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Spectral analysis of N-body Schrödinger operators*

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Abstract

For a large class of two body potentials, we solve two of the main problems in the spectral analysis of multiparticle quantum Hamiltonians: explicitly, we prove that the point spectrum lies in a closed countable set (and describe that set in terms of the eigenvalues of Hamiltonians of subsystems) and that there is no singular continuous spectrum. We accomplish this by extending Mourre's work on three body problems to N-body problems.

1. Introduction

In this paper, we study the spectral properties of the Hamiltonian operators of multiparticle nonrelativistic quantum mechanics (Schrödinger operators). To describe N particles moving in ν space dimensions, we write a point in $R^{N\nu}$ as $(r_1, \ldots, r_N)$ with $r_i \in R^\nu$. If Δ_i is the Laplacian with respect to r_i and if the masses are $m_1, \ldots, m_N$, then the kinetic energy is (in units with $\hbar = 1$)

$$\tilde{H}_0 = -\sum_{i=1}^{N} (2m_i)^{-1}\Delta_i. \tag{1.1}$$

For each pair $\gamma = \{i, j\} \subset \{1, \ldots, N\}$, we let $r_\gamma = r_i - r_j$ and suppose we are given a measurable function V_γ on R^ν. We will use V_γ interchangeably for the function and for the corresponding multiplication operator $V_\gamma(r_\gamma)$ on $L^2(R^{N\nu})$. In addition, after we remove the center of mass motion or restrict to various clusterings (see below), we will continue to use the symbol V_γ for the obvious function or multiplication operator even though the underlying space changes.

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The full Hamiltonian is

$$\tilde{H} = \tilde{H}_0 + V, \tag{1.2}$$

$$V = \sum_{\gamma} V_{\gamma}. \tag{1.3}$$

Conventionally, one does not study $\tilde{H}$ but rather an operator H which has the trivial center of mass motion removed. Explicitly, one writes

$$L^2(R^{N\nu}) = \mathcal{H} \otimes \mathcal{H}_{\mathrm{CM}} \tag{1.4}$$

where $\mathcal{H}$ consists of functions of the r_γ and $\mathcal{H}_{\mathrm{CM}}$ consists of functions of $R = \sum_i m_i r_i / \sum m_i$. Under this decomposition

$$\tilde{H}_0 = H_0 \otimes 1 + 1 \otimes T_0, \tag{1.5}$$

$$V = V \otimes 1. \tag{1.6}$$

((1.6) is an example of our convention of using V_γ and V for the "same" operator on different spaces.) In (1.5)

$$T_0 = -\left(2 \sum m_i\right)^{-1} \Delta_R$$

and H_0 has various forms depending on the precise coordinate system used; e.g. if we use the $N - 1$ coordinates $x_i = r_i - r_N$, $i = 1, \ldots, N - 1$, then H_0 has the form

$$H_0 = -\sum_{i=1}^{N} (2\mu_i)^{-1} \Delta_{x_i} + (2m_N)^{-1} \sum_{i<j} \nabla_{x_i} \cdot \nabla_{x_j}. \tag{1.7}$$

Clearly

$$\tilde{H} = H \otimes 1 + 1 \otimes T_0$$

where H is the operator on $\mathcal{H}$ given by

$$H = H_0 + V. \tag{1.8}$$

It is operators of the form H we will study here. Notice that as $m_N \to \infty$, (1.7) has a nice limit; while we will not be explicit about this, our methods below extend to treat operators with one mass infinite without any significant change.

(Parenthetically, we remark that there is an alternative way of describing reduction of the center of mass [35, 4]: let π be the plane of codimension ν given by the condition $\sum_i m_i r_i = 0$. Then $\mathcal{H}$ is just $L^2(\pi, d^{(N-1)\nu}x)$, V is the obvious function and H_0 is the Laplace-Beltrami operator associated with the metric obtained by restricting the metric $\sum (2m_i)(dx_i)^2$ on $L^2(R^{N\nu})$ to π.)

There are three main problems associated with the general spectral analysis of H:

(1) Prove that the point spectrum of H can only accumulate at thresholds.

(2) Prove that the singular continuous spectrum of H is empty.

(3) Prove asymptotic completeness.

(Below, after introducing some notation, we will define thresholds and asymptotic completeness; we note here that proving (1) for H and for the H's associated to subsystems proves that the thresholds and the union of thresholds with point spectrum are closed countable sets.)

In this paper, we will solve problems (1) and (2) for very general classes of potentials V_γ. We note that prior to our work, these problems were only solved when $N \geq 4$ for V's with very special analyticity properties or with restrictions on the sign or size of V's. (We give a more complete history at the conclusion of this introduction.) For example, even if all the V_γ are in C_0^∞, there have been no results on these problems when $N \geq 4$; our conditions on V allow arbitrary C_0^∞ functions as well as much broader classes.

Our work was motivated by and depends heavily on ideas in a remarkable paper by Eric Mourre [24]. Mourre develops an abstract theory and then shows its applications include Schrödinger operators with $N = 2$ or 3. To describe Mourre's abstract theory, we define the scale of spaces $\mathcal{H}_{+2}, \mathcal{H}_{+1}, \mathcal{H}, \mathcal{H}_{-1}, \mathcal{H}_{-2}$ associated to a self-adjoint operator H on a Hilbert space, $\mathcal{H}$. $\mathcal{H}_{+2}$ is just $D(H)$ with the graph norm

$$\|\psi\|_{+2} = \left[\|H\psi\|^2 + \|\psi\|^2\right]^{1/2}.$$

$\mathcal{H}_{+1}$ is $D(|H|^{1/2})$ with its graph norm and $\mathcal{H}_{-1}, \mathcal{H}_{-2}$ are the duals of $\mathcal{H}_{+1}, \mathcal{H}_{+2}$ defined via the $\mathcal{H}$-duality so that

$$\mathcal{H}_{+2} \subset \mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2}$$

(thus $\phi \in \mathcal{H}$ is associated to a linear functional in $\mathcal{H}_{-1}$ by $l_\phi(\eta) = (\phi, \eta)$ with the $\mathcal{H}$-inner product; when $H = -\Delta$ on $L^2(R^\nu)$, these are the familiar Sobolev spaces). Mourre proves the following abstract theorem:

THEOREM 1.1 ([24]). *Let H be a self-adjoint operator which is bounded from below on a Hilbert-space $\mathcal{H}$ and let $\mathcal{H}_i$ be its scale of spaces. Let A be a second self-adjoint operator so that:*

(a) *$D(A) \cap \mathcal{H}_{+2}$ is dense in $\mathcal{H}_{+2}$;*

(b) *$e^{i\alpha A}$ leaves $\mathcal{H}_{+2}$ invariant and for each $\psi \in \mathcal{H}_{+2}$, $\sup_{|\alpha|\leq 1}\|e^{i\alpha A}\psi\|_{+2} < \infty$.*

(c) *The quadratic form $i[H, A]$, defined on $D(A) \cap \mathcal{H}_{+2}$, is bounded from below and extends to a bounded operator, B, from $\mathcal{H}_{+2}$ to $\mathcal{H}$.*

(d) *The form defined on* $D(A) \cap \mathcal{H}_{+2}$ *by* $[B, A]$ *extends to a bounded operator from* $\mathcal{H}_{+2}$ *to* $\mathcal{H}_{-2}$.

(e) *For some open interval* Δ, *there exists* $\alpha > 0$ *and a compact operator* K *so that*

$$E_\Delta B E_\Delta \geq \alpha E_\Delta^2 + E_\Delta K E_\Delta \tag{1.9}$$

where E_Δ *is the spectral projection for the interval* Δ *associated to* H. *Then*

(i) *Any point spectrum of* H *in* Δ *has finite multiplicity.*

(ii) *There is no accumulation point of point spectrum of* H *in* Δ.

(iii) *There is no singular continuous spectrum of* H *in* Δ.

Among other things, we want to extend Mourre's result in what appears to be a partly technical way but what is in fact a particularly significant way for applications:

THEOREM 1.2. *Let* H, H_0 *be two self-adjoint operators which are bounded from below on a Hilbert space* $\mathcal{H}$. *Suppose that* $D(H) = D(H_0)$ *so that the scale of spaces* $\mathcal{H}_i$ *is the same for* H *and* H_0. *Suppose that hypotheses* (a), (b) *of Theorem* 1.1 *hold and that*

(c′) *The quadratic form* $i[H_0, A]$ *defined in* $D(A) \cap \mathcal{H}_{+2}$ *is bounded below and extends to a bounded operator*, B_0, *from* $\mathcal{H}_{+2}$ *to* $\mathcal{H}$.

(c″) *The quadratic form* $i[H, A]$ *defined on* $D(A) \cap \mathcal{H}_{+2}$ *extends to a bounded operator*, B, *from* $\mathcal{H}_{+2}$ *to* $\mathcal{H}_{-1}$.

(a′) *There is a common core* C *for* A *and* H_0 *so that* A *maps* C *into* $\mathcal{H}_{+1}$.

(d′) *The quadratic form defined on* C *by* $[B, A]$ *extends to a bounded operator from* $\mathcal{H}_{+2}$ *to* $\mathcal{H}_{-2}$.

Moreover suppose that (e) *holds for* $E_\Delta(H)$, B. *Then the conclusions* (i)–(iii) *of Theorem* 1.1 *remain true.*

In our applications, H and H_0 are N-body Schrödinger operators and A is the generator of dilations, i.e.

$$\left(e^{i\alpha A} f\right)(x_i) = e^{\alpha(N-1)\nu/2} f\left(e^{\alpha} x_i\right) \tag{1.10}$$

so that

$$iA = \left(\sum_{i=1}^{N-1} x_i \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} \cdot x_i\right). \tag{1.10′}$$

In sections 6 and 7, we will prove Theorem 1.2 in this special case. Where we use the special properties of this particular A and H_0 (which allows for some simplifications), we will explain how to use results in [24] to handle the general abstract case.

We can see the distinction between Theorems 1.1 and 1.2 by considering the two body case. Then $[iA, V] = (x \cdot \nabla)V$ (distributional gradient) and a necessary condition (sufficient if $\nu \leq 3$) for a multiplication operator to be bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$ is that the corresponding function be uniformly locally square integrable. Thus, Theorem 1.1 requires that ∇V at least be locally L^2. However, one can write

$$[iA, V] = \left[\frac{\partial}{\partial x}, xV\right] - \nu V.$$

Since $\partial/\partial x$ is bounded from $\mathcal{H}_{-1}$ to $\mathcal{H}_{-2}$ and from $\mathcal{H}_{+2}$ to $\mathcal{H}_{+1}$, the latter will be bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-1}$ if xV is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$ (it is then automatically bounded from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$). Thus:

PROPOSITION 1.3. *Let V be a function on R^ν and let $\mathcal{H}_i$ be the* (*Sobolev*) *scale associated to $-\Delta$. Suppose that $V = V^{(1)} + V^{(2)}$ with $x \cdot (\nabla V^{(1)})$ bounded* (*resp. compact*) *from $\mathcal{H}_{+2}$ to $\mathcal{H}$, $xV^{(2)}$ bounded* (*resp. compact*) *from $\mathcal{H}_{+2}$ to $\mathcal{H}$ and $V^{(2)}$ bounded* (*resp. compact*) *from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-1}$. Then $[iA, V]$ is bounded* (*resp. compact*) *from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-1}$.*

We can now describe the hypotheses we will place on our potentials:

Definition. We will say that a potential V obeys *Hypothesis* M if and only if $V = V^{(1)} + V^{(2)} + V^{(3)}$ where:

(i) Each $V^{(i)}$ is compact from $\mathcal{H}_{+2}$ to $\mathcal{H}$;
(ii) $xV^{(1)}$ is compact from $\mathcal{H}_{+2}$ to $\mathcal{H}$;
(iii) $x^2V^{(1)}$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$;
(iv) $x \nabla V^{(2)}$ is compact from $\mathcal{H}_{+2}$ to $\mathcal{H}$;
(v) $x^2 \nabla V^{(2)}$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$;
(vi) $x \nabla V^{(3)}$ is compact from $\mathcal{H}_{+2}$ to $\mathcal{H}$;
(vii) $x^2 \nabla \nabla V^{(3)}$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}$.

Remarks. 1. (i) and (iii) imply (ii).

2. We emphasize that this is a compactness condition on $L^2(R^\nu)$. If we think of $V_\gamma^{(i)}$ as a function on $L^2(R^{\nu(N-1)})$ with $N \geq 3$, we will no longer have compactness.

3. In only one place do we require the double commutator $[A, [A, V]]$. If that could be removed, then one could set $V^{(3)} = 0$ and dispense with all conditions except (i), (ii), (iv). Of course these hypotheses are chosen precisely to have:

PROPOSITION 1.4. *If V obeys hypothesis M, then as operators on $L^2(R^\nu)$, $i[V, A]$ is compact from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-1}$ and $i[[V, A], iA]$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$.*

The proof is easy.

We can now state our main theorem:

THEOREM 1.5. *Let H be an N-body Hamiltonian with potentials, V_γ, obeying hypothesis M. Then:*

(i) *The set of thresholds of H (defined below) is a closed countable set.*

(ii) *Any non-threshold eigenvalue is of finite multiplicity, and these eigenvalues can only accumulate at thresholds.*

(iii) *H has no singular continuous spectrum.*

Conclusion (i) of this theorem will hold inductively given conclusion (ii) for subsystems and the definition of threshold. Given Theorem 1.2, this focuses attention on basic estimate (1.9) which we will call a "Mourre-type" estimate. Theorem 1.5 will follow from Theorem 1.2 if we prove that for any real number λ_0 which is not a threshold, (1.9) holds for Δ, some open interval containing λ_0. We will prove (1.9) in Section 4, under the hypothesis that every subsystem has finitely many point eigenvalues, each of finite multiplicity. We prove (1.9) in the general case, which is considerably more complicated, in Section 5. Our proof will require two special technical devices we develop here: the notion of a-compact operators in Section 2 and a systematic expansion in subsystems in Section 3. We base this expansion on geometric ideas from [39]; for the case discussed in Section 4, one can choose a combinatorial expansion described in an appendix. In addition to these devices, we exploit some of the ideas Mourre used in his study of three-body systems and we use the virial theorem which we prove in Section 6 (the virial theorem is also crucial in the proof of Theorem 1.2).

Next, we would like to describe the combinatorial notation we will need associated with breakups of the N particle into clusters. We will use the symbols $a, b, c, \ldots$ for partitions of $\{1, \ldots, N\}$, i.e. $a = \{A_1, \ldots, A_k\}$ where the A's are disjoint non-empty subsets of $\{1, \ldots, N\}$ whose union is all of $\{1, \ldots, N\}$. When we write a_k we intend to mean a general partition with k elements. Thus a_1 stands for the unique partition of one cluster and a_N the unique partition with N, one element, clusters.

