TILINGS DEFINED BY AFFINE WEYL GROUPS

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Abstract. Let W be a Weyl group, presented as a reflection group on a Euclidean vector space V, and $C \subset V$ an open Weyl chamber. In a recent paper, Waldspurger proved that the images $(\mathrm{id} - w)(C)$ for $w \in W$ are all disjoint, with union the closed cone spanned by the positive roots. We prove that similarly, the images $(\mathrm{id} - w)(A)$ of the open Weyl alcove A, for $w \in W^{\mathrm{a}}$ in the affine Weyl group, are disjoint and their union is V.

1. Introduction

Let W be the Weyl group of a simple Lie algebra, presented as a crystallographic reflection group in a finite-dimensional Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$. Choose a fundamental Weyl chamber $C \subset V$, and let D be its dual cone, i.e. the open cone spanned by the corresponding positive roots. In his recent paper [2], Waldspurger proved the following remarkable result. Consider the linear transformations $(\mathrm{id} - w)\colon V \to V$ defined by elements $w \in W$.

Theorem 1.1 (Waldspurger). *The images $D_w := (\mathrm{id} - w)(C)$, $w \in W$ are all disjoint, and their union is the closed cone spanned by the positive roots:*

$$\overline{D} = \bigcup_{w \in W} D_w.$$

For instance, the identity transformation $w = \mathrm{id}$ corresponds to $D_{\mathrm{id}} = \{0\}$ in this decomposition, while the reflection s_α defined by a positive root α corresponds to the open half-line $D_{s_\alpha} = \mathbb{R}_{>0} \cdot \alpha$.

The aim of this note is to prove a similar result for the *affine* Weyl group W^{a}. Recall that $W^{\mathrm{a}} = \Lambda \rtimes W$ where the co-root lattice $\Lambda \subset V$ acts by translations. Let $A \subset C$ be the Weyl alcove, with $0 \in \overline{A}$.

Theorem 1.2. *The images $V_w = (\mathrm{id} - w)(A)$, $w \in W^{\mathrm{a}}$ are all disjoint, and their union is V:*

$$V = \bigcup_{w \in W^{\mathrm{a}}} V_w.$$

Figure 1 is a picture of the resulting tiling of V for the root system $\mathbf{G}_2$. Up to translation by elements of the lattice Λ, there are five 2-dimensional tiles, corresponding to the five Weyl group elements with trivial fixed point set. Letting s_1, s_2 denote the simple reflections, the lightly shaded polytopes are labeled by the Coxeter elements s_1s_2, s_2s_1, the medium shaded polytopes by $(s_1s_2)^2$, $(s_2s_1)^2$, and the darkly shaded polytope by the longest Weyl group element $w_0 = (s_1s_2)^3$.

One also has the following related statement.

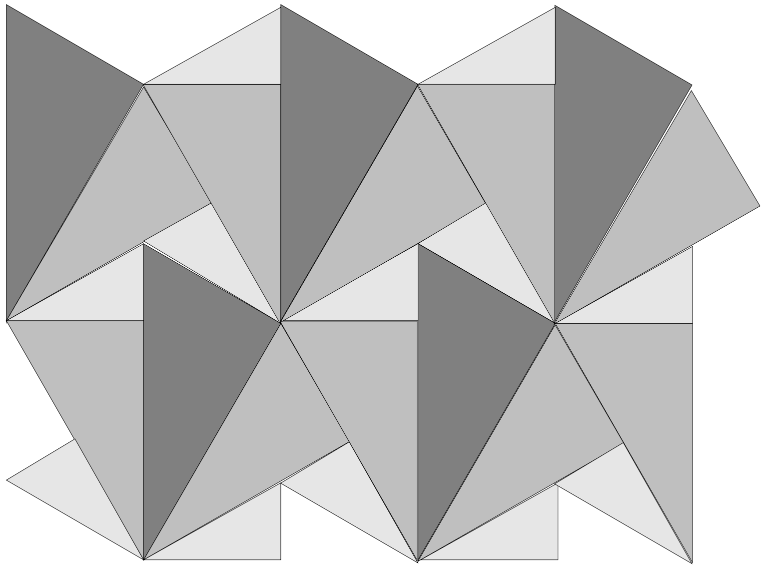

FIGURE 1. The tiling for the root system $\mathbf{G_2}$

Theorem 1.3. *Suppose $S \in \mathrm{End}(V)$ with $||S|| < 1$. Then the sets $V_w^{(S)} = (S-w)(A),\ w \in W^{\mathrm{a}}$ are all disjoint, and their closures cover V:*

$$V = \bigcup_{w \in W^{\mathrm{a}}} \overline{V}_w^{(S)}.$$

Note that for $S = 0$ the resulting decomposition of V is just the Stiefel diagram, while for $S = \tau\,\mathrm{id}$ with $\tau \to 1$ one recovers the decomposition from Theorem 1.2.

