{pt}, \mathcal{B} \otimes \mathrm{Cl}(\mathfrak{g}/\mathfrak{h})) \to K_0^G(G/H, \mathrm{ind}_H^G(\mathcal{B}))$ if $\mathcal{B}$ has finite rank. As remarked in Section 2.1, all H-Dixmier-Douady bundles over pt are Morita isomorphic to finite rank ones.

References

[1] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. **48** (1998), no. 3, 445–495.

[2] A. Alekseev and E. Meinrenken, in preparation.

[3] A. Alekseev, E. Meinrenken, and C. Woodward, *The Verlinde formulas as fixed point formulas*, J. Symplectic Geom. **1** (2001), no. 1, 1–46.

[4] M. Atiyah and G. Segal, *Twisted K-theory*, Ukr. Mat. Visn. **1** (2004), no. 3, 287–330.

[5] B. Blackadar, *K-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.

[6] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, and D. Stevenson, *Twisted K-theory and K-theory of bundle gerbes*, Comm. Math. Phys. **228** (2002), no. 1, 17–45.

[7] V. Braun, *Twisted K-theory of Lie groups*, J. High Energy Phys. (2004), no. 3, 029, 15 pp. (electronic).

[8] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[9] J. Brodzki, V. Mathai, J. Rosenberg, and R. Szabo, *D-Branes, RR-Fields and Duality on Noncommutative Manifolds*, ESI-1813, HWM-06-30, EMPG-06-06.

[10] J. Dixmier and A. Douady, *Champs continus d'espaces hilbertiens et de C^*-algèbres*, Bull. Soc. Math. France **91** (1963), 227–284.

[11] P. Donovan and M. Karoubi, *Graded Brauer groups and K-theory with local coefficients*, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 5–25.

[12] C. L. Douglas, *On the twisted K-homology of simple Lie groups*, Topology **45** (2006), no. 6, 955–988.

[13] D. Freed, *The Verlinde algebra is twisted equivariant K-theory*, Turkish J. Math. **25** (2001), no. 1, 159–167.

[14] ______, *Twisted K-theory and loop groups*, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 419–430.

[15] D. Freed, M. Hopkins, and C. Teleman, *Loop Groups and Twisted K-Theory III*, arXiv:math.AT/0312155.

[16] ______, *Loop Groups and Twisted K-Theory II*, arXiv:math.AT/0511232.

[17] ______, *Twisted K-theory and Loop Group Representations*, arXiv:math.AT/0312155.

[18] K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. J. **89** (1997), no. 2, 377–412.

[19] N. Higson and J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, Oxford Science Publications.

[20] V. Kac, *Infinite-dimensional Lie algebras*, second ed., Cambridge University Press, Cambridge, 1985.

[21] J. Kalkman, *Cohomology rings of symplectic quotients*, J. Reine Angew. Math. **485** (1995), 37–52.

[22] G. G. Kasparov, *Topological invariants of elliptic operators. I. K-homology*, Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no. 4, 796–838.
[23] ———, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), no. 1, 147–201.
[24] ———, *K-theory, group C^*-algebras, and higher signatures (conspectus)*, Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., vol. 226, Cambridge Univ. Press, Cambridge, 1995, pp. 101–146.
[25] N. Kitchloo and J. Morava, *Thom prospectra for loopgroup representations*, Elliptic Cohomology (D. Ravenel H. Miller, ed.), London Mathematical Society Lecture Note Series, vol. 342, Cambridge University Press, 2007, arXiv:math.AT/0404541, pp. 214–238.
[26] V. Mathai, R. Melrose, and I. Singer, *Fractional analytic index*, J. Differential Geom. **74** (2006), 265–292.
[27] V. Mathai, R. Melrose, and I.M. Singer, *The index of projective families of elliptic operators*, Geom. Topol. **9** (2005), 341–373, arXiv:math.DG/0206002.
[28] E. Meinrenken, *The basic gerbe over a compact simple Lie group*, Enseign. Math. (2) **49** (2003), no. 3-4, 307–333.
[29] E. Meinrenken and R. Sjamaar, in preparation.
[30] E. Meinrenken and C. Woodward, *Canonical bundles for Hamiltonian loop group manifolds*, Pacific J. Math. **198** (2001), no. 2, 477–487.
[31] M. Mostow and J. Perchik, *Notes on Gelfand-Fuks cohomology and characteristic classes (lectures delivered by R. Bott)*, Proceedings of the eleventh annual holiday symposium, New Mexico State University, 1973, pp. 1–126.
[32] E. M. Parker, *The Brauer group of graded continuous trace C^*-algebras*, Trans. Amer. Math. Soc. **308** (1988), no. 1, 115–132.
[33] H. Pittie, *Homogeneous vector bundles on homogeneous spaces*, Topology **11** (1972), 199–203.
[34] R. J. Plymen, *Strong Morita equivalence, spinors and symplectic spinors*, J. Operator Theory **16** (1986), no. 2, 305–324.
[35] I. Raeburn and D. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.
[36] J. Rosenberg, *Continuous-trace algebras from the bundle theoretic point of view*, J. Austral. Math. Soc. Ser. A **47** (1989), no. 3, 368–381.
[37] G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 105–112.
[38] E. H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint.
[39] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173–177.
[40] S. Sternberg, *Lie algebras*, Lecture notes, available at http://www.math.harvard.edu/ shlomo/.
[41] J.-L. Tu, *Twisted K-theory and Poincare duality*, Trans. Amer. Math. Soc. (to appear).
[42] J.-L. Tu and P. Xu, *The ring structure for equivariant twisted K-theory*, J. Reine Angew. Math. (to appear).
[43] J.-L. Tu, P. Xu, and C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 6, 841–910.

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