The partitions have a natural lattice structure; we write $a \subset b$ if each cluster $A_i \in a$ is a subset of some cluster $B_j \in b$. Thus a_N is a minimal element and a_1 a maximal element. The set of all partitions is a lattice with this order: the glb $a \cap b$ is the family of non-empty intersections $A_i \cap B_j$ with $A_i \in a$, $B_j \in b$. The lub $a \cup b$ can be described as follows: draw lines between each pair in $\{1, \ldots, N\}$ in a common cluster of a and then between each pair in a common cluster of b. The connected components of $\{1, \ldots, N\}$ after this is done are the clusters in

$a \cup b$. (We note that in the combinatorial literature, what we call $a \subset b$ is often written $a \supset b$ thereby interchanging our $\cup$ and $\cap$; in [36], [30], one of us has used the symbol $a \triangleright b$ where we now use $a \subset b$).

The symbol γ will be used as already indicated for a pair $\{i, j\}$. We will also use it for the partition with $N-1$ clusters with γ as one cluster and $N-2$ singlets. A symbol like $\gamma \equiv \{i, j\} \subset a$ therefore means that i and j are in the same cluster of a.

Later we will have rather involved combinatorial sums. A symbol like

$$\sum_{a_1 \supset a_2 \supset \cdots \supset \boxed{a_k}}$$

means that a_k is a fixed "variable" and one is to sum over all a_1 (there is only one a_1!), $a_2, \ldots, a_{k-1}$ obeying $a_1 \supset a_2 \supset \cdots \supset a_{k-1} \supset a_k$, i.e. variables within boxes are fixed; others are summed.

We define, as usual, for each a:

$$V(a) = \sum_{\gamma \subset \boxed{a}} V_\gamma, \tag{1.11}$$

$$I(a) = V - V(a) = \sum_{\gamma \not\subset \boxed{a}} V_\gamma, \tag{1.12}$$

$$H(a) = H - I(a) = H_0 + V(a); \tag{1.13}$$

i.e. $I(a)$ is the potential between clusters of a.

There is a natural tensor product decomposition

$$\mathcal{H} = \mathcal{H}^a \otimes \mathcal{H}_a \tag{1.14}$$

associated to each a (we should really write $\otimes_a$ but don't). $\mathcal{H}^a$, the *a-internal* space, is a square integrable function of the r_γ with $\gamma \subset a$ and $\mathcal{H}_a$, the *a-external* space consists of functions of $R_i - R_j$ where for each $A_i \in a$,

$$R_i = \sum_{k \in A_i} m_k r_k \Big/ \sum_{k \in A_i} m_k$$

is the center of mass of the cluster A_i. The reason for taking $\mathcal{H}_a$ in this way is that with this choice,

$$H_0 = H_0^a \otimes I + I \otimes T_a \tag{1.15}$$

for suitable operators H_0^a on $\mathcal{H}^a$ and T_a on $\mathcal{H}_a$. Since

$$V(a) = V^a \otimes I, \tag{1.16}$$

we see that

$$H(a) = H^a \otimes I + I \otimes T_a. \tag{1.17}$$

Occasionally, we will use the symbol

$$T(a) = I \otimes T_a.$$

The celebrated HVZ theorem ([44], [40], [13], [7], [32]) asserts that

$$\sigma_{\text{ess}}(H) = \bigcup_{a \neq a_1} \sigma(H(a)). \tag{1.18}$$

Eigenvalues of some H^a with $a \neq a_1$ are called *thresholds*. Notice that the thresholds for H^a are eigenvalues of some H^b with $b \underset{\neq}{\subset} a$ so that these thresholds are also thresholds for H. As a result, conclusion (i) of Theorem 1.5 holds inductively if one proves (ii) for the H^a. Therefore, by induction, in proving Theorem 1.5, we can *suppose* that (i) is true rather than prove it!

If $b \underset{\neq}{\supset} a$, then any $\gamma \subset a$ is a $\gamma \subset b$, so that we can write

$$\mathcal{H}^b = \mathcal{H}^a \otimes \mathcal{H}^b_a \tag{1.19}$$

where $\mathcal{H}^b_a$ consists of the functions of $R_i - R_j$ for those clusters $A_i, A_j \in a$, which are subsets of the same cluster in b. Consider the Hamiltonian $H(a)$, i.e. a Hamiltonian with interactions between distinct $A_i \in a$ removed. Then, as above

$$H(a) = H(a)^b \otimes 1 + 1 \otimes T_b \tag{1.20}$$

under the decomposition $\mathcal{H}^b \otimes \mathcal{H}_b$. One fact about (1.19) we will need is that the operator $H(a)^b$ on $\mathcal{H}^b$, defined by (1.20), has the form

$$H(a)^b = H^a \otimes 1 + 1 \otimes T^a_b$$

under the decomposition (1.19). Since T^a_b has purely absolutely continuous spectrum if $a \neq b$, we conclude that spectral measures of $H(a)^b$ are convolutions of spectral measures of H^a and absolutely continuous measures. Hence:

PROPOSITION 1.6. *Let $a \underset{\neq}{\subset} b$. The operator $H(a)^b$ on $\mathcal{H}^b$, given by* (1.20), *is purely absolutely continuous.*

There is one last piece of notation we wish to introduce now. For each $a \neq a_1$, P^a is the projection on $\mathcal{H}^a$ onto the point spectrum of H^a. We let

$$\overline{P^a} = 1 - P^a \tag{1.21}$$

$$P(a) = P^a \otimes I; \ \overline{P(a)} \doteq \overline{P^a} \otimes I \tag{1.22}$$

Thus, since $\mathcal{H}^{a_N} = \mathbf{C}$, we have that $P(a_N) \equiv 1$ and

$$\overline{P(a_N)} = 0. \tag{1.22a}$$

For the special case $a = a_1$, we make a special convention

$$P(a_1) = 0. \tag{1.22b}$$

With this notation, we can describe the asymptotic completeness problem referred to above; for each $a \neq a_1$, one can prove the existence of the limits

$$\Omega_a^{\pm} = s - \lim_{t \to \mp\infty} e^{itH} e^{-itH(a)} P(a)$$

under rather general conditions on the potentials [20] and moreover [20], for $a \neq b$, $\operatorname{Ran} \Omega_a^+ \perp \operatorname{Ran} \Omega_b^+$. One defines

$$\mathcal{H}_{\text{in}} = \bigoplus_{a \neq a_1} \operatorname{Ran} \Omega_a^+$$

and similarly with (in, $+$) replaced by (out, $-$). $\mathcal{H}_{\text{in}}$ describes those (scattering) states which in the past break into bound fragments $A_1, \ldots, A_k$ moving asymptotically freely. *Asymptotic completeness* is the assertion

$$\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}} = \mathcal{H}_{\text{ac}}(H) \tag{1.23}$$

with $\mathcal{H}_{\text{ac}}(H)$ the absolutely continuous subspace for H. The stronger assertion with $\mathcal{H}_{\text{ac}}$ in (1.23) replaced by the orthogonal complement of the span of the eigenvectors of H is often called "asymptotic completeness". This stronger assertion is equivalent to a positive solution of problems (2) and (3) listed at the start of this section. We emphasize that in this paper, we only solve problems (1) and (2) and have nothing to report on problem (3). In the two-body case (with $V_\gamma^{(3)} = 0$) and slightly strengthened falloff hypotheses, our estimates solve (3) as is well-known (e.g. [32]) but additional estimates will probably be needed before one can solve (3) in the general N-body case.

The structure of "particles" is not really essential to our work. For example, Morawetz [23] has considered the "Union Jack" problem in two dimensions: let $V(x, y)$ be 0 (resp. 1) if $|x| < 1$ or $|y| < 1$ or $|x - y| < 1$ or $|x + y| < 1$ (otherwise) and consider $-\Delta + V$. While this is not a three-body problem, it is very similar in structure. Indeed, if V_α are functions on R^{μ_α} obeying hypothesis M and if T_α are a finite family of linear maps from R^ν to R^{μ_α}, one can analyze

$$-\Delta + \sum_\alpha V_\alpha(T_\alpha x).$$

The methods of this paper (with thresholds suitably defined) also prove the analog of Theorem 1.5 for such operators. The three-body case corresponds to the situation where any *two* $\operatorname{Ker} T_\alpha$ intersect in $\{0\}$, as happens in the Union Jack problem. In general, if there is a set of $k - 1$ $\operatorname{Ker} T_\alpha$'s whose intersection is not $\{0\}$, but any set of k $\operatorname{Ker} T_\alpha$'s intersect in $\{0\}$, the problem will have the structure of an N-body problem with $N = k + 1$.

Finally, we close this introduction with a *brief* discussion of some previous work on the problems solved here and the related problem of asymptotic completeness which remains open.

The solution of the main problems for the case $N = 2$, often with very strong hypotheses on V, was elucidated in the late 1950's by T. Kato and his students and by a number of Russian mathematicians, most notably Povzner and Birman. There have been twenty years of development and refinement culminating in the weighted L^2-space analysis of Kuroda [19] and Agmon [1]; see [30], [32] and their notes for references. (Very recently, Enss [8] has invented an intriguing and elegant new approach to the problems.) In Section 8 of this paper, we will obtain weighted L^2 estimates for N-body systems with potentials obeying hypothesis M, which are essentially as good as those obtained by Agmon and Kuroda in the two-body case.

The first general results for $N = 3$ were obtained by Faddeev in his celebrated book [9] which gave solutions of problems (2) and (3) for potentials which roughly had $r^{-2-\epsilon}$ falloff at infinity (there is a gap in his solution of problem (2) filled in later by Sigal [33] and Yafeev [43]). Faddeev also assumed a technical condition that only held for "almost all V_γ's": explicitly, that for each γ, the dimension of the negative spectral subspace for $H^\gamma + \alpha V^\gamma$ was independent of α for $|\alpha|$ small. (There has recently been work on removing this condition [22].) In our work (as in Mourre's [24] in the case $N = 3$), no assumptions of this type are needed for any subsystem.

In Faddeev's work, he used a great deal of information about the $N = 2$ problem to solve the $N = 3$ problem. In general, one needs to know a lot about subsystem Hamiltonians H^a, $a \neq a_1$, to analyze H using an extended Faddeev approach. Some results along these lines, i.e. analyzing H *assuming* various features of H^a, can be found in the work of Hepp [12] and Sigal [33].

We will not attempt to describe all the work on embedded eigenvalues (problem (1) for discrete eigenvalues is solved by the HVZ theorem) yielding partial solutions, but see [10], [42].

For general N, the solutions of problems (1)–(3) fall into three classes: (i) Weak coupling, (ii) Repulsive potentials, (iii) Analytic potentials.

For weak coupling all three problems have been solved: indeed, H and H_0 are unitarily equivalent under unitaries given by $s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$. Weak coupling results of this type exist only if $\nu \geq 3$. The potentials are required to have roughly $r^{-2-\epsilon}$ falloff, explicitly for some p, q with $p < \nu/2 < q$, all $V_\gamma \in L^p \cap L^q$ and all $\|V_\gamma\|_p + \|V_\gamma\|_q$ sufficiently small (how small depends on ν, p, q, N). There are a number of results of this sort for $N = 2$ of which we mention Schwartz [32], Prosser [27] and Kato [17]; Hunziker [14] obtained the

$N = 3$ case and Iorio-O'Carroll [15] the general N case. In this last paper, a major role is played by ideas of Kato [17].

Solutions of Problems (1) and (2) (and, with extra hypotheses, also of problem (3)) were obtained by Lavine [20], [21] under hypotheses that *each* V_γ is repulsive, i.e. $i[A, V_\gamma] \geq 0$. Again ideas of Kato [17], [18] played an important role.

In both the weak coupling and repulsive potential cases, H has only one channel, i.e. no H^a with $a \neq a_N$ has eigenvalues. The only previous results about (1)–(3) from first principles for many channel systems are for "dilation analytic" potentials introduced by Combes [5] and analyzed in the N-body case by Balslev-Combes [3] (see [37], [38] for further results) who solved problems (1) and (2). Solutions of problem (3) for this class of potentials with extra hypotheses ("generic couplings", $r^{-2-\epsilon}$ falloff) were found independently by Hagedorn [11] for $N = 3,4$ and by Sigal [34] for all N. Rather than be precise about the definition of "dilation analytic" we note that it requires a kind of analyticity of the potential away from $\vec{r} = 0$; e.g. if V is spherically symmetric and dilation analytic then $V(\vec{r}) = f(|\vec{r}|)$ where f is analytic in a sector $|\arg z| < \delta$ (> 0) and $\lim_{|z| \to \infty, |\arg z| < \delta} |f(z)| = 0$. Coulomb and Yukawa potentials are dilation analytic; C_0^∞ functions are not.

Our summary would not be complete without mentioning Mourre's work [24] which we have already explained was our major motivation. Mourre only handled the case $N = 2,3$. In the latter case he solved problems (1) and (2) allowing rather long range potentials although he did require smoothness (e.g. $V_\gamma(\vec{r}) = [\ln(2 + |r|^2]^{-2}$ is allowed!) and no hypotheses on generic coupling were needed. We should also mention that Mourre relies in part on his own earlier work and on a train of ideas involving positive commutators including the work of Putnam [28], Kato [18] and especially Lavine [20], [21] on this subject.

It is a pleasure to thank E. Mourre for sending us his paper, for encouraging this work and for many discussions.

2. *a*-compact operators

In the kind of analysis we will make, and, in particular, in getting rid of the second term on the right of (1.9), a useful role is played by the fact that if K is compact, H is any operator and λ_0 is not an eigenvalue of H, then

$$\lim_{\epsilon \downarrow 0} \| K E_{(\lambda_0 - \epsilon, \lambda_0 + \epsilon)}(H) \| = 0. \tag{2.1}$$

This is easy because $s - \lim E_{(\lambda_0 - \epsilon, \lambda_0 + \epsilon)}(H) = 0$.

Here, we will deal with a slightly more subtle but related idea. Let V_γ be a two-body potential and let H_1 be an N-body free energy operator (with center of mass motion removed). We claim that

$$\lim_{\varepsilon \downarrow 0} \| V_\gamma E_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(H_1) \| = 0. \tag{2.2}$$

If $N = 2$, (2.2) follows from (2.1) if we note that $V_\gamma(H_1 + 1)^{-1}$ is compact when $N = 2$. To prove (2.2), one notes that $H_1 = H_1^\gamma \otimes I + I \otimes T_\gamma$ in the usual way described in Section 1. Then by "diagonalizing" T_γ, one can view $V_\gamma E_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(H_1)$ as a "fibered operator", whose fibers are $V_\gamma E_{(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)}(H_1^\gamma + t_\gamma(p_\gamma))$ and then (2.2) follows from the fact that there is a uniformity in λ_0 in (2.1); we describe the details below.