The proof of Theorem 1.2 is in large parts parallel to Waldspurger's [2] proof of Theorem 1.1. We will nevertheless give full details in order to make the paper self-contained.

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2. Notation

With no loss of generality we will take W to be irreducible. Let $\mathfrak{R} \subset V$ be the set of roots, $\{\alpha_1, \ldots, \alpha_l\} \subset \mathfrak{R}$ a set of simple roots, and

$$C = \{x|\ \langle \alpha_i, x\rangle > 0,\ i = 1, \ldots, l\}$$

the corresponding Weyl chamber. We denote by $\alpha_{\max} \in \mathfrak{R}$ the highest root, and $\alpha_0 = -\alpha_{\max}$ the lowest root. The open Weyl alcove is the l-dimensional simplex defined as

$$A = \{x|\ \langle \alpha_i, x\rangle + \delta_{i,0} > 0,\ i = 0, \ldots, l\}.$$

Its faces are indexed by the proper subsets $I \subset \{0, \ldots, l\}$, where A_I is given by inequalities $\langle \alpha_i, x\rangle + \delta_{i,0} > 0$ for $i \notin I$ and equalities $\langle \alpha_i, x\rangle + \delta_{i,0} = 0$ for $i \in I$. Each A_I has codimension

$|I|$. In particular, $A_i = A_{\{i\}}$ are the codimension 1 faces, with α_i as inward-pointing normal vectors. Let s_i be the affine reflections across the affine hyperplanes supporting A_i,

$$s_i\colon x \mapsto x - (\langle \alpha_i, x\rangle + \delta_{i,0})\alpha_i^\vee, \quad i = 0, \ldots, l,$$

where $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i\rangle$ is the simple co-root corresponding to α_i. The Weyl group W is generated by the reflections $s_1, \ldots, s_l$, while the affine Weyl group W^{a} is generated by the affine reflections $s_0, \ldots, s_l$. The affine Weyl group is a semi-direct product

$$W^{\mathrm{a}} = \Lambda \rtimes W$$

where the co-root lattice $\Lambda = \mathbb{Z}[\alpha_1^\vee, \ldots, \alpha_l^\vee] \subset V$ acts on V by translations. For any $w \in W^{\mathrm{a}}$, we will denote by $\tilde{w} \in W$ its image under the quotient map $W^{\mathrm{a}} \to W$, i.e. $\tilde{w}(x) = w(x) - w(0)$, and by $\lambda_w = w(0) \in \Lambda$ the corresponding lattice vector.

The stabilizer of any given element of A_I is the subgroup W_I^{a} generated by s_i, $i \in I$. It is a finite subgroup of W^{a}, and the map $w \mapsto \tilde{w}$ induces an isomorphism onto the subgroup W_I generated by $\tilde{s}_i$, $i \in I$. Recall that W_I is itself a Weyl group (not necessarily irreducible): its Dynkin diagram is obtained from the extended Dynkin diagram of the root system $\mathfrak{R}$ by removing all vertices that are in I.

3. The top-dimensional polytopes

For any $w \in W^{\mathrm{a}}$, the subset

$$V_w = (\mathrm{id} - w)(A)$$

is the relative interior of a convex polytope in the affine subspace $\mathrm{ran}(\mathrm{id} - w)$. Let

$$W^{\mathrm{a}}_{\mathrm{reg}} = \{w \in W^{\mathrm{a}} |\ (\mathrm{id} - w) \text{ is invertible}\}$$

and $W_{\mathrm{reg}} = W \cap W^{\mathrm{a}}_{\mathrm{reg}}$, so that $w \in W^{\mathrm{a}}_{\mathrm{reg}} \Leftrightarrow \tilde{w} \in W_{\mathrm{reg}}$. The top dimensional polytopes V_w are those indexed by $w \in W^{\mathrm{a}}_{\mathrm{reg}}$, and the faces of these polytopes are $V_{w,I} := (\mathrm{id} - w)(A_I)$. For $w \in W_{\mathrm{reg}}$ and $i = 0, \ldots, l$ let

$$n_{w,i} := (\mathrm{id} - \tilde{w}^{-1})^{-1}(\alpha_i).$$

Lemma 3.1. *For all $w \in W^{\mathrm{a}}_{\mathrm{reg}}$, the open polytope V_w is given by the inequalities*

$$\langle n_{w,i}, \xi + \lambda_w\rangle + \delta_{i,0} > 0$$

for $i = 0, \ldots, l$. The face $V_{w,I} = (\mathrm{id} - w)(A_I)$ is obtained by replacing the inequalities for $i \in I$ by equalities.