In this section, we will abstract and formalize these ideas; throughout, the reader should think of (2.2) as a motivating example. We will use freely the notion of "constant fiber direct integral" and "decomposable operator", described, e.g. in Section XIII.16 of [32].

Fix a partition a and let $U_a(t)$ denote the family of translations of the clusters in a (with the center of mass motion removed); i.e. if a has k-clusters $A_1, \ldots, A_k$, then t runs through sets of k vectors $t_j \in R^\nu$ with

$$\sum_{j=1}^{k} t_j \Bigl(\sum_{i \in A_j} m_i \Bigr) = 0$$

and

$$U_a(t) = \prod_{j=1}^{k} \prod_{i \in A_j} U^{(i)}(t_j)$$

where

$$\bigl(U^{(i)}(t) f\bigr)(x_j) = f(x_j + \delta_{ji} t)$$

translates particle i. We denote by p_a the infinitesimal generator of U_a. p_a should be viewed as the set of differences of momenta of the clusters in a.

The various momenta in p_a commute and so a simultaneous diagonalization is possible by the spectral theorem. By the Plancherel theorem, the spectral measure is just Lebesque measure on $\mathbf{R}^{(k-1)\nu}$ and the corresponding fibers in a direct integral decomposition are constant. Indeed, under the decomposition $\mathcal{H}^a \otimes \mathcal{H}_a$, each $U_a(t)$ is of the form $I \otimes \tilde{U}_a(t)$ and the $\tilde{U}_a(t)$ generates a maximal Abelian algebra on $\mathcal{H}_a$, so that the fibers of the above decomposition are naturally associated to $\mathcal{H}^a$. We therefore write

$$\mathcal{H} = \int^{\oplus} \mathcal{H}^a \, dp_a. \tag{2.3}$$

As is usual, we do not distinguish between the operators p_a and the variables p_a in the integral decomposition since the operator is multiplication by the variable. We note that the kinetic energy T_a is a function, t_a of the operators p_a, i.e.

$$T_a = t_a(p_a). \tag{2.4}$$

Definition. A bounded operator, A, on $\mathcal{H}$ is called *a-fibered* (written $A \in \mathcal{F}(a)$) if and only if A commutes with all the translations $U_a(t)$.

On account of the decomposition (2.3), any a-fibered operator, A, is decomposable (Thm. XIII.84 of [32]) with fibers $A(p_a)$ bounded operators on $\mathcal{H}^a$. Note that

$$\|A\|_{\mathcal{H}} = \sup_{p_a} \|A(p_a)\|_{\mathcal{H}^a}. \tag{2.5}$$

Definition. Let $A \in \mathcal{F}(a)$. We write $A \in \mathrm{Cont}(a)$, (resp. $A \in \mathrm{Cont}_\infty(a)$) (resp. $A \in C_0^\infty(a)$) if the fibers, $A(p_a)$, are *norm* continuous (resp., if also $\lim_{p_a \to \infty} \|A(p_a)\| = 0$) (resp. if $p_a \to A(p_a)$ is a norm-C^∞ operator-valued function of compact support).

PROPOSITION 2.1. *Let $a \subset b$ and let $A \in \mathcal{F}(a)$. Then $A \in \mathcal{F}(b)$. The same remains true if $\mathcal{F}(\cdot)$ is replaced by* $\mathrm{Cont}_\infty(\cdot)$ *or* $C_0^\infty(\cdot)$.

Proof. Every operator $U_b(t)$ is a $U_a(t)$ so $\mathcal{F}(b) \subset \mathcal{F}(a)$ is obvious. For the other results, we note that one can write $p_a = (p_a^b, p_b)$ corresponding to the decomposition $\mathcal{H}_a = \mathcal{H}_a^b \otimes \mathcal{H}_b$ (complementary to (1.19)). If $A_a(p_a)$ denote the a-fibers of A and $A(p_b)$, the b-fibers of A, then $A_b(p_b)$ as an operator on $\mathcal{H}^b = \mathcal{H}^a \otimes \mathcal{H}_a^b \equiv \int^\oplus \mathcal{H}^a \, dp_a^b$ is fibered with fibers

$$[A_b(p_b)](p_a^b) = A_a(p_a^b, p_b). \tag{2.6}$$

From this and (2.5), the results on $C_0^\infty(\cdot)$ and $\mathrm{Cont}_\infty(\cdot)$ follow. □

Remark. The proposition is false if $\mathcal{F}(\cdot)$ is replaced by $\mathrm{Cont}(\cdot)$, for (2.6) and (2.5) show that norm continuity of A_b in p_b requires continuity of A_a *uniformly* in p_a^b.

COROLLARY 2.2. *If $A \in \mathcal{F}(a)$ and $B \in \mathcal{F}(b)$, then $AB \in \mathcal{F}(a \cup b)$. The same remains true if $\mathcal{F}(\cdot)$ is replaced by* $\mathrm{Cont}_\infty(\cdot)$ *or* $C_0^\infty(\cdot)$.

Our key definition is

Definition. An operator A is called *a-compact* (written $A \in \mathrm{Com}(a)$) if and only if $A \in \mathrm{Cont}_\infty(a)$ and each fiber $A(p_a)$ is a compact operator on $\mathcal{H}^a$).

A related but distinct notion of a-connectivity has been introduced recently by Polyzou [26]; a-compact operators are a-connected in his sense (see Lemma A-4).

We note that if C is a compact operator on $\mathcal{H}^a$ and $\tilde{C} = C \otimes I$ according to $\mathcal{H} = \mathcal{H}^a \otimes \mathcal{H}_a$, then by our definition, $\tilde{C}$ is *not* a-compact (although $\tilde{C}(H_0 + 1)^{-1}$ will be a-compact). This seems a strange choice to make. Below we will explain our reason for it. Clearly, by (2.5):

Proposition 2.3. *Com*(a) *is norm closed.*

Proposition 2.4. *If* $A \in \mathrm{Com}(a)$ *and* $B \in \mathrm{Cont}(a)$, *the* $AB \in \mathrm{Com}(a)$.

Definition. An operator A is called a-finite (written $A \in \mathrm{Fin}(a)$) if and only if $A \in C_0^\infty(a)$ and for a *fixed* finite rank (orthogonal) projection P on $\mathcal{H}^a$,

$$(P \otimes I)A = A(P \otimes I) = A \tag{2.7}$$

(or, equivalently,

$$PA(p_a) = A(p_a)P = A(p_a) \tag{2.8}$$

for all p_a).

The following approximation result is very important in the development of the theory:

Proposition 2.5. *Fin*(a) *is norm dense in* Com(a).

Proof. By a standard (functional analytic) approximation argument $C_0^\infty(a) \cap \mathrm{Com}(a)$ is dense in $\mathrm{Com}(a)$. Given $A \in C_0^\infty(a) \cap \mathrm{Com}(a)$ and ε, find by continuity, points $p_a^{(1)}, \ldots, p_a^{(n)} \in \mathbf{R}^{\nu(k-1)}$ so that for any p_a

$$\max_{i=1,\ldots,n} \| A(p_a) - A(p_a^{(i)})\| \leq \frac{\varepsilon}{3}.$$

By compactness of each $A(p_a^{(i)})$ we can find a finite rank orthogonal projection P on $\mathcal{H}^a$ with

$$\| A(p_a^{(i)}) - PA(p_a^{(i)})P\| \leq \frac{\varepsilon}{3}$$

for all i. Then $(P \otimes I)A(P \otimes I) \in \mathrm{Fin}(a)$ and

$$\|A - (P \otimes I)A(P \otimes I)\| \leq \varepsilon. \qquad \square$$

Remark. We can now explain why we define a-compact in such a way that $\{\tilde{C} = C \otimes I\}$ are not a-compact. For a-compact operators to be approximable by operators obeying (2.7) for a fixed finite rank projection, we cannot require only that $A \in \mathrm{Cont}(a)$ with compact fibers. We need something like a requirement that $A(p_a)$ have a limit at infinity. But with only that requirement, Proposition

2.4 would not hold. We note that in the applications, operators $\tilde{C} = C \otimes I$ often enter and we will have to arrange for some extra $(H_0 + 1)^{-1}$ factors to make them a-compact. These are a nuisance to have to add but present no serious problem.

The following result is a version of an idea that has run through much of the N-body literature; the earliest version we know is in a paper of Combes [4].

THEOREM 2.6. *If* $A \in \mathrm{Com}(a)$ *and* $B \in \mathrm{Com}(b)$, *then* $AB \in \mathrm{Com}(a \cup b)$.

Proof. By Proposition 2.5, it suffices to prove the result when $A \in \mathrm{Fin}(a)$ and $B \in \mathrm{Fin}(b)$. That $AB \in \mathrm{Cont}_\infty(a \cup b)$ is then just Corollary 2.2. We must therefore only show that the fibers are compact. Writing $A = A(P \otimes_a I)$ and $B = (Q \otimes_b I)B$, with P (resp. Q) a finite rank operator on $\mathcal{H}^a$ (resp. $\mathcal{H}^b$) and using Proposition 2.3, we see that it suffices to show that $(P \otimes_a I)(Q \otimes_b I)$ have compact $a \cup b$-fibers.

Clearly, we can suppose that P and Q are both rank 1 and without loss of generality we can suppose that $\mathrm{Ran}\, Q$ lies in some convenient total set. We now proceed to pick that total set. We claim we can pick distinct $\gamma_1, \ldots, \gamma_l \subset b$ so that (a) each $\gamma_i \not\subset a$, (b) $a \cup b = a \cup \gamma_1 \cdots \cup \gamma_l$, (c) For any γ_i, $a \cup \gamma_1 \cdots \cup \gamma_{i-1} \cup \gamma_{i+1} \cup \cdots \gamma_l \neq a \cup b$ (i.e. for each cluster C_i in $a \cup b$, we write $C_i = A_{j_1} \cup \cdots \cup A_{j_m}$ (A_j distinct clusters of a) and then pick $m - 1$ pairs contained in b and in C_i so that joining those pairs with pairs in some A_j connects all of C_i). Let ζ be a set of coordinates internal to b so that $r_{\gamma_l}, \ldots, r_{\gamma_l}, \zeta$ is a complete set of coordinates for $\mathcal{H}^b$ with ζ orthogonal to r_{γ_i}. Suppose that $\mathrm{Ran}\, Q$ consists of multiples of a product function

$$f_1(r_{\gamma_1}) f_2(r_{\gamma_2}) \ldots f_l(r_{\gamma_l}) g(\zeta) \equiv f(r_\gamma) g(\zeta).$$

Then, we can write $Q = Q_f \tilde{Q}_g$ where $(Q_f h)(r_\gamma, \zeta) = \int f(r_\gamma)\overline{f(r'_\gamma)}\, h(r'_\gamma, \zeta)\, dr'_\gamma$ and $\tilde{Q}_g$ is similar. By an elementary change of variables, $(P \otimes_a I)(Q_f \otimes_b I)$ is seen to have Hilbert-Schmidt $a \cup b$-fibers. $\square$

Remark. Polyzou [26] proves an analog of this fact for his "a-connected" operators.

Example. Let $N = 3$, with $m_1 = \infty$, $m_2 = m_3$. Let $a = \{\{1,2\},3\}$, $b = \{1,\{2,3\}\}$. Use coordinates r_2, r_3. Then if Q is a projection onto a function f, we have that $Q \otimes_b I$ has an "integral kernel":

$$f(r_2 - r_3)\, \overline{f(r'_2 - r'_3)}\, \delta\big(\tfrac{1}{2}(r_2 + r_3) - \tfrac{1}{2}(r'_2 + r'_3)\big)$$

with δ a Dirac-delta function. If P is the projection onto a function j, then $P \otimes_a I$

has integral kernel

$$j(r_2)\overline{j(r_2')}\,\delta(r_3 - r_3')$$

and $(P \otimes_a I)(Q \otimes_b I)$ has kernel:

$$2j(r_2)\overline{j(r_2' + r_3' - r_3)}\,f(r_2' + r_3' - 2r_3)\overline{f(r_2' - r_3')},$$

which, by a change of variables, is easily seen to be L^2; but notice that this kernel is not separable, i.e. a finite sum $\Sigma h_\alpha(r_2, r_3)\overline{h_\alpha(r_2', r_3')}$. Thus, it is not true that a product of $A \in \mathrm{Fin}(a)$ and $B \in \mathrm{Fin}(b)$ lies in $\mathrm{Fin}(a \cup b)$.

The point of introducing the notion of a-compact operators is to obtain various extensions of (2.2):

THEOREM 2.7. *Let $a \neq a_1$. Suppose that $A \in \mathrm{Com}(a)$. Then as Δ runs through all intervals*

$$\lim_{|\Delta| \to 0} \|A\,\overline{P(a)}\,E_\Delta(H(a))\| = 0 \tag{2.9}$$

where $|\Delta| =$ Lebesque measure of Δ.

Remark. It is only because of our convention that $\overline{P(a_1)} = 1$ that we need to say $a \neq a_1$. If we replaced $\overline{P(a_1)}$ by the projection onto the absolutely continuous subspace for H $(= H(a_1))$, (2.9) would remain true.

Proof. By the simple approximation argument, Proposition 2.5, we may suppose that $A \in \mathrm{Fin}(a)$. Since $A = A(P \otimes I)$, it suffices that

$$\lim_{|\Delta| \to 0} \|(P \otimes I)\,\overline{P(a)}\,E_\Delta(H(a))\| = 0 \tag{2.10}$$

for any finite rank operator P on $\mathcal{H}^a$. Clearly, it suffices that P be rank 1, say the projection onto a vector $\psi \in \mathcal{H}^a$. But the operator in (2.10) is fibered under the decomposition $\mathcal{H} = \int^\oplus \mathcal{H}^a\, dp_a$ and its fibers are $P\overline{P^a}E_\Delta(H^a + t_a(p_a))$. Using (2.5) and letting $\phi = \overline{P^a}\psi$, we see that (2.10) is implied by:

$$\lim_{\alpha \downarrow 0} \sup\{\|E_\Delta(H^a)\phi\| : |\Delta| = \alpha\} = 0. \tag{2.11}$$

Let $d\mu$ be the spectral measure for ϕ associated to H^a. Since $\|E_\Delta(H^a)\phi\|^2 = \int_\Delta d\mu(y)$, we see that (2.11) is equivalent to

$$\lim_{|\Delta| \to 0} \int_\Delta d\mu(y) = 0. \tag{2.12}$$

But, since Δ is an interval, (2.12) is equivalent to the uniform continuity of

$$F(y) = \int_{-\infty}^{y} d\mu(x).$$

Since F is continuous (since $\phi \in \operatorname{Ran} \overline{P^a}$ and P^a contains all eigenvectors) on the extended reals $[-\infty, \infty]$, this uniform continuity is obvious. $\square$

THEOREM 2.8. *Let $a \subsetneq b$ and let $A \in \operatorname{Com}(b)$. Then*

$$\lim_{|\Delta| \to 0} \| A E_\Delta(H(a)) \| = 0. \tag{2.13}$$

Proof. Think of $H(a)$ as an N-body Hamiltonian $\tilde{H}$. By Proposition 1.6, the operator $P^b(\tilde{H})$ associated to $\tilde{H}$ is 0 so $\overline{P^b(\tilde{H})} = 1$. Thus (2.9) for $\tilde{H}$ is (2.13). $\square$

Remarks. 1. If $b = a_1$, then, as we have already noted, (2.9) only fails because of the definition we gave $P(a_1)$. Thus (2.13) remains true even when $b = a_1$.

2. (2.2) is implied by (2.13) if we note that

$$V_\gamma E(H_0) = \left[V_\gamma (H_0 + 1)^{-1} E(H_0) \right] \left[(H_0 + 1) E(H_0) \right].$$

THEOREM 2.9. *Let $A \in \operatorname{Com}(a)$ and let C_n^a be a sequence of (bounded) self-adjoint operators on $\mathcal{H}^a$ going strongly to zero. Let*

$$C_n(a) = C_n^a \otimes 1.$$

Then

$$\lim_{n \to \infty} \| A C_n(a) \| = 0. \tag{2.14}$$

Proof. By a limiting argument, we need only prove (2.14) for $A = P \otimes I$ with P rank one. But then (2.14) is equivalent to

$$\lim_{n \to \infty} \| P C_n^a \|_{\mathcal{H}^a} = 0,$$

which is a direct consequence of the strong convergence of $(C_n^a)^* = C_n^a$. $\square$

3. Expanding in subsystems: The geometric method

For fixed partition a, the Hamiltonians $H(b)$ with $a \supset b$ have the potentials V_γ with $\gamma \subset a$ but $\gamma \not\subset b$ dropped, and thus they can be viewed as describing the interactions of subclusterings of b. An important element in our analysis will be the possibility of expanding functions, $f(H(a))$, into combinations of the $f(H(b))$ with $b \subsetneq a$. Of course, one cannot expect to write $f(H(a))$ in terms of the $f(H(b))$ with no error but one can hope for the error to be small; the natural notion might be compact errors. Since, for all $b \subset a$, $f(H(b))$ is a-fibered (see Prop. 2.1), it is too much to hope for a compact error in general; the natural thing to look for is an a-compact error. Thus, our goal is an expansion of the form:

$$f(H(a)) = K^f(a) + \sum_{b \subsetneq \boxed{a}} c(b,a) f(H(b)) \tag{3.1}$$

where $K_f(a)$ is a-compact. (The symbol $\sum_{b \subsetneq \boxed{a}}$ is defined in §1.)

The existence of an expansion like (3.1), even with operator-valued c's is very significant; for example, it implies the hard part of the HVZ theorem, i.e.

$$\sigma_{\mathrm{ess}}(H) \subset \bigcup_{a \neq a_1} \sigma(H(a)). \tag{3.2}$$

For (following [39]) note that if f has support disjoint from $\cup_{a \neq a_1} \sigma(H(a))$, then (3.1) and compactness of $K_f(a_1)$ show that $f(H(a_1))$ is compact and thus (3.2) follows.

The simplest version of (3.1) has the $c(b,a)$ as real numbers. As we will show in the appendix, this requirement determines the expansion uniquely. In that appendix, we will also construct such an expansion. The proof of HVZ that results can be viewed as a variant of the Weinberg-van Winter equation proof of Hunziker and van Winter. The big disadvantage of this "combinatorial" approach is that *all* b with $b \subsetneq a$ enter in the sum in (3.1). For our purpose, this is a severe defect: we could use the method of the appendix to provide an alternative to the arguments in Section 4 but we have not succeeded in doing what we do in Section 5 without the geometric expansion of this section.

Our goal in this section will be to construct an expansion of the form (3.1) with the property that the only b's which enter in the sum are ones that have only one additional cluster. The price one pays for this desirable feature is that the c's are now (multiplication) operators. We follow the construction of Simon [39] who developed (3.1) in the case $a = a_1$ (so $K_f(a)$ is compact). The name "geometric method" comes from the fact that the basic input is the observation that $H(a)$ and $H(b)$ look alike in the region where all r_γ with $\gamma \subset a$, $\gamma \not\subset b$ or $\gamma \subset b$, $\gamma \not\subset a$ are large and one exploits regions in r-space.

LEMMA 3.1. *For all pairs $a_{k+1} \subset a_k$ where a_k (resp. a_{k+1}) has k (resp. $k+1$) clusters, there exists a function $j(a_{k+1}, a_k)$ so that the following hold:*

(i) $0 \leq j \leq 1$; $j \in C^\infty(R^{\nu(N-1)})$;

(ii) $j(a_{k+1}, a_k)$ *is a_k-fibered;*

(iii) $(\nabla j)(a_{k+1}, a_k)(H_0 + 1)^{-1/2}$ *is a_k-compact;*

(iv) *If $\gamma \subset a_k$ but $\gamma \not\subset a_{k+1}$ then $(H_0 + 1)^{-1} V_\gamma j(a_{k+1}, a_k)(H_0 + 1)^{-1}$ is a_k-compact if V_γ is compact from the two body $\mathcal{H}_{+2}$ to the two body $\mathcal{H}$.*

(v) *We have that*

$$\sum_{a_{k+1} \subset \boxed{a_k}} j(a_{k+1}, a_k) = 1 \tag{3.3}$$

for each a_k.

Proof. Pick a coordinate system for $R^{(N-1)\nu}$ consisting of some r_γ's with $\gamma \subset a_k$ (call them r_α) and some ζ which are differences of centers of mass of clusters in a_k. We will take the j's to be only functions of the r_α so that (ii) holds. We will initially define the j's on the "sphere" where $\Sigma_\alpha r_\alpha^2 = 1$. We will extend to r's with $\Sigma_\alpha r_\alpha^2 > 1$ by requiring j to be homogeneous of degree zero there and, in the interior of the sphere, we will extend in an arbitrary way so that (i) and (v) hold. By construction, ∇j will be homogeneous of degree -1 in the r_α so $|\nabla j| \leq C(1 + \Sigma_\alpha r_\alpha^2)^{-1/2}$ from which (iii) follows. Thus, it only remains to prove that the j's can be chosen on the sphere in such a way that (iv) holds. We will make a choice so that $\gamma \subset a_k$ and $\gamma \not\subset a_{k+1}$ and $j(a_{k+1}, a_k)(r) \neq 0$ (with $\Sigma r_\alpha^2 = 1$) implies that $r_\gamma \geq d_N > 0$ where d_N is a constant depending only on N.

Given j's with this property, verify (iv) as follows: Since V_γ is h_0-compact, we can find for each ε, an R so that

$$\| F(|r_\gamma| \geq R) V_\gamma (H_0 + 1)^{-1} \| \leq \varepsilon$$

where $F(A)$ is the characteristic function of the set A. But by the above property of the j's, $F(|r_\gamma| < R) j(a_{k+1}, a_k)$ is supported in the set of r's with $(\Sigma_\alpha r_\alpha^2)^{1/2} \leq \min(1, R d_N^{-1})$. From this we obtain the required compactness condition (iv).

Finally, we note that the existence of j's on the unit sphere with the requisite properties follows from a geometric argument identical to that presented in [39] (see also [32], Section XIII.4). □

Remark. The above choice has the property that $(\Delta j)(H_0 + 1)^{-1}$ is a_k-compact so that

$$\begin{aligned}(H_0 + 1)^{-1}[H_0, j] = &-2(\nabla)(H_0 + 1)^{-1/2}\left[(H_0 + 1)^{-1/2} \nabla j\right] \\ &-2(H_0 + 1)^{-1} \Delta j\end{aligned}$$

is a_k-compact.

LEMMA 3.2. *Let $f \in C_0^\infty(R)$. Let the j obey the hypotheses of Lemma 3.1 and let each V_γ be compact from the two body $\mathcal{H}_{+2}$ to the two body $\mathcal{H}$. Then*

$$j(a_{k+1}, a_k)\left[f(H(a_k)) - f(H(a_{k+1}))\right]$$

is a_k-compact.

Proof (following [39]). Suppose we prove this if $f(x) = (x - z)^{-1}$ with $\operatorname{Re} z$ large in absolute value and negative. Then, by writing out derivatives as Cauchy integrals we obtain the a_k-compactness for $f(x) = (x - z)^{-k}$. By use of the Stone-Weierstrass theorem any $f \in C_0^\infty$ is a uniform limit on $[\alpha, \infty)$ of polynomials in $(x - \alpha + 1)^{-1}$. Therefore, we are reduced to the case where $f(x) = (x - z)^{-1}$. But, then:

$$j\left[(H(a_k) - z)^{-1} - (H(a_{k+1}) - z)^{-1}\right] = S_1 + S_2$$

where

$$S_1 = (H(a_k) - z)^{-1} j \left(\sum_{\substack{\gamma \subset a_k \\ \gamma \not\subset a_{k+1}}} V_\gamma \right) (H(a_{k+1}) - z)^{-1}$$

and

$$S_2 = (H(a_k) - z)^{-1}[H(a_k), j]\left\{(H(a_k) - z)^{-1} - (H(a_{k+1}) - z)^{-1}\right\}.$$

Since

$$[H(a_k), j] = [H_0, j]$$

and $\nabla j (H_0 + 1)^{-1}$ is a_k-compact (see (iii) of Lemma 3.1), S_2 is a_k-compact. By (iv) of Lemma 3.1, S_1 is a_k-compact. □

Theorem 3.3. *Let $f \in C_0^\infty(R)$ and let the j's obey Lemma* 3.1. *Then*

$$f(H(a_k)) = K_f(a_k) + \sum_{a_{k+1} \subset \boxed{a_k}} j(a_{k+1}, a_k) f(H(a_{k+1})) \tag{3.4}$$

where:

$$K_f(a_k) = \sum_{a_{k+1} \subset \boxed{a_k}} j(a_{k+1}, a_k)\left[f(H(a_k)) - f(H(a_{k+1}))\right] \tag{3.5}$$

is a_k-compact.

Proof. The validity of the expansion follows from (3.3); the a_k-compactness from Lemma 3.2. □

The final fact we will need about the expansion is that K_f is actually a_k-compact from $\mathcal{H}_{+2}$ to $\mathcal{H}$. As a preliminary to this, we note that:

Lemma 3.4. *Let $b \subseteq a_k$. Then for all $f \in C_0^\infty$*:

$$[j(a_{k+1}, a_k), f(H(b))]$$

is a_k-compact.

Proof. As in the proof of Lemma 3.2, we need only consider the case $f(x) = (x - z)^{-1}$. Thus, since $(H(b) - z)^{-1}$ is a_k-fibered with continuous fibers, we only need that

$$(H_0 + 1)^{-1}[H_0, j](H_0 + 1)^{-1}$$

is a_k-compact. This is noted already in the proof of Lemma 3.2. □

THEOREM 3.5. *The operator*, $K_f(a_k)$, *of* (3.5) *obeys*: $K_f(a_k)[H_0 + 1]$ *is* a_k*-compact.*

Proof. Write $f = gh$ with $h(x) = (x - z)^{-1}$ and write

$$j(a_{k+1}, a_k)\big[f(H(a_k)) - f(H(a_{k+1}))\big] = U_1 + U_2 + U_3,$$

with

$$\begin{aligned} U_1 &= j\big[g(H(a_k)) - g(H(a_{k+1}))\big]h(H(a_k)),\\ U_2 &= \big[j, g(H(a_{k+1}))\big]\big[h(H(a_k)) - h(H(a_{k+1}))\big],\\ U_3 &= g(H(a_{k+1}))j\big[h(H(a_k)) - h(H(a_{k+1}))\big]. \end{aligned}$$

Now $U_1[H_0 + 1]$ is a_k-compact, by Lemma 3.2 and $U_2[H_0 + 1]$ is a_k-compact by Lemma 3.4. To see that $U_3[H_0 + 1]$ is a_k-compact, we follow the proof of Lemma 3.2, using the remark following the proof of Lemma 3.1. □

4. Mourre-type estimates for N-body systems: finitely many channels

In this section, we will prove the key estimate (1.9) under the extra hypotheses that each P^a is of finite rank (since $P^{a_1} = 0$ *by convention*, this is a hypothesis on subsystems only). Given Theorem 1.2 (which we prove in §6, 7) and Proposition 1.4, this proves Theorem 1.5 under this extra hypothesis. In the next section, we remove the finite rank hypothesis.

Alas, it will be useful to have two more pieces of notation. For functions f, g we write

$$f \subset g$$

to indicate that $0 \le f \le g \le 1$ and $g(x) \equiv 1$ on the set where $f(x) \neq 0$ so $fg = f$.

Below, we will have sequences of functions

$$f_1 \subset f_2 \subset \cdots \subset f_N$$

with f_1 identically 1 near some fixed point λ_0 and we will have operators O_1 and O_2 depending on $f_1, \ldots, f_n$. We write

$$O_1 \doteqdot O_2$$

to mean that for any ε, one can find $f_1, \dots, f_N$ (with f_1 identically one near λ_0) and a compact operator K so that

$$\|O_1 - O_2 - K\| \leq \varepsilon.$$

$\doteq$ is an equivalence relation. We use the symbol

$$O_1 \underset{*}{\doteq} O_2$$

to mean that

$$O_1(H_0 + 1) \doteq O_2(H_0 + 1).$$

The symbol $O_1 \underset{\cdot}{\geq} O_2$ means that given ε, one can choose f's, K compact and Q with $\|Q\| < \varepsilon$ so that $O_1 + K + Q \geq O_2$.

The following will be needed in both this section and the next:

PROPOSITION 4.1. *Let λ_0 be fixed and not a threshold of H and let $\alpha' < 2\,\mathrm{dist}(\lambda_0,$ thresholds). If*

$$f_1(H)Bf_1(H) \underset{\cdot}{\geq} \alpha' f_1(H)^2 \tag{4.1}$$

then (1.9) *holds for any $\alpha < \alpha'$.*

Remark. The proof does not use the restriction on α'. We state it above, since it is only under that restriction that we will prove (4.1) (and only under that restriction that we believe (4.1) will hold). A careful look at our proof shows only the distance to lower lying thresholds matters.