Proof. For any $\xi = (\mathrm{id} - w)x \in V$, we have

$$\langle \alpha_i, x\rangle = \langle (\mathrm{id} - \tilde{w}^{-1})^{-1}\alpha_i, (\mathrm{id} - \tilde{w})x\rangle = \langle n_{w,i}, (\mathrm{id} - \tilde{w})x\rangle = \langle n_{w,i}, \xi + \lambda_w\rangle,$$

since $\tilde{w}^{-1}$ is the transpose of $\tilde{w}$ under the inner product $\langle\cdot,\cdot\rangle$. This gives the description of V_w and of its faces $V_{w,I}$. □

Lemma 3.2. *Suppose $w \in W^{\mathrm{a}}_{\mathrm{reg}}$, $i \in \{0, \ldots, l\}$. Then*

$$V_{w,i} = V_{\sigma,i} \subset \mathrm{ran}(\mathrm{id} - \sigma)$$

with $\sigma = ws_i$. In particular, σ is an affine reflection, and $n_{w,i}$ is a normal vector to the affine hyperplane $\mathrm{ran}(\mathrm{id} - \sigma)$*. One has $\langle n_{w,i}, \alpha_i^\vee\rangle = 1$.*

Proof. For any orthogonal transformation $g \in \mathrm{O}(V)$ and any reflection $s \in \mathrm{O}(V)$, the dimension of the fixed point set of the orthogonal transformations g, gs differ by ± 1. Since $\tilde{w}$ fixes only the origin, it follows that $\tilde{\sigma}$ has a 1-dimensional fixed point set. Hence $\mathrm{ran}(\mathrm{id}-\sigma)$ is an affine hyperplane, and σ is the affine reflection across that hyperplane. Since s_i fixes A_i, we have $V_{w,i} = (\mathrm{id}-w)(A_i) = (\mathrm{id}-ws_i)(A_i) = V_{\sigma,i} \subset \mathrm{ran}(\mathrm{id}-\sigma)$. By definition $n_{w,i} - \tilde{w}^{-1}n_{w,i} = \alpha_i$. Hence

$$-2\langle n_{w,i}, \alpha_i\rangle + \langle \alpha_i, \alpha_i\rangle = ||n_{w,i} - \alpha_i||^2 - ||n_{w,i}||^2 = ||\tilde{w}^{-1}n_{w,i}||^2 - ||n_{w,i}||^2 = 0. \qquad \square$$

The following Proposition indicates how the top-dimensional polytopes $V_{w,i}$ are glued along the polytopes of codimension 1.

Proposition 3.3. *Let $\sigma \in W^{\mathrm{a}}$ be an affine reflection, i.e.* $\mathrm{ran}(\mathrm{id}-\sigma)$ *is an affine hyperplane. Consider*

$$\xi \in V_\sigma \backslash \bigcup_{|I|\geq 2} V_{\sigma,I}. \tag{1}$$

Then there are two distinct indices $i, i' \in \{0, \ldots, l\}$ such that $\xi \in V_{\sigma,i} \cap V_{\sigma,i'}$. Furthermore, $w = \sigma s_i$ and $w' = \sigma s_{i'}$ are both in $W^{\mathrm{a}}_{\mathrm{reg}}$, so that $V_{w,i} = V_{\sigma,i}$ and $V_{w',i'} = V_{\sigma,i'}$, and the polytopes $V_w, V_{w'}$ are on opposite sides of the affine hyperplane $\mathrm{ran}(\mathrm{id}-\sigma)$.

Proof. Let n be a generator of the 1-dimensional subspace $\ker(\mathrm{id}-\tilde{\sigma})$. Then n is a normal vector to $\mathrm{ran}(\mathrm{id}-\sigma)$. The pre-image $(\mathrm{id}-\sigma)^{-1}(\xi) \subset V$ is an affine line in the direction of n. Since $\xi \in V_\sigma$, this line intersects A, hence it intersects the boundary $\partial\overline{A}$ in exactly two points x, x'. By (1), x, x' are contained in two distinct codimension 1 boundary faces $A_i,\ A_{i'}$. Since n is 'inward-pointing' at one of the boundary faces, and 'outward-pointing' at the other, the inner products $\langle n, \alpha_i\rangle,\ \langle n, \alpha_{i'}\rangle$ are both non-zero, with opposite signs. Let $w = \sigma s_i$ and $w' = \sigma s_{i'}$. We will show that $w \in W^{\mathrm{a}}_{\mathrm{reg}}$, i.e. $\tilde{w} \in W_{\mathrm{reg}}$ (the proof for w' is similar). Let $z \in V$ with $\tilde{w}z = z$. Then $\tilde{\sigma}^{-1}z = \tilde{s}_i z$, so

$$(\mathrm{id}-\tilde{\sigma}^{-1})(z) = (\mathrm{id}-\tilde{s}_i)(z) = \langle \alpha_i, z\rangle \alpha_i^\vee.$$

The left hand side lies in $\mathrm{ran}(\mathrm{id}-\tilde{\sigma})$, which is orthogonal to n, while the right hand side is proportional to α_i. Since $\langle n, \alpha_i\rangle \neq 0$ this is only possible if both sides are 0. Thus z is fixed under $\tilde{\sigma}$, and hence a multiple of n. On the other hand we have $\langle \alpha_i, z\rangle = 0$, hence using again that $\langle n, \alpha_i\rangle \neq 0$ we obtain $z = 0$. This shows $\ker(\mathrm{id}-\tilde{w}) = 0$.