Proof. By the meaning of $\underset{\cdot}{\geq}$, we can find an open interval Δ about λ_0 and f_1 identically one on Δ so that

$$f_1 B f_1 \geq \alpha' f_1^2 + K + Q$$

with $\|Q\| \leq \alpha' - \alpha$. Multiply on both sides by $E_\Delta(H)$ and use $f_1(H)E_\Delta(H) = E_\Delta(H)$ to see that

$$E_\Delta B E_\Delta \geq \alpha' E_\Delta + E_\Delta K E_\Delta + E_\Delta Q E_\Delta.$$

But since $\|Q\| \leq \alpha' - \alpha$, we have that

$$E_\Delta Q E_\Delta \leq (\alpha' - \alpha) E_\Delta,$$

so (1.9) holds. $\square$

Now consider $f_1 \subset \cdots \subset f_N$. Fix a j-cluster partition a_j and expand:

$$f_j(H(a_j)) = f_j(H(a_j))P(a_j) + f_j(H(a_j))\bar{P}(a_j)$$

$$= f_j(H(a_j))P(a_j) + f_j(H(a_j))\bar{P}(a_j)f_{j+1}(H(a_j)) \tag{4.2}$$

$$= f_j(H(a_j))P(a_j) + f_j(H(a_j))\bar{P}(a_j)K_{f_{j+1}}(a_j)$$

$$+ \sum_{a_{j+1} \subset \boxed{a_j}} f_j(H(a_j))\bar{P}(a_j)j(a_{j+1}, a_j)f_{j+1}(H(a_{j+1})). \tag{4.3}$$

In (4.2), we use $f_j f_{j+1} = f_j$ since $f_j \subset f_{j+1}$ and in (4.3), we use the expansion (3.4). Iterating this and using the fact that $\bar{P}(a_N) = 0$, we find that

PROPOSITION 4.2. *For any* $f_1 \subset f_2 \cdots \subset f_N$:

$$f_1(H(a_1)) = \sum_{i=1}^{N} \sum_{a_i} \Big[N(a_i)f_i(H(a_i))P(a_i) + N(a_i)f_i(H(a_i))\bar{P}(a_i)K_{f_{i+1}}(a_i) \Big] \tag{4.4}$$

where $N(a_1) = 1$ *and for* $i \geq 2$:

$$N(a_i) = \sum_{a_1 \supset a_2 \supset \cdots \supset \boxed{a_i}} f_1(H(a_1))\bar{P}(a_1)j(a_2, a_1) \times f_2(H(a_2)\bar{P}(a_2)j(a_3, a_2) \cdots j(a_i, a_{i-1}) \tag{4.5}$$

and $K_f(a)$ *is given by* (3.5)

An important point about (4.4) is that there are a priori bounds on N and K which come from combinatorial factors alone since each term in the defining sums (3.5) and (4.5) is bounded with norm at most 1 for N and at most 2 for K. Explicitly:

$$\|K_f(a_i)\| \leq 2 \sum_{l=1}^{i} (2^{n_l - 1} - 1) \tag{4.6}$$

if a_i has clusters with $n_1, \ldots, n_i$ particles, and

$$\|N(a_i)\| \leq 2^{-(i-1)} i!\,(i-1)!. \tag{4.7}$$

Now given any f_N, we can use the fact that $K_{f_N}(a_{N-1})$ is a_{N-1}-compact (Thm. 3.3) and Theorem 2.7 to choose f_{N-1} with such a small support that $f_{N-1}(H(a_{N-1}))\bar{P}(a_{N-1})K_{f_N}(a_{N-1})$ is as small as we like. Proceeding to choose $f_{N-1}, \ldots, f_2$ successively, we can arrange that all $K_{f_{i+1}}(a_i)$ terms with $i \geq 2$ are as small as we wish (using the a priori bounds on N). Because of the requirement

$a \neq a_1$ in Theorem 2.7 (coming from our convention that $\bar{P}(a_1) = 1$), we cannot assume that this term is small but it will be compact since "a_1-compact" is the same as compact. Thus, noting that $K_f(a_i)(H_0 + 1)$ is a_i-compact (Thm. 3.5), we have:

PROPOSITION 4.3. *For any fixed f_N,*

$$f_1(H(a_1)) \underset{*}{=} \sum_{i=1}^{N} \sum_{a_i} N(a_i) f_i(H(a_i)) P(a_i). \tag{4.8}$$

Since $B = i[H, A]$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$, (4.8) implies that

$$f_1(H(a_1))^2 \underset{*}{=} \sum_{i,j} \sum_{a_i, b_j} N(a_i) f_i(H(a_i)) P(a_i) P(b_j) f_j(H(b_j)) N(b_j)^*, \tag{4.9}$$

$$f_1(H(a_1)) B f_1(H(a_1)) \underset{*}{=} \sum_{i,j} \sum_{a_i, b_j} N(a_i) f_i(H(a_i)) \times P(a_i) B P(b_j) f_j(H(b_j)) N(b_j)^*. \tag{4.10}$$

Next note that $Q(a_j) \equiv (H(a_j) + i)^{-1} P(a_j)$ is a_j-compact and thus by Theorem 2.6, $Q(a_i)Q(b_j)^*$ is $a_i \cup b_j$-compact. Suppose that $a_i \neq a_i \cup b_j$. Then, by Theorem 2.8, we know that $E_\Delta(H(a_i))Q(a_i)Q(b_j)$ will have small norm so long as Δ is small enough. Moreover, since $Q(a_i)Q(b_j)$ is fixed and independent of the choice of $f_{N-1}, \ldots, f_1$, we can initially choose f_N with such a small support that $E_\Delta f_N = f_N$. Thus we are guaranteed that

$$\begin{aligned} &f_i(H(a_i)) P(a_i) P(b_j) f_j(H(b_j)) \\ &\quad \equiv (H(a_i) + i) f_i(H(a_i)) E_\Delta(H(a)) Q(a_i) Q(b_j) f_j(H(b_j))(H(b_j) + i) \end{aligned}$$

has any a priori desired degree of smallness for those a, b with $a_i \neq a_i \cup b_j$. Similarly, we can handle pairs with $b_j \neq a_i \cup b_j$. Since $a = a \cup b$ and $b = a \cup b$, imply $a = b$, we have proved

PROPOSITION 4.4.

$$f_1(H(a_1))^2 \underset{*}{=} \sum_{i=1}^{N} \sum_{a_i} N(a_i) f_i(H(a_i)) P(a_i) f_i(H(a_i)) N(a_i)^*,$$

and given any a priori requirement on smallness, the only requirement on f_N is that its support is sufficiently small.

Now let $W_\gamma \equiv i[V_\gamma, A]$ and write

$$B = 2H(a) + \sum_{\gamma \subset a} (W_\gamma - 2V_\gamma) + \sum_{\gamma \not\subset a} W_\gamma \equiv 2H(a) + \sum_\gamma C_\gamma,$$

and note that

$$(H_0 + i)^{-1} C_\gamma (H_0 + i)^{-1}$$

is γ-compact. Thus

$$(H(a) + i)^{-1} Q(a) B Q(b) (H(b) + i)^{-1}$$

is a sum of terms which are either $a \cup b$-compact or $a \cup b \cup \gamma$ compact for some $\gamma \subset a$. Therefore since $(x + i)^2 f_j(x)$ is bounded only depending on $\operatorname{supp} f_N$, we have, by mimicking the proof of Proposition 4.4:

PROPOSITION 4.5.

$$f_1(H(a_1)) B f_1(H(a_1)) \doteq \sum_{i=1}^{N} \sum_{a_i} N(a_i) f_i(H(a)) P(a_i) B P(a_i) f_i(H(a_i)) N(a_i)^*.$$

Each P^{a_i} is finite rank, so we can write

$$P^{a_i} = \sum_{j=1}^{n(a_i)} \rho_j^{a_i}, \tag{4.11}$$

where the ρ's are projections onto eigenspaces for distinct eigenvalues (we will change this convention slightly in the next section), $\lambda_j^{a_i}$. Let $\rho_j(a_i) = \rho_j^{a_i} \otimes 1$. Notice that by (4.11) and Proposition 4.4, we clearly have

PROPOSITION 4.6.

$$f_1(H(a_1))^2 \doteq \sum_{i=1}^{N} \sum_{a_i} \sum_{j=1}^{n(a_i)} N(a_i) f_i(H(a_i)) \rho_j(a_i) f_i(H(a_i)) N(a_i)^*.$$

Next suppose that we have picked $\operatorname{supp} f_N$ so small that

$$\lambda_j^{a_i} + x \neq \lambda_k^{a_i} + x \tag{4.12}$$

for all $x \in \operatorname{supp} f_N$, all a_i and all $j \neq k$. This is possible since $\inf_{a_i, j \neq k} |\lambda_j^{a_i} - \lambda_k^{a_i}| \equiv d > 0$ and we can choose the support of f_N contained in $\left(\lambda_0 - \frac{d}{2}, \lambda_0 + \frac{d}{2}\right)$. (4.12) assures us that

$$f_i\big(\lambda_j^{a_i} + T(a_i)\big) f_i\big(\lambda_k^{a_i} + T(a_i)\big) = 0 \tag{4.13}$$

which is relevant since

$$f_i(H(a_i)) \rho_j(a_i) = \rho_j(a_i) f_i\big(\lambda_j^{a_i} + T(a_i)\big). \tag{4.14}$$

Now write

$$B = B(a_j) + \sum_{\gamma \not\subset a_j} W_\gamma$$

with $B(a_j) = i[H(a_j), A]$. Since

$$Q(a_j)\big(H(a_j)+i\big)^{-1}W_\gamma\big(H(a_j)+i\big)^{-1}$$

is $\gamma \cup a_j(\underset{\neq}{\supset} a_j)$-compact, we can (as in the proof of Prop. 4.5) replace B by $B(a_i)$ in the sum in Proposition 4.5. For $j \neq k$, we write:

$$\begin{aligned}
f_i(H(a_i))\rho_j(a_i)B(a_i)\rho_k(a_i)f_i(H(a_i))
&= \rho_j(a_i)f_i\big(\lambda_j^{a_i}+T(a_i)\big)B(a_i)f_i\big(\lambda_k^{a_i}+T(a_i)\big)\rho_k(a_i)\\
&= \rho_j(a_i)B(a_i)f_i\big(\lambda_j^{a_i}+T(a_i)\big)f_i\big(\lambda_k^{a_i}+T(a_i)\big)\rho_k(a_i)\\
&= 0. \qquad (4.15)
\end{aligned}$$

In the first step we used (4.14), in the last step (4.13) and in the middle step we wrote ($B^{a_i}\otimes I = i[H^{a_i}\otimes I, A]$),

$$B(a_i) = B^{a_i}\otimes I + 2T(a_i) \tag{4.16}$$

which commutes with $f_i(\lambda_j^{a_i}+T(a_i))$. Finally, we note that the Virial theorem for H^{a_i} (see Section 6) implies that

$$\rho_j^{a_i}B^{a_i}\rho_j^{a_i} = 0$$

so, by (4.16) we have that

$$\rho_j(a_i)B(a_i)\rho_j(a_i) = 2\rho_j(a_i)T(a_i)\rho_j(a_i). \tag{4.17}$$

Putting all this together (i.e. Prop. 4.5, the replacement of B by $B(a_i)$ in that proposition and then (4.15) and (4.17)), we have

PROPOSITION 4.7.

$$f_1(H(a_1))Bf_1(H(a_1)) \doteq \sum_{i=1}^{N}\sum_{a_i}\sum_{j=1}^{n(a_i)} N(a_i)f_i(H(a_i))\rho_j(a_i)2T(a_i)\rho_j(a_i)f_i(H(a_i))N(a_i)^*.$$

As the final step, we note that since $T(a_i)\ge 0$

$$T(a_i)f_i\big(\lambda_j^{a_i}+T(a_i)\big)^2 \ge c_j^{a_i}f_i\big(\lambda_j^{a_i}+T(a_i)\big)^2$$

where

$$\begin{aligned}
c_j^{a_i} &= \min\{x\ge 0 \mid (\lambda_j^{a_i}+x)\in \operatorname{supp} f_i\}\\
&\ge \operatorname{dist}(\operatorname{supp} f_i, \lambda_j^{a_i}).
\end{aligned}$$

By shrinking $\operatorname{supp} f_N \supset \operatorname{supp} f_i$ enough, we can be sure that $c \ge \frac{1}{2}\alpha'$ for any fixed $\alpha' < 2\operatorname{dist}(\lambda_0, \text{thresholds})$ (since $\lambda_j^{a_i}$ is a threshold). As a result we have proved:

THEOREM 4.8. *If each ρ^{a_i} $(i \geq 2)$ is of finite rank, then* (1.9) *holds in the following sense: there is for any non-threshold λ_0, and any $\alpha < 2\,\mathrm{dist}(\lambda_0,$ thresholds), an open interval Δ about λ_0 for which* (1.9) *holds.*

5. Mourre-type estimates for N-body systems: General case

In this section, we will prove the key estimate (1.9) without the hypothesis that each P^a is finite. Let us begin by explaining why the simple device exploited by Mourre [24] to handle an infinite number of bound states in *his* analysis of the three body case will not work. Let us change notation slightly from the last section and pick orthogonal *rank* 1 projections ρ_i^a $(i = 1, \ldots, n(a)$; $n(a)$ may be infinite) so that

$$P^a = \sum \rho_i^a,$$

$$\rho_i^a H^a = H^a \rho_i^a = \lambda_i^a H^a$$

with λ_i^a eigenvalues of H^a. It is no longer true that the λ_i^a are distinct if a is fixed. Our shift from the last section is made since we want each operator

$$P_n^a \equiv \sum_{i=1}^{n} \rho_i^a$$

with $n < \infty$ to be of finite rank and some eigenvalue of H^a might have infinite multiplicity (this could only happen if it is at a threshold of H^a; we don't know if such bizarre behavior can actually occur!).

We define

$$P_n(a) = P_n^a \otimes 1,$$

$$\bar{P}_n^a = 1 - P_n^a; \quad \bar{P}_n(a) = 1 - P_n(a)$$

as usual. For any $f_1 \subset f_2 \subset \cdots \subset f_N$ and any $n_1, \ldots, n_N$, there is an expansion analogous to (4.4) with $P(a_i)$ replaced by $P_{n_i}(a_i)$ in both (4.4) and (4.5). One can show (see below) that (4.8) still holds with $P(a_i)$ replaced by $P_{n_i}(a_i)$ if one chooses the f's and n's successively, i.e. n_N, f_N, then n_{N-1}, f_{N-1}, etc. The problem comes with controlling $P_{n_i}(a_i)$, $P_{n_j}(b_j)$ terms. To explain the problem suppose that $a_i \subsetneq b_j$ so that $j < i$. Suppose $i = j + 1$. In Section 4, we controlled

$$f_i(H(a_i))P_{n_i}(a_i)P_{n_j}(b_j)f_j(H(b_j)) \tag{5.1}$$

by shrinking the support of f_i. This could be accomplished since $P(b_j)$ was fixed, so we arranged for (5.1) to be small at the stage where f_i was chosen independent of any later choice. The problem now is that $P_{n_j}(b_j)$ depends on the choice of n_j. n_j must be chosen to be certain that various $\bar{P}_{n_j}(b_j)K_{f_j}$ terms are small and so n_j

depends on f_i. Thus, we must pick f_i after n_j to arrange for (5.1) to be small but we must pick n_j after f_i for the $\bar{P}K$ terms to be small.

The only way we have found out of this conundrum is to give up on expanding f_1 alone as we do in Section 4, but instead expanding f_1^2 and $f_1 B f_1$.

We begin by noting that the final steps in Section 4 go over without change, that is:

PROPOSITION 5.1. *If (with $\doteq$ meaning, now, for suitable choice of f_i and n_i)*

$$(5.2)\qquad f_1(H(a_1))^2 \doteq \sum_{i=1}^{N} \sum_{a_i} N(a_i) f_i(H(a_i)) P_{n_i}(a_i) f_i(H(a_i)) N(a_i)^*$$

and

$$(5.2')\quad f_1(H(a_1)) B f_1(H(a_1)) \doteq 2 \sum_{i=1}^{N} \sum_{a_i} N(a_i) f_i(H(a_i)) \times P_{n_i}(a_i) T(a_i) P_{n_i}(a_i) f_i(H(a_i)) N(a_i)^*$$

for suitable operators N (perhaps different from 4.5) and for f's obeying:

$$(5.3)\qquad f_i(x+\lambda) f_i(x+\mu) = 0$$

for all x and all λ, μ which are distinct eigenvalues among the numbers $\{\lambda_k^{a_i} \mid k = 1, \ldots, n_i\}$, then (1.9) *holds in the precise sense stated in Theorem* 4.8.

Remark. (5.3) allows us to eliminate the terms in (5.2′), $\rho_j(a_i) T(a_i) \rho_k(a_i)$ with $\lambda_j^{a_i} \neq \lambda_k^{a_i}$ and then to follow the proof of Theorem 4.8.

To save having to report too many formulae, we introduce the symbol $B^{\#}$ to denote either B or 1. We begin with an expansion based on repeated use of Theorem 3.3.

PROPOSITION 5.2. *For any $n_1, \ldots, n_N$ and $f_1 \subset \cdots \subset f_N$:*

$$(5.4)\qquad f_1(H(a_1)) B^{\#} f_1(H(a_1)) = \alpha_1 + \alpha_2 + \alpha_3$$

where:

(5.5)

$$\alpha_1 = \sum_{l=1}^{N} \sum_{a_l, b_l} N(a_l) f_l(H(a_l)) P_{n_l}(a_l) B^{\#} P_{n_l}(b_l) f_l(H(b_l)) N(b_l)^*,$$

$$\alpha_2 = \alpha_{2a} + \alpha_{2b},$$

$$\alpha_{2a} = \sum_{l=1}^{N-1} \sum_{a_l, b_l} N(a_l) f_l(H(a_l)) \overline{P_{n_l}}(a_l) K_{f_{l+1}}(a_l) B^{\#} \overline{P_{n_l}}(b_l) f_l(H(b_l)) N(b_l)^*,$$

$$\alpha_{2b} = \sum_{l=1}^{N-1} \sum_{a_{l+1}, b_l} N(a_{l+1}) f_{l+1}(H(a_{l+1})) B^{\#} K_{f_{l+1}}(b_l)^* \overline{P_{n_l}}(b_l) f_l(H(b_l)) N(b_l)^*,$$

$$\alpha_3 = \sum_{l=1}^{N} \sum_{a_l, b_l} N(a_l) f_l(H(a_l)) \left[P_{n_l}(a_l) B^{\#} \bar{P}_{n_l}(b_l) + \overline{P_{n_l}(a_l)}\, B^{\#} P_{n_l}(b_l) \right]$$
$$\times f_l(H(b_l)) N(b_l)^*$$

with $K_f(a)$ *given by* (3.5) *and with* $N(a_i)$ *given by* (4.5) *except that* $\bar{P}(a_1), \ldots, \bar{P}(a_{i-1})$ *are replaced by* $\bar{P}_{n_1}(a_1), \ldots, \bar{P}_{n_{i-1}}(a_{i-1})$.

Proof. Basically, we have, after some expansion, a term like (5.5) but without the $P_{n_l}(a_l)$ and $P_{n_l}(b_l)$; we insert $P_{n_l}(a_l) + \bar{P}_{n_l}(a_l)$ and a similar b_l sum. The $P - P$ terms are put into α_1, the $P - \bar{P}$ and $\bar{P} - P$ terms into α_3, and we expand the $\bar{P} - \bar{P}$ term using (3.4). The terms with K's are put into α_2 and the leftover term is of the form we started with. Having explained the strategy, let us give a formal proof.

Define α_i^L, $i = 1,2,3$, by the formula for α_i except that $\sum_{i=1}^{N}$ (or $\sum_{i=1}^{N-1}$) are replaced by $\sum_{i=1}^{L}$ (or $\sum_{i=1}^{L-1}$). Let

$$\alpha_4^L = \sum_{a_L, b_L} N(a_L) f_L(H(a_L)) \bar{P}_{n_L}(a_L) B^{\#} \bar{P}_{n_L}(b_L) f_L(H(b_L)) N(b_L).$$

We claim that

$$f_1(H(a_1)) B^{\#} f_1(H(a_1)) = \alpha_1^L + \alpha_2^L + \alpha_3^L + \alpha_4^L \tag{5.6}$$

for each L. (5.4) is just (5.6) for $L = N$, since $\bar{P}_{n_N}(a_N) = 0$ (of necessity we take $n_N = 1$; recall that $\mathcal{H}^{a_N} = \mathbf{C}$) which implies that $\alpha_4^N = 0$. (5.6) is proved inductively. Since the sum in $\alpha_2^{(L=1)}$ is empty, $\alpha_2^{(L=1)} = 0$. Since we will still take $P(a_1) = 0$, $\alpha_1^{(L=1)} = \alpha_3^{(L=1)} = 0$; and trivially $\alpha_4^{(L=1)} =$ LHS of (5.6). Thus suppose that (5.6) holds for a given L and let us prove it for $L + 1$.

Writing

$$f_L(H(a_L)) \bar{P}_{n_L}(a_L) = f_L(H(a_L)) \bar{P}_{n_L}(a_L) f_{L+1}(H(a_L))$$

and expanding $f_{L+1}(H(a_L))$ via (3.4), we see that

$$\alpha_4^L = \left(\alpha_{2a}^{L+1} - \alpha_{2a}^L\right) + \alpha_5^L$$

where

$$\alpha_5^L = \sum_{a_L, b_L} \sum_{a_{L+1}, b_L} N(a_{L+1}) f_{L+1}(H(a_{L+1})) B^{\#} \bar{P}_{n_L}(b_L) f_L(H(b_L)) N(b_L)^*.$$

Now write

$$\overline{P_{n_L}}(b_L) f_L(H(b_L)) = f_{L+1}(H(b_L)) \bar{P}_{n_L}(b_L) f_L(H(b_L))$$

and expand $f_{L+1}(H(b_L))$ via (3.4) to obtain

$$\alpha_5^L = \left(\alpha_{2b}^{L+1} - \alpha_{2b}^L\right) + \alpha_6^L$$

where

$$\alpha_6^L = \sum_{a_{L+1}, b_{L+1}} N(a_{L+1}) f_{L+1}(H(a_{L+1})) B^\# f_{L+1}(H(b_{L+1})) N(b_{L+1})^*.$$

By induction, we have that

$$\text{LHS of (5.6)} = \alpha_1^L + \alpha_2^{L+1} + \alpha_3^L + \alpha_6^L. \tag{5.7}$$

Now, in α_6^L, replace $B^\#$ by

$$\left(P_{n_{L+1}}(a_{L+1}) + \overline{P_{n_{L+1}}}(a_{L+1})\right) B^\# \left(P_{n_{L+1}}(b_L) + \overline{P_{n_{L+1}}}(b_L)\right)$$

and note that the $P - P$ term is $\alpha_1^{L+1} - \alpha_1^L$, that the $P - \bar{P}$ plus $\bar{P} - P$ term is $\alpha_3^{L+1} - \alpha_3^L$ and that the $\bar{P} - \bar{P}$ term is α_4^{L+1} so

$$\alpha_6^L = \left(\alpha_1^{L+1} - \alpha_1^L\right) + \left(\alpha_3^{L+1} - \alpha_3^L\right) + \alpha_4^{L+1}. \tag{5.8}$$

(5.7) and (5.8) yield (5.6) for $L + 1$. □

If we expanded the $\bar{P}$ terms in α_3 in the way we did in Section 4, we would obtain the $P_{n_l}(a_l)P_{n_k}(b_k)$ terms with $k \neq l$ that we indicated give one trouble. Thus, the key to our proof is a more careful expansion of α_3. We will write down the expansion and informally indicate its proof; a formal inductive argument of the type just given is left to the reader.

PROPOSITION 5.3. *Given functions $\phi_l^{(l)} \subset \phi_{l+1}^{(l)} \subset \cdots \subset \phi_N^{(l)}$ with $f_l = \phi_l^{(l)}$ and integers $m_l(l), \ldots, m_l(N-1)$ with $m_l(l) \geq n_l$ we have:*

$$\begin{aligned} f_l(H(a_l)) P_{n_l}(a_l) B^\# \overline{P_{n_l}}(b_l) f_l(H(b_L)) \\ = f_l(H(a_l)) P_{n_l}(a_l) B^\# \left[\beta_1 + \beta_2 + \beta_3 + \beta_4\right] f_l(H(b_l)), \end{aligned} \tag{5.9}$$

where

$$\beta_1 = \left[P_{m_l(l)}(b_l) - P_{n_l}(b_l)\right],$$

$$\beta_2 = \sum_{k=1}^{N-l} \sum_{\substack{b_{l+k} \subset \boxed{b_l} \\ b_{l+k} \not\subset \boxed{a_l}}} P_{m_l(l+k)}(b_{l+k}) \phi_{l+k}^{(l)}(H(b_{l+k})) A(b_{l+k}, b_l),$$

where

$$A(b_{l+k}, b_l) = \sum_{\boxed{b_{l+k}} \subset b_{l+k-1} \subset \cdots \subset \boxed{b_l}} \Big[\bar{P}_{m_l(l+k-1)}(b_{l+k-1}) \times \phi^{(l)}_{l+k-1}(H(b_{l+k-1})) \ldots \bar{P}_{m_{l(l)}}(b_l) \Big],$$

$$\beta_3 = \sum_{k=1}^{N-l} \sum_{\substack{b_{l+k} \subset \boxed{b_l} \\ b_{l+k} \subset \boxed{a_l}}} \phi^{(l)}_{l+k}(H(b_{l+k})) \tilde{A}(b_{l+k}, b_l, a_l),$$

where for $k \geq 2$, $\tilde{A}$ *is defined as is* A *except for the added condition in the sum* $b_{l+k-1} \subset a_l$. *For* $k = 1$,

$$\tilde{A}(b_{l+1}, b_l, a_l) = A(b_{l+1}, b_l) = \bar{P}_{m_l(l)}(b_l).$$

Finally

$$\beta_4 = K_{\phi^{(l)}_{l+1}}(b_l) \bar{P}_{m_l(l)}(b_l) + \sum_{k=1}^{N-l-1} \sum_{\substack{b_{l+k} \subset \boxed{b_l} \\ b_{l+k} \not\subset \boxed{a_l}}} \Big[K_{\phi^{(l)}_{l+k+1}}(b_{l+k}) \bar{P}_{m_l(l+k)}(b_{l+k}) \times \phi^{(l)}_{l+k}(H(b_{l+k})) A(b_{l+k}, b_l) \Big]$$

with K *given by* (3.5).

Proof. First replace $\bar{P}_{n_l}(b_l)$ by $\bar{P}_{m(l)}(b_l)$ noting that the difference is exactly the β_1-term. Now write

$$\bar{P}_{m_l(l)}(b_l) \phi^{(l)}_l(H(b_l)) = \phi^{(l)}_{l+1}(H(b_l)) \bar{P}_{m_l(l)}(b_l) \phi^{(l)}_l(H(b_l))$$

and expand $\phi^{(l)}_{l+1}(H(b_l))$ by (3.4). The K-term is the first term in β_4 and we have in addition

$$\sum_{b_{l+1} \subset \boxed{b_l}} \phi^{(l)}_{l+1}(H(b_l)) A(b_{l+1}, b_l).$$

Those terms associated to b_{l+1}'s with $b_{l+1} \subset a_l$ we place in β_3; the others we multiply by $\bar{P}_{m_l(l+1)}(H(b_{l+1})) + P_{m_l(l+1)}(H(b_l))$. The P-terms are put into the sum β_2 and the $\bar{P}$ terms are expanded to give the $k = 1$ contribution to β_4 and a remaining sum indexed by b_{l+2}. These terms are treated as above. Since eventually some $b_{l+k} \subset a_l$ (for $b_N \subset a_l$), the procedure terminates. □

We have gained two things by the last two expansions:

(a) We have decoupled the functions, $\phi_i^{(l)}$, entering in the $P - \bar{P}$ expansion from the functions f_i entering in the $P - P$ expansion. Thus, for example, the need to pick f_{l+1}, then n_l and then f_{l+1} again (!) will be met by picking f_{l+1}, then n_l and then $\phi_{l+1}^{(l)}$.

(b) By expanding both parts of $f_1 B^{\#} f_1$ we can have conditions like $b_{l+k} \not\subset a_l$ which will play a major role.

We can now state the major new technical result of this paper:

THEOREM 5.4. *For N-body systems obeying Hypothesis* M, (1.9) *holds in the sense that given any non-threshold point* λ_0, *and any* $\alpha < 2\operatorname{dist}(\lambda_0, \textit{thresholds})$, (1.9) *holds for* Δ *a small neighborhood about* λ_0.

Proof. By Proposition 5.1, we need only verify (5.2), (5.2′). We begin with the expansion (5.4) and successively pick $n_N = 1, f_N, n_{N-1}, \ldots$. Suppose that $l \geq 2$ and we have picked all n's and f's up to f_{l+1}. We describe how to pick n_l and f_l so that various terms are at some desired level of smallness. (For $l = 1$, $P \equiv 0$, so only the α_2 terms are present and these are compact.) By Theorems 2.9 and 3.5,

$$\left\|(H_0 + 1)K_{f_{l+1}}(b_l)\left[\overline{P_\infty(b_l)} - \overline{P_{n_l}}(b_l)\right]\right\| \to 0$$

as $n_l \to \infty$, so we can pick n_l so that the α_2 terms at level l with $\bar{P}_{n_l}$ replaced by $\bar{P}_{n_l} - \bar{P}_{n_\infty}$ have a desired smallness. As usual, using Theorem 2.7, we can, by choosing f_l to have sufficiently small support, be sure the rest of the α_2-terms at level l are small. When we finally pick f_l we will be sure to have at least this small support. Since n_l is already chosen, we can be certain that f_l is chosen so that (5.3) holds for $i = l$.