As we had seen above, $n_{w,i}$ is a normal vector to $\mathrm{ran}(\mathrm{id}-\sigma)$, hence it is a multiple of n. By Lemma 3.2, it is a positive multiple if and only if $\langle n, \alpha_i\rangle > 0$. But then $\langle n, \alpha_{i'}\rangle < 0$, and so $n_{w',i'}$ is a negative multiple of n. This shows that $V_w, V_{w'}$ are on opposite sides of the hyperplane $\mathrm{ran}(\mathrm{id}-\sigma)$. $\square$

Consider the union over $W \subset W^{\mathrm{a}}$,

$$X := \bigcup_{w \in W} V_w. \tag{2}$$

Thus $\bigcup_{w\in W^{\mathrm{a}}} V_w = \bigcup_{\lambda\in\Lambda}(\lambda + X)$. The statement of Theorem 1.2 means in particular that X is a fundamental domain for the action of Λ. Figures 2 and 3 give pictures of X for the root systems $\mathbf{B_2}$ and $\mathbf{G_2}$. The shaded regions are the top-dimensional polytopes (i.e. the sets V_w for $\mathrm{id}-w$ invertible), the dark lines are the 1-dimensional polytopes (corresponding to reflections), and the origin corresponds to $w = \mathrm{id}$.

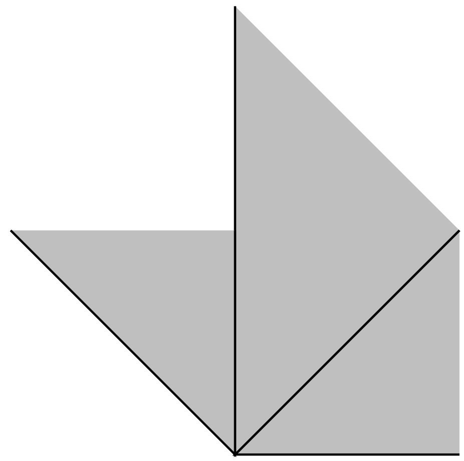

FIGURE 2. The set X for the root system $\mathbf{B_2}$

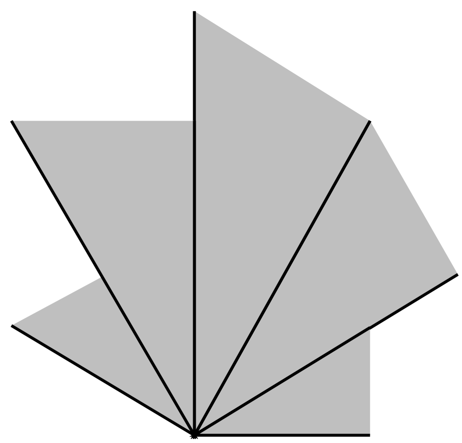

FIGURE 3. The set X for the root system $\mathbf{G_2}$

Proposition 3.4. *(a) The sets* $\lambda + \mathrm{int}(\overline{X})$, $\lambda \in \Lambda$ *are disjoint, and* $\bigcup_{\lambda\in\Lambda} \lambda + \overline{X} = V$. *(b) The open polytopes* V_w *for* $w \in W^{\mathrm{a}}_{\mathrm{reg}}$ *are disjoint, and* $\bigcup_{w\in W^{\mathrm{a}}_{\mathrm{reg}}} \overline{V}_w = V$.

Proof. Since the collection of closed polytopes $\overline{V}_w$, $w \in W_{\mathrm{reg}}$ is locally finite, the union $\bigcup_{w\in W^{\mathrm{a}}_{\mathrm{reg}}} \overline{V}_w$ is a closed polyhedral subset of V. Proposition 3.3 shows that a point $\xi \in V_{w,i}$ cannot contribute to the boundary of this subset unless it lies in $\bigcup_{\sigma\in W^{\mathrm{a}}} \bigcup_{|I|\geq 2} V_{\sigma,I}$. We therefore see that the boundary has codimension ≥ 2, and hence is empty since $\bigcup_{w\in W^{\mathrm{a}}_{\mathrm{reg}}} \overline{V}_w$ is a closed polyhedron. This proves $\bigcup_{w\in W^{\mathrm{a}}_{\mathrm{reg}}} \overline{V}_w = V$, and also $\bigcup_{\lambda\in\Lambda}(\lambda + \overline{X}) = V$ with X as defined in (2). Hence the volume $\mathrm{vol}(X)$ (for the Riemannian measure on V defined by the inner product) must be at least the volume of a fundamental domain for the action of Λ:

$$\mathrm{vol}(X) \geq |W|\,\mathrm{vol}(A). \tag{3}$$

On the other hand, $\mathrm{vol}(V_w) = \mathrm{vol}((\mathrm{id} - w)(A)) = \det(\mathrm{id} - w)\,\mathrm{vol}(A)$, so

$$\mathrm{vol}(X) \le \sum_{w\in W} \mathrm{vol}(V_w) = \mathrm{vol}(A) \sum_{w\in W} \det(\mathrm{id} - w) = |W|\,\mathrm{vol}(A) \tag{4}$$

where we used the identity [1, p.134] $\sum_{w\in W} \det(\mathrm{id} - w) = |W|$. This confirms $\mathrm{vol}(X) = |W|\,\mathrm{vol}(A)$. It follows that the sets $\lambda + \mathrm{int}(\overline{X})$ are pairwise disjoint, or else the inquality (3) would be strict. Similarly that the sets $V_w,\ w \in W_{\mathrm{reg}}$ are disjoint, or else the inequality (4) would be strict. (Of course, this also follows from Waldpurger's Theorem 1.1 since $C_w \subset D_w$.) Hence all $V_w,\ w \in W^{\mathrm{a}}_{\mathrm{reg}}$ are disjoint. □

To proceed, we quote the following result from Waldspurger's paper, where it is stated in greater generality [2, "Lemme"].

Proposition 3.5 (Waldspurger)**.** *Given $w \in W$ and a proper subset $I \subset \{0, \ldots, l\}$ there exists a unique $q \in W_I$ such that*

$$\ker(\mathrm{id} - wq) \cap \{x \in V |\ \langle \alpha_i, x\rangle > 0 \textit{ for all } i \in I\} \neq \emptyset.$$

Following [2] we use this to prove,

Proposition 3.6. *Every element of V is contained in some $V_w,\ w \in W^{\mathrm{a}}$:*

$$\bigcup_{w\in W^{\mathrm{a}}} V_w = V. \tag{5}$$

Proof. Let $\xi \in V$ be given. Pick $w \in W^{\mathrm{a}}_{\mathrm{reg}}$ with $\xi \in \overline{V}_w$, and let $I \subset \{0, \ldots, l\}$ with $\xi \in V_{w,I}$. Then $x := (\mathrm{id} - w)^{-1}(\xi) \in A_I$ is fixed under W^{a}_I. Using Proposition 3.5 we may choose $\tilde{q} \in W_I$ and $n \in V$ such that

(a) $\tilde{w}\tilde{q}(n) = n$,
(b) $\langle \alpha_i, n\rangle > 0$ for all $i \in I$

Taking $||n||$ sufficiently small we have $x + n \in A$, and

$$(\mathrm{id} - wq)(x + n) = (\mathrm{id} - wq)(x) + (\mathrm{id} - \tilde{w}\tilde{q})n = (\mathrm{id} - w)(x) = \xi.$$

This shows $\xi \in V_{wq}$. □

4. Disjointness of the sets $\lambda + X$

To finish the proof of Theorem 1.2, we have to show that the union (5) is disjoint. Waldspurger's Theorem 1.1 shows that all $D_w = (\mathrm{id} - w)(C),\ w \in W$ are disjoint. (We refer to his paper for a very simple proof of this fact.) Hence the same is true for $V_w \subset D_w,\ w \in W$. It remains to show that the sets $\lambda + X,\ \lambda \in \Lambda$, with X given by (2), are disjoint.

The following Lemma shows that the closure $\overline{X} = \bigcup_{w\in W} \overline{V}_w$ only involves the top-dimensional polytopes.

Lemma 4.1. *The closure of the set X is a union over W_{reg},*

$$\overline{X} = \bigcup_{w\in W_{\mathrm{reg}}} \overline{V}_w.$$

Furthermore, $\mathrm{int}(\overline{X}) = \mathrm{int}(X)$.

Proof. We must show that for any $\xi \in \overline{V}_\sigma$, $\sigma \in W \backslash W_{\mathrm{reg}}$, there exists $w \in W_{\mathrm{reg}}$ such that $\xi \in \overline{V}_w$. Using induction, it is enough to find $\sigma' \in W$ such that $\xi \in \overline{V}_{\sigma'}$ and $\dim(\ker(\mathrm{id} - \sigma')) = \dim(\ker(\mathrm{id} - \sigma)) - 1$. Let $\pi \colon V \to \ker(\mathrm{id} - \sigma)^\perp = \mathrm{ran}(\mathrm{id} - \sigma)$ denote the orthogonal projection. Then $\mathrm{id} - \sigma$ restricts to an invertible transformation of $\pi(V)$, and $\overline{V}_\sigma$ is the image of $\pi(\overline{A})$ under this transformation. We have

$$\pi(\overline{A}) = \pi(\partial \overline{A}) = \bigcup_{i=0}^{l} \pi(\overline{A}_i),$$

and this continues to hold if we remove the index $i = 0$ from the right hand side, as well as all indices i for which $\dim \pi(A_i) < \dim \pi(V)$. That is, for each point $x \in \pi(\overline{A})$ there exists an index $i \neq 0$ such that $x \in \pi(\overline{A}_i)$, with $\dim \pi(A_i) = \dim \pi(V)$. Taking x to be the pre-image of ξ under $(\mathrm{id} - \sigma)|_{\pi(V)}$, we have $\xi \in \overline{V}_{\sigma,i}$ with $i \neq 0$ and $\dim V_{\sigma,i} = \dim \mathrm{ran}(\mathrm{id} - \sigma)$. Let $\sigma' = \sigma s_i \in W$. Then $V_{\sigma,i} = V_{\sigma',i}$, hence $\dim(\mathrm{ran}(\mathrm{id} - \sigma')) \geq \dim V_{\sigma,i} = \dim(\mathrm{ran}(\mathrm{id} - \sigma))$, which shows $\dim \ker(\mathrm{id} - \sigma') \leq \dim \ker(\mathrm{id} - \sigma)$. By elementary properties of reflection groups, the dimensions of the fixed point sets of σ, σ' differ by either $+1$ or -1. Hence $\dim(\ker(\mathrm{id} - \sigma')) = \dim(\ker(\mathrm{id} - \sigma)) - 1$, proving the first assertion of the Lemma.

It follows in particular that the closure of $\mathrm{int}(\overline{X})$ equals that of X. Suppose $\xi \in \mathrm{int}(\overline{X})$. By Proposition 3.6 there exists $\lambda \in \Lambda$ with $\xi \in \lambda + X$. It follows that $\mathrm{int}(\overline{X})$ meets $\lambda + X$, and hence also meets $\lambda + \mathrm{int}(\overline{X})$. Since the Λ-translates of $\mathrm{int}(\overline{X})$ are pairwise disjoint (see Proposition 3.4), it follows that $\lambda = 0$, i.e. $\xi \in X$. This shows $\xi \in X \cap \mathrm{int}(\overline{X}) = \mathrm{int}(X)$, hence $\mathrm{int}(\overline{X}) \subset \mathrm{int}(X)$. The opposite inclusion is obvious. □

Since we already know that the sets $\lambda + \mathrm{int}(X)$ are disjoint, we are interested in $X \backslash \mathrm{int}(X) \subset \partial X = \overline{X} \backslash \mathrm{int}(X)$. Let us call a closed codimension 1 boundary face of the polyhedron $\overline{X}$ 'horizontal' if its supporting hyperplane contains $V_{w,0}$ for some $w \in W_{\mathrm{reg}}$, and 'vertical' if its supporting hyperplane contains $V_{w,i}$ for some $w \in W_{\mathrm{reg}}$ and $i \neq 0$. These two cases are exclusive:

Lemma 4.2. *Let n be the inward-pointing normal vector to a codimension 1 face of $\overline{X}$. Then $\langle n, \alpha_{\max} \rangle \neq 0$. In fact, $\langle n, \alpha_{\max} \rangle < 0$ for the horizontal faces and $\langle n, \alpha_{\max} \rangle > 0$ for the vertical faces.*

Proof. Given a codimension 1 boundary face of $\overline{X}$, pick any point ξ in that boundary face, not lying in $\bigcup_{w \in W^{\mathrm{a}}} \bigcup_{|I| \geq 2} V_{w,I}$. Let $w \in W_{\mathrm{reg}}$ and $i \in \{0, \ldots, l\}$ such that $\xi \in V_{w,i}$, and $n_{w,i}$ is an inward-pointing normal vector. By Proposition 3.3 there is a unique $i' \neq i$ such that $\xi \in V_{w',i'}$, where $w' = w s_i s_{i'}$. Since $V_w, V_{w'}$ lie on opposite sides of the affine hyperplane spanned by $V_{w,i}$, and ξ is a boundary point of $\overline{X}$, we have $w' \notin W$. Thus one of i, i' must be zero. If $i = 0$ (so that the given boundary face is horizontal) we obtain $\langle n_{w,0}, \alpha_{\max} \rangle = -\langle n_{w,0}, \alpha_0 \rangle < 0$. If $i' = 0$ we similarly obtain $\langle n_{w',0}, \alpha_{\max} \rangle < 0$, hence $\langle n_{w,i}, \alpha_{\max} \rangle > 0$. □

Lemma 4.3. *Let $\xi \in X \backslash \mathrm{int}(X)$. Then there exists a vertical boundary face of $\overline{X}$ containing ξ. Equivalently, the complement $\partial \overline{X} \backslash (X \backslash \mathrm{int}(X))$ is contained in the union of horizontal boundary faces.*