In a moment, we will show how to pick f_l so the α_3 terms are small. Having done this, we control the off-diagonal terms in α_1 ($a_l \neq b_l$) by noting that since a_l and b_l have the same number of clusters $a_l \neq a_l \cup b_l$ if $a_l \neq b_l$. Thus, as in the proof of Section 4, the off-diagonal terms in (5.5) can be made small by shrinking f_l further. Moreover, we can write

$$P_{n_l}(a_l)BP_{n_l}(a_l) = 2P_{n_l}(a_l)\left[2T(a_l) + \gamma_1 + \gamma_2\right]P_{n_l}(a_l)$$

with

$$\gamma_1 = \sum_{\gamma \not\subset a_l} W_\gamma,$$

$$\gamma_2 = B^{a_l} \otimes I.$$

By (5.3) and the Virial theorem, the γ_2-term is zero as in Section 4. By shrinking support of f_l we can be sure that $f_l(H(a_l))P_{n_l}(a_l)\gamma_1 P_{n_l}(a_l)f_l(H(a_l))$ is small by exploiting $\gamma \cup a_l$-compactness.

Thus, to prove (5.2), (5.2′) we need only show that the lth-level terms in α_3 can be made small by shrinking the support of f_l. To do this, we expand these terms using Proposition 5.3.

By the arguments above, we can arrange, while making successive choices $m_l(N), \phi_N^{(l)}, m_l(N-1), \phi_{N-1}^{(l)}, \dots, \phi_l^{(l)}$, to have the β_4-terms be small. If $b_{l+k} \subset a_l$, then

$$(H(a_l)+i)^{-2}P_{n_l}(a_l)B^{\#}(H(b_{l+k})+i)^{-2}$$

is a_l-compact. Since $b_{l+k} \neq a_l$ ($k \geq 1$!), we can, by choosing $\phi_{l+k}^{(l)}$, arrange for the smallness of the β_3-terms (recall that n_l is picked before the ϕ's) by Theorem 2.8.

If $b_{l+k} \not\subset a_l$, then

$$(H(a_l)+i)^{-2}P_{n_l}(a_l)B^{\#}P_{m(l+k)}(b_{l+k})(H(b_{l+k}))+i)^{-2}$$

is $a_l \cup b_{l+k} \supsetneq a_l$-compact, so the β_2-terms can be arranged small at the time f_l is chosen. Thus, by choice of the ϕ_l's and m_l's and a preliminary choice on the size of supp f_l, we can arrange for all the $\beta_2, \beta_3, \beta_4$ terms to be small. The off-diagonal β_1 terms, i.e. with $a_l \neq b_l$, are controlled as we controlled the off-diagonal α_1 terms (recall that $m_l(l)$ is picked before f_l).

Thus, we must show that by shrinking the support of f_l, we can arrange for the

$$P_{n_l}(a_l)B^{\#}\Big[P_{m_l(l)}(a_l) - P_{n_l}(a_l)\Big]$$

terms to be small. When $B^{\#} = 1$, this product is zero since $m_l(l) \geq n_l$. Since $T(a_l)$ commutes with $P_{n_l}(a_l)$, this contribution in

$$B = 2T(a_l) + B^{a_l} \otimes I + \sum_{\gamma \not\subset a_l} W_\gamma$$

is similarly zero. The W_γ terms are treated by the same $\gamma \cup a_l$-compactness arguments used before. Finally, if we pick supp f_l so that (5.3) holds for $i = l$ and for λ, μ distinct elements of $\{\lambda_k^{a_l} \mid k = 1, \dots, m_l(l)\}$, then the $B^{a_l} \otimes I$ are zero; for distinct eigenvalues, we use $f_l(T(a_l)+\mu)f_l(T(a_l)+\lambda) = 0$ and we use the Virial theorem for identical eigenvalues. □

6. The Virial theorem

As we already noted (Prop. 1.4), if H is a Schrödinger operator with potentials, V_γ, obeying hypothesis M and if A is given by (1.10′), then $B \equiv i[H, A]$

is bounded from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-2}$. Our main goal in this section is the proof of:

THEOREM 6.1 (Virial theorem). *Assume V is Δ-bounded and the distributional derivative $X \cdot \nabla V$ is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$ and let $H\phi = E\phi$; $\phi \in \mathcal{H}_{+2}$. Then $(\phi, B\phi) = 0$.*

Since formally,

$$(\phi, B\phi) = i(\phi, [H - E, A]\phi) = i[((H - E)\phi, A\phi) - (A\phi, (H - E)\phi)] = 0$$

(the only problem is ϕ may not be in $D(A)$), this result is formally obvious and was used extensively in the physics literature for many years without any attempt at rigorous proof. The first rigorous proof was given by Weidmann [41] for potentials V where one could justify taking derivatives of $e^{i\alpha A}Ve^{-i\alpha A}$. Another approach of controlling boundary terms in an integration by parts was exploited by Kalf [16]. Here we will use the new approach of Mourre [24] of approximating ϕ's by vectors in $D(A) \cap \mathcal{H}_{+2}$. Our only claim to originality is in the fact that considerable simplication is possible if special properties of A, H_0 are exploited.

We begin with a technical point: namely, most commutator calculations are done on $\mathcal{S}$ and B should be viewed as being defined by density from an a priori definition on $\mathcal{S}$, but we want to know that for any $\phi, \psi \in D(A) \cap \mathcal{H}_{+2}$:

$$(\phi, B\psi) = i[(H\phi, A\psi) - (A\phi, H\psi)]. \tag{6.1}$$

We define for λ real and non-zero:

$$R_\lambda \equiv i\lambda(A + i\lambda)^{-1}. \tag{6.2}$$

LEMMA 6.2. *For $\lambda \neq 2$, R_λ maps $\mathcal{H}_{+2}$ to $\mathcal{H}_{+2}$ and as maps from $\mathcal{H}_{+2}$ to $\mathcal{H}_{+2}$,*

$$s - \lim_{\lambda \to \infty} R_\lambda = I. \tag{6.3}$$

Proof. As maps from $\mathcal{S}$ to $\mathcal{S}$:

$$i[H_0, A] = 2H_0, \tag{6.4}$$

so

$$H_0(A + i\lambda) = [A + (\lambda - 2)i]H_0.$$

Thus, if $\lambda \neq 0, 2$:

$$H_0(A + i\lambda)^{-1} = (A + (\lambda - 2)i)^{-1}H_0,$$

so

$$H_0R_\lambda = \lambda(\lambda - 2)^{-1}(R_{\lambda-2})H_0. \tag{6.5}$$

Since

$$\|R_\lambda\phi\| \leq \|\phi\|,$$

we have that

$$\|H_0R_\lambda\phi\| \leq \lambda(\lambda - 2)^{-1}\|H_0\phi\|. \tag{6.6}$$

This implies the boundedness of R on $\mathcal{H}_{+2}$ and, by the duality, on $\mathcal{H}_{-2}$. From

$$1 - R_\lambda = (i\lambda)^{-1}AR_\lambda, \tag{6.7}$$

(6.4), (6.5) and simple commutation relations with $(M_0 + 1)^{-1}$, we see that $\|(1 - R_\lambda)\phi\|_{\pm 2} \to 0$ for $\phi \in \mathcal{S} \subset \{\psi | A\psi \in \mathcal{H}_{\pm 2}\}$ from which (6.3) follows. □

Proof of Theorem 6.1. We define the mollified dilation generator

$$A_\lambda = AR_\lambda.$$

We will show that

$$i[H, A_\lambda] \to B \quad \text{as } \lambda \to \infty \tag{6.8}$$

as maps from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$. Indeed, compute

$$A_\lambda = i\lambda + \lambda^2(A + i\lambda)^{-1}.$$

Therefore

$$i[H, A_\lambda] = R_\lambda BR_\lambda. \tag{6.9}$$

This implies (6.8) in virtue of Lemma 6.2 and the condition on the potential (the latter implies that B is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$). Since $(\phi, [H, A_\lambda]\phi) = 0$ for each eigenfunction ϕ of H, $(\phi, B\phi) = 0$ follows. □

The above proof used the special commutation relation $i[H_0, A] = 2H_0$. Theorem 1.2 requires a Virial theorem under the hypothesis of that theorem. Such a Virial theorem can be proved by mimicking Mourre's paper: the hypothesis on $[H_0, A]$ yields a bound similar to (6.3), i.e. that R_λ is $\mathcal{H}_{+2} \to \mathcal{H}_{+2}$ bounded uniformly as $\lambda \to \infty$. This is proved in Mourre's paper [24]. Since the $\mathcal{H}_{+2}$ associated to H is the same as that associated to H_0, we have the same bound for H even though we have less regularity than Mourre. Now the proof is identical.

We have already used the Virial theorem for subsystems in the proof of (1.9). In addition, we have, following Mourre [24]:

Proof of (i) *and* (ii) *of Theorem 1.2,* By (1.9) and the Virial theorem, if $H\phi = E\phi$ with $E \in \Delta$, we have that

$$(\phi, K\phi) \leq -\alpha(\phi, \phi). \tag{6.10}$$

Since K is compact, the set of ϕ's obeying (6.10) is a compact set. Such a set cannot include an infinite orthonormal set so (i) and (ii) follow. □

7. The absence of singular continuous spectrum

In this section, we will prove conclusion (iii) of Theorem 1.2 (and thereby complete the proof of Thm. 1.5). We emphasize that not only is the strategy of this proof taken from Mourre's paper [24], so is much of the tactics. We provide the details in part for the reader's convenience and, in part, because of some small changes necessitated by our weaker hypotheses.

Since we have already proved (ii), Δ has only a discrete set, D, of eigenvalues. We will prove for $\lambda \in \Delta \setminus D$ that

$$\sup_{0<\mu<1} \left\| (|A| + 1)^{-1}(H - \lambda - i\mu)^{-1}(|A| + 1)^{-1} \right\| < \infty \tag{7.1}$$

with a bound uniform as λ runs through compacts of $\Delta \setminus D$. Given the conditions we have for (1.9) to hold, we will have proved:

THEOREM 7.1. *For Schrödinger operators with hypothesis M,* (7.1) *holds for any λ which is neither a threshold nor an eigenvalue and it holds uniformly on compact subsets of the allowed set of λ.*

Suppose that (1.9) holds for some set Δ. Then for any $\Delta' \subset \Delta$:

$$E_{\Delta'} B E_{\Delta'} \geq \alpha E_{\Delta'} + E_{\Delta'} K E_{\Delta'}, \tag{7.2}$$

since (1.9) can be multiplied by $E_{\Delta'}$ on each side. Now let $\lambda_0 \in \Delta$ not be an eigenvalue. Then as $|\Delta'| \to 0$ with Δ' centered about λ_0, we have $E_{\Delta'} \to 0$ strongly. As a result $\|E_{\Delta'} K E_{\Delta'}\| \to 0$, so by picking Δ' small, $\|E_{\Delta'} K_{\Delta'}\| \leq \alpha/2$ so that $\alpha E_{\Delta'} + E_{\Delta'} K E_{\Delta'} \geq \frac{1}{2}\alpha E_{\Delta'}$. Now change the meaning of α by a factor of 2 and multiply by a function $f(H)$ supported in Δ'. We conclude that:

LEMMA 7.2. *For any $\lambda \in \Delta \setminus D$, we can find $f \in C_0^\infty$, f identically* 1 *near λ so that*

$$f(H) B f(H) \geq \alpha f(H)^2. \tag{7.3}$$

Henceforth, we fix f and suppose that $0 \leq f \leq 1$ and $f \equiv 1$ on $[\lambda - \delta_0, \lambda + \delta_0] \equiv \tilde{I}$. We will let $I \equiv \left[\lambda - \frac{\delta_0}{2}, \lambda + \frac{\delta_0}{2}\right]$.

We define $D = (|A| + 1)^{-1}$, $M^2 = f(H)Bf(H)$ (this is non-negative by (7.3)),

$$G_\varepsilon(z) = (H - i\varepsilon M^2 - z)^{-1} \tag{7.4}$$

(we will prove this exists below when $\varepsilon > 0$, $\operatorname{Im} z > 0$) and

$$F_\varepsilon(z) = DG_\varepsilon(z)D. \tag{7.5}$$

Following Mourre [24], we will prove (7.1) by showing that for suitable ε_0,

$$\sup_{\substack{\operatorname{Re} z \in I,\, 0 < \operatorname{Im} z < 1 \\ 0 \leq \varepsilon < \varepsilon_0}} \|F_\varepsilon(z)\| < \infty, \tag{7.6}$$

and this will be accomplished by proving the differential inequality

$$\left\|\frac{dF_\varepsilon}{d\varepsilon}\right\| \leq c\left(\|F_\varepsilon\| + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2} + 1\right) \tag{7.7}$$

and then integrating.

Even though there is still (to us at least) a somewhat magical aspect to Mourre's proof, we feel a word of explanation may make the consideration of $G_\varepsilon(z)$ and $dG_\varepsilon/d\varepsilon$ a little less mysterious. Let us recall one mechanism for exploiting positive commutators of bounded operators: Let us imagine, for a moment, that $i[H, A] \geq \alpha$ with A bounded and $\alpha > 0$ (this is not possible but this argument is only heuristic). Then

$$\begin{aligned} \alpha\left\|(H - \lambda \pm i\varepsilon)^{-1}\phi\right\|^2 &= \left(\phi, (H - \lambda \mp i\varepsilon)^{-1} i[H, A](H - \lambda \pm i\varepsilon)^{-1}\phi\right) \\ &= a_1 + a_2, \end{aligned}$$

where

$$\begin{aligned} a_1 &= i\left(A\phi, (H - \lambda \pm i\varepsilon)^{-1}\phi\right) - i\left((H - \lambda \pm i\varepsilon)^{-1}\phi, A\phi\right), \\ a_2 &= \mp 2\varepsilon\left(\phi, (H - \lambda \mp i\varepsilon)^{-1} A (H - \lambda \mp i\varepsilon)^{-1}\phi\right). \end{aligned}$$

The a_1 term can be bounded by the square root of the quantity to be bounded while the a_2 term can be bounded by the quantity to be bounded times $2|\varepsilon|\|A\|$. (This incidentally shows that $i[H, A] \geq \alpha$ cannot hold for A bounded and $\alpha > 0$ since H would then have empty spectrum.) The point is that A need not be bounded to control a_1, only $\|A\phi\| < \infty$ is needed for that. The bad term is a_2 and this comes from the fact that it is $(H - \lambda \mp i\varepsilon)^{-1} i[H, A](H - \lambda \pm i\varepsilon)^{-1}$ that occurs, not $(H - \lambda \pm i\varepsilon)^{-1} i[H, A](H - \lambda \pm i\varepsilon)^{-1}$. But it is a term just like this that occurs in $dG_\varepsilon/d\varepsilon$! With this motivation, we begin the formal proof. The next lemma uses mainly ideas of Mourre [24].