Proof. The alcove A is invariant under multiplication by any scalar in $(0, 1)$. Hence, the same is true for the sets V_w for $w \in W$, as well as for X and $\mathrm{int}(X)$. Hence, if $\xi \in X \backslash \mathrm{int}(X)$ there exists $t_0 > 1$ such that $t\xi \in X \backslash \mathrm{int}(X)$ for $1 \leq t < t_0$. The closed codimension 1 boundary face containing this line segment is necessarily vertical, since a line through the origin intersects the affine hyperplane $\{x|\ \langle n_{w,0}, x-\xi\rangle = 0\}$ in at most one point. □

Proposition 4.4. *For any $\xi \in X$, there exists $\epsilon > 0$ such that $\xi + s\alpha_{\max} \in \mathrm{int}(X)$ for $0 < s < \epsilon$.*

Proof. If $\xi \in \mathrm{int}(X)$ there is nothing to show, hence suppose $\xi \in X\setminus \mathrm{int}(X)$. Suppose first that ξ is not in the union of horizontal boundary faces of $\overline{X}$. Then there exists an open neighborhood U of ξ such that $U \cap X = U \cap \overline{X}$. All boundary faces of $\overline{X}$ meeting ξ are vertical, and their inward-pointing normal vectors n all satisfy $\langle n, \alpha_{\max}\rangle > 0$. Hence, $\xi + s\alpha_{\max} \in \mathrm{int}(U \cap \overline{X}) = \mathrm{int}(U \cap X) \subset X$ for $s > 0$ sufficiently small.

For the general case, suppose that for all $\epsilon > 0$, there is $s \in (0, \epsilon)$ with $\xi + s\alpha_{\max} \notin \mathrm{int}(X)$. We will obtain a contradiction. Since ξ is contained in some vertical boundary face, one can choose $t > 1$ so that $\xi' := t\xi \in X\setminus \mathrm{int}(X)$, but ξ' is not in the closure of the union of horizontal boundary faces. Given $\epsilon > 0$, pick $s \in (0, \epsilon)$ such that $\xi + \frac{s}{t}\alpha_{\max} \notin \mathrm{int}(X)$. Since $\mathrm{int}(X)$ is invariant under multiplication by scalars in $(0, 1)$, the complement $V\setminus \mathrm{int}(X)$ is invariant under multiplication by scalars in $(1,\infty)$, hence we obtain $\xi' + s\alpha_{\max} \notin \mathrm{int}(X)$. This contradicts what we have shown above, and completes the proof. □

Proposition 4.5. *The sets $\lambda + X$ for $\lambda \in \Lambda$ are disjoint.*

Proof. Suppose $\xi \in (\lambda + X) \cap (\lambda' + X)$. By Proposition 4.4, we can choose $s > 0$ so that $\xi + s\alpha_{\max} \in (\lambda + \mathrm{int}(X)) \cap (\lambda' + \mathrm{int}(X))$. Since the Λ-translates of $\mathrm{int}(X)$ are disjoint, it follows that $\lambda = \lambda'$. □

This completes the proof of Theorem 1.2. We conclude with some remarks on the properties of the decomposition $V = \bigcup_{w\in W^{\mathrm{a}}} V_w$.

Remarks 4.6. (a) The group of symmetries τ of the extended Dynkin diagram (i.e. the outer automorphisms of the corresponding affine Lie algebra) acts by symmetries of the decomposition $V = \bigcup_{w\in W^{\mathrm{a}}} V_w$, as follows. Identify the nodes of the extended Dynkin diagram with the simple affine reflections $s_0, \ldots, s_l$. Then τ extends to a group automorphism of W^{a}, taking s_i to $\tau(s_i)$. This automorphism is implemented by a unique Euclidean transformation $g\colon V \to V$ i.e. $gwg^{-1} = \tau(w)$ for all $w \in W^{\mathrm{a}}$. Then g preserves A, and consequently

$$g\,V_w = g(\mathrm{id} - w)(A) = (\mathrm{id} - \tau(w))(A) = V_{\tau(w)},\quad w \in W^{\mathrm{a}}.$$

(b) It is immediate from the definition that the Euclidean transformation $-w\colon V \to V$, $x \mapsto -wx$ takes $V_{w^{-1}}$ into V_w:

$$-w(V_{w^{-1}}) = V_w.$$

(c) For any positive root α, let s_α be the corresponding reflection. Then $(\mathrm{id} - s_\alpha)(\xi) = \langle\alpha, \xi\rangle\alpha^\vee$, where $\alpha^\vee$ is the co-root corresponding to α. Hence D_{s_α} is the relative interior of the line segment from 0 to $\lambda\alpha^\vee$, where λ is the maximum value of the linear functional $\xi \mapsto \langle\alpha, \xi\rangle$ on the closed alcove $\overline{A}$. This maximum is achieved at one of the vertices. Let $\varpi_1^\vee, \ldots, \varpi_l^\vee$ be the fundamental co-weights, defined by $\langle\alpha_i,\, \varpi_j^\vee\rangle = \delta_{ij}$ for $i, j = 1, \ldots, l$. Let $c_i \in \mathbb{N}$ be the coefficients of $\alpha_{\max}$ relative to the simple roots: $\alpha_{\max} = \sum_{i=1}^{l} c_i\alpha_i$. Then the non-zero vertices of A are $\varpi_i^\vee/c_i$. Similarly let $a_i \in \mathbb{Z}_{\geq 0}$ be the coefficients of α, so that $\alpha = \sum_{i=1}^{l} a_i\alpha_i$. Then the value of α at the i-th vertex of $\overline{A}$ is a_i/c_i, and λ is the maximum of those values. Two interesting cases are: (i) If $\alpha = \alpha_{\max}$, then all

$a_i/c_i = 1$, and $\alpha^\vee = \alpha$. That is, the open line segment from the origin to the highest root always appears in the decomposition. (ii) If $\alpha = \alpha_i$, then $a_i = 1$ while all other a_j vanish. In this case, one obtains the open line segment from the origin to $\frac{1}{c_i}\alpha_i^\vee$.

(d) Every V_w contains a distinguished 'base point'. Indeed, let $\rho \in V$ be the half-sum of positive roots, and $h^\vee = 1 + \langle \alpha_{\max}, \rho \rangle$ the dual Coxeter number. Then $\rho/h^\vee \in A$, and consequently $\rho/h^\vee - w(\rho/h^\vee) \in V_w$.

5. Proof of Theorem 1.3

The proof is very similar to the proof of Proposition 3.4, hence we will be brief. Each $V_w^{(S)} = (S-w)(A)$ is the interior of a simplex in V, with codimension 1 faces $V_{w,i}^{(S)} = (S-w)(A_i)$. As in the proof of Lemma 3.1, we see that

$$n_{w,i}^{(S)} = (S - \tilde{w}^{-1})^{-1}\alpha_i$$

is an inward-pointing normal vector to the i-th face $V_{w,i}^{(S)}$. For $S = 0$ this simplifies to

$$n_{w,i}^{(0)} = -w\alpha_i$$

If $w' = ws_i$ we have $V_{w,i}^{(S)} = V_{w',i}^{(S)}$, so that $n_{w,i}^{(S)}$ and $n_{w',i}^{(S)}$ are proportional. Since $n_{w,i}^{(0)} = -n_{w',i}^{(0)}$, it follows by continuity that $n_{w,i}^{(S)}$ is a negative multiple of $n_{w',i}^{(S)}$. As a consequence, we see that $V_w^{(S)}$, $V_{w'}^{(S)}$ are on opposite sides of affine hyperplane supporting $V_{w,i}^{(S)} = V_{w',i}^{(S)}$. Arguing as in the proof of Proposition 3.4, this shows that

$$\bigcup_{w \in W^{\mathrm{a}}} \overline{V}_w^{(S)} = V.$$

Letting $X^{(S)} = \bigcup_{w\in W} V_w^{(S)}$, it follows that $V = \bigcup_{\lambda\in\Lambda}(\lambda + \overline{X}^{(S)})$. Hence $\mathrm{vol}(X^{(S)}) \geq |W| \,\mathrm{vol}(A)$. But

$$\begin{aligned} \mathrm{vol}(X^{(S)}) &\leq \sum_{w\in W} \mathrm{vol}\left((S-w)(A)\right) \\ &= \mathrm{vol}(A) \sum_{w\in W} |\det(S-w)| \\ &= \mathrm{vol}(A) \sum_{w\in W} \det(\mathrm{id} - Sw^{-1}) = |W| \,\mathrm{vol}(A), \end{aligned}$$

using [1, p.134]. It follows that $\mathrm{vol}(X^{(S)}) = |W|\,\mathrm{vol}(A)$, which implies (as in the proof of Proposition 3.4) that all $\mathrm{int}(\overline{V}_w^{(S)}) = V_w^{(S)}$ are disjoint. This completes the proof.

Remark 5.1. Theorem 1.3, and its proof, go through for any S in the component of 0 in the set $\{S \in \mathrm{End}(V) |\ \det(S-w) \neq 0\ \forall w \in W\}$. For instance, the fact that $\det(\mathrm{id} - Sw^{-1}) > 0$ follows by continuity from $S = 0$. On the other hand, if e.g. S is a positive matrix with $S > 2\,\mathrm{id}$, the result becomes false, since then (cf. [1, p. 134]) $\sum_{w\in W} |\det(S-w)| = \sum_{w\in W} \det(S-w) = \det(S)|W|$.

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