LEMMA 7.3. (a) *For* $\varepsilon \geq 0$ *and* $\operatorname{Im} z > 0$, $(H - i\varepsilon M^2 - z)$ *is invertible and* $G_\varepsilon(z)$ *is* C^1 *in* ε *on* $(0, \infty)$ *and continuous on* $[0, \infty)$.

(b) *The following estimate holds for suitable* $\varepsilon_0 > 0$:

$$\|f(H)G_\varepsilon(z)\phi\| \leq C\varepsilon^{-1/2}|(\phi, G_\varepsilon(z)\phi)|^{1/2} \tag{7.8}$$

uniformly for all z *with* $\operatorname{Re} z \in I$ *and all* ε *with* $0 < \varepsilon < \varepsilon_0$.

(c) *For* z, ε *with the properties listed in* (b)

$$\|G_\varepsilon(z)\| \leq C\varepsilon^{-1}; \ \|(1 - f(H))G_\varepsilon(z)\| \leq C. \tag{7.9}$$

(d) (7.9) *remains true if the* $\mathcal{H} \to \mathcal{H}$ *operator norm is replaced by the norm as operators from* $\mathcal{H}$ *to* $\mathcal{H}_{+2}$.

Proof. (a) Let $z = \mu + i\delta$. Then

$$\|(H - i\varepsilon M^2 - z)\phi\|^2 = \|(H - i\varepsilon M^2 - \mu)\phi\|^2 + \delta^2\|\phi\|^2 + 2\delta\varepsilon\|M\phi\|^2.$$

This proves that $H - i\varepsilon M^2 - z$ is invertible from its automatically closed range to $D(H)$ so long as $\delta > -2\varepsilon\|M\|^2$. Since $(H - i\varepsilon M^2 - z)^* = H + i\varepsilon M^2 - z^*$ obeys an identical estimate, $\operatorname{Ker}(H - i\varepsilon M^2 - z) = \{0\}$, so the range is all of H. Since M^2 is bounded, $H - i\varepsilon M^2 - z$ is an analytic family of type (A) and thus $G_\varepsilon(z)$ is, for z fixed, analytic in a neighborhood of $(-\frac{1}{2}\delta\|M\|^{-2}, \infty)$ from which the required smoothness holds. For later purposes, we note that this analyticity implies that

$$\frac{dG_\varepsilon}{d\varepsilon} = iG_\varepsilon M^2 G_\varepsilon. \tag{7.10}$$

(b) It is in this step only that the basic estimate (7.3) enters. By (7.3)

$$\begin{aligned}\|f(H)G_\varepsilon(z)\phi\|^2 &= \langle\phi, G_\varepsilon(z)^* f(H)^2 G_\varepsilon(z)\phi\rangle \\ &\leq (2\alpha\varepsilon)^{-1}(\phi, G_\varepsilon(z)^*(2\varepsilon M^2)G_\varepsilon(z)\phi) \\ &\leq (\alpha\varepsilon)^{-1}|(\phi, G_\varepsilon(z)\phi)|\end{aligned}$$

which is (7.8). The last step uses

$$G_\varepsilon(z)^*(2\varepsilon M^2)G_\varepsilon(z) \leq G_\varepsilon(z)^*(2\varepsilon M^2 + 2\operatorname{Im} z)G_\varepsilon(z) = i[G_\varepsilon(z)^* - G_\varepsilon(z)]$$

and $|(\phi, (T^* - T)\phi)| \leq 2|(\phi, T\phi)|$.

(c) We begin by writing

$$(1 - f(H))G_\varepsilon(z) = (1 - f(H))G_0(z)[1 + i\varepsilon M^2 G_\varepsilon(z)].$$

Since $(1 - f(H))G_0(z)$ is bounded for $\operatorname{Re} z \in I$ (by $2/\delta_0$ with δ_0 given in the definition of I), we conclude that

$$\|(1 - f(H))G_\varepsilon(z)\| \leq C_0(1 + \varepsilon\|G_\varepsilon(z)\|) \tag{7.11}$$

since M is bounded. In particular, the first estimate in (7.9) implies the second. To prove the first estimate, we write

$$\begin{aligned}\|G_\varepsilon(z)\| + 1 &\le \|f(H)G_\varepsilon(z)\| + \|(1-f(H))G_\varepsilon(z)\| + 1\\ &\le C\varepsilon^{-1/2}\|G_\varepsilon(z)\|^{1/2} + C_0(1+\varepsilon\|G_\varepsilon(z)\|) + 1\\ &\qquad\qquad\text{(by (7.8) and (7.11))}\\ &\le 2C\varepsilon^{-1/2}(\|G_\varepsilon(z)\|+1)^{1/2} + \tfrac{1}{2}(\|G_\varepsilon(z)\|+1),\end{aligned}$$

so long as $C_0\varepsilon \le \frac{1}{2}$ and $C_0 + \frac{1}{2} \le C\varepsilon^{-1/2}$. Thus if $\varepsilon < \varepsilon_0 \equiv \min(\frac{1}{2}C_0, C^2(C_0 + \frac{1}{2})^{-2})$, we have that

$$\|G_\varepsilon(z)\| \le 16C^2\varepsilon^{-1}$$

as required.

(d) Returning to the proof of (7.11) and noting that $\|(H+i)(1-f(H))G_0(z)\|$ is bounded, we see that

$$\|(H+i)[1-f(H)]G_\varepsilon(z)\| \le C_0(1+\varepsilon\|G_\varepsilon(z)\|),$$

which given (7.9) for $\mathcal{H}\to\mathcal{H}$ norms implies the second estimate in (7.9) for $\mathcal{H}\to\mathcal{H}_{+2}$ norms. Since, $f\in C_0^\infty$,

$$\|(H+i)f(H)G_\varepsilon(z)\| \le C\|G_\varepsilon(z)\|,$$

so the first estimate holds for $\mathcal{H}\to\mathcal{H}_{+2}$ norms. □

The next pair of lemmas are the main estimates in this section which go beyond Mourre's paper and are the key to our being able to use weakened regularity hypotheses on $[A, H]$.

LEMMA 7.4. *Let $f\in C_0^\infty$. Then $[A, f(H)]$ is bounded as a map from $\mathcal{H}_{-1}$ to $\mathcal{H}_{+1}$.*

Proof. In the calculations below, we ignore domain questions resulting from the fact that A is unbounded. To handle this, one can replace A by AR_λ as in Section 6, use $i[H, AR_\lambda] = R_\lambda BR_\lambda$ and obtain bounds on $[AR_\lambda, f(H)]$ as maps from $\mathcal{H}_{-1}$ to $\mathcal{H}_{+1}$ uniform as $\lambda\to\infty$.

Write, as forms on $D(A)\cap D(H)$:

$$e^{+itH}A - Ae^{itH} = [e^{itH}Ae^{-itH} - A]e^{itH} = \left[\int_0^t e^{isH}Be^{-isH}\,ds\right]e^{itH}$$

and conclude by the hypothesis on B ((c″) in Thm. 1.2) that

$$\|[A, e^{itH}]\|_{2,-1} \le ct$$

where $\|\cdot\|_{j,k}$ is the norm as a map from $\mathcal{H}_j$ to $\mathcal{H}_k$. Since

$$g(H) = (2\pi)^{-1/2}\int \hat{g}(s)e^{isH}\,ds$$

we conclude that for any $g \in C_0^\infty$,

$$\|[A, g(H)]\|_{2,-1} < \infty \tag{7.12a}$$

Next, note that

$$[A, (H+i)^{-1}] = -i(H+i)^{-1}B(H+i)^{-1}$$

so that

$$\|[A, (H+i)^{-1}]\|_{0,1} < \infty, \ \|[A, (H+i)^{-1}]\|_{-1,0} < \infty. \tag{7.12b}$$

Now, any $f(H)$ with $f \in C_0^\infty$ can be written as

$$f(H) = (H+i)^{-1}g(H)(H+i)^{-1}$$

for $g \in C_0^\infty$ and

$$\begin{aligned}[A, f(H)] = {}&[A, (H+i)^{-1}]g(H)(H+i)^{-1} + (H+i)^{-1} \\ &\times [A, g(H)](H+i)^{-1} + (H+i)^{-1}g(H)[A, (H+i)^{-1}].\end{aligned}$$

By (7.12b), the first and third terms take $\mathcal{H}_{-1}$ to $\mathcal{H}_{+1}$. By (7.12a), the middle term takes $\mathcal{H}$ to $\mathcal{H}_{+1}$ and then by iteration, we obtain the $\mathcal{H}_{-1}$ to $\mathcal{H}_{+1}$ result. □

LEMMA 7.5. $[A, M^2]$ *is bounded (as an operator on* $\mathcal{H}$*).*

Proof.

$$[A, M^2] = [A, f(H)]Bf(H) + f(H)[A, B]f(H) + f(H)B[A, f(H)].$$

By hypothesis (d′) of Theorem 1.2, the middle term is bounded. In the first term, $f(H)$ takes $\mathcal{H}$ to $\mathcal{H}_{+2}$, B takes $\mathcal{H}_{+2}$ to $\mathcal{H}_{-1}$ and by Lemma 7.4, $[A, f(H)]$ takes $\mathcal{H}_{-1}$ to $\mathcal{H}$. □

Remark. $[A, M^2]$ can be shown to map $\mathcal{H}_{-1}$ to $\mathcal{H}_{+1}$.

LEMMA 7.6. $\|G_\varepsilon D\| \le C(1 + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2})$.

Proof. Taking $\phi = D\psi$ in (7.8), we find that ($f \equiv f(H)$)

$$\|fG_\varepsilon D\| \le c\varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}.$$

Since

$$\|(1-f)G_\varepsilon D\| \le c\|(1-f)G_\varepsilon\|$$

is bounded by (7.9), the result is proved. □

LEMMA 7.7. *The differential inequality* (7.7) *holds.*

Proof. By (7.10)

$$(-i)\frac{dF_\varepsilon}{d\varepsilon} = DG_\varepsilon M^2 G_\varepsilon D = Q_1 + Q_2 + Q_3$$

where ($f \equiv f(H)$),

$$Q_1 = -DG_\varepsilon(1-f)B(1-f)G_\varepsilon D,$$
$$Q_2 = -DG_\varepsilon(1-f)BfG_\varepsilon D - DG_\varepsilon fB(1-f)G_\varepsilon D,$$
$$Q_3 = DG_\varepsilon BG_\varepsilon D.$$

Since B is bounded from $\mathcal{H}_{+2}$ to $\mathcal{H}_{-2}$, and $(1-f)G_\varepsilon$ is bounded from $\mathcal{H}$ to $\mathcal{H}_{+2}$, $\|Q_1\| < \infty$ and

$$\|Q_2\| \le c\|(H+i)fG_\varepsilon D\| \le c'\|G_\varepsilon D\| \le c''\big(1 + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}\big)$$

by Lemma 7.6.

To control Q_3, we write

$$Q_3 = Q_4 + Q_5,$$
$$Q_4 = DG_\varepsilon[H - i\varepsilon M^2 - z, A]G_\varepsilon D,$$
$$Q_5 = i\varepsilon DG_\varepsilon[M^2, A]G_\varepsilon D.$$

Expanding the commutator in Q_4, we get two similar terms, one of which has a norm

$$\|DAG_\varepsilon D\| \le \|DA\|\,\|G_\varepsilon D\| \le c\big(1 + \varepsilon^{-1/2}\|F_\varepsilon\|^{1/2}\big) \tag{7.13}$$

by Lemma 7.6. Finally, by Lemma 7.5:

$$\|Q_5\| \le c_0\varepsilon\|G_\varepsilon D\|^2 \le c^2\big(\varepsilon^{1/2} + \|F_\varepsilon\|^{1/2}\big)^2 \le c'(1 + \|F_\varepsilon\|).$$

Putting all these estimates together, we get (7.7). □

Conclusion of the proof of Theorem 7.2. As indicated, all we need to prove is (7.6). By (7.9), we begin with the bound

$$\|F_\varepsilon(z)\| \le C\varepsilon^{-1}.$$

Putting this in the differential inequality, we find

$$\left\|\frac{dF}{d\varepsilon}\right\| \le C\varepsilon^{-1},$$

so that integrating towards $\varepsilon = 0$ from $\varepsilon = \varepsilon_0$ ($\|F_{\varepsilon_0}(z)\|$ is bounded by (7.9)), we obtain

$$\|F_\varepsilon(t)\| \le C\ln\varepsilon.$$

Putting this into (7.7) and again integrating, we find (7.6). □

E. Mourre [25] kindly showed us the following method of improving (7.1) to obtain:

THEOREM 7.8 (MOURRE [25]). *Under the hypotheses of Theorem* 1.2,

$$\sup_{0<\mu<1} \|(|A|+1)^{-\alpha}(H-\lambda-i\mu)^{-1}(|A|+1)^{-\alpha}\| < \infty \tag{7.14}$$

for any fixed $\alpha > \frac{1}{2}$. (7.14) *holds uniformly as* λ *runs through compacts of* $\Delta \setminus D$.

Proof. We include here Mourre's proof with his kind permission. Let $D_\varepsilon = (|A|+1)^{-\alpha}(\varepsilon|A|+1)^{\alpha-1}$ and replace F_ε by

$$\tilde{F}_\varepsilon = D_\varepsilon G_\varepsilon D_\varepsilon.$$

We claim that

$$\left\|\frac{d\tilde{F}_\varepsilon}{d\varepsilon}\right\| \le C\varepsilon^{-(1-\alpha)}\left(1 + \varepsilon^{-1/2}\|\tilde{F}_\varepsilon\|^{1/2} + \|\tilde{F}_\varepsilon\|\right) \tag{7.15}$$

To prove this, we first note that the proof of Lemma 7.6 used no properties of D so that

$$\|G_\varepsilon D_\varepsilon\| \le c\left(1 + \varepsilon^{-1/2}\|\tilde{F}_\varepsilon\|^{1/2}\right),$$

Since

$$\left\|\frac{dD_\varepsilon}{d\varepsilon}\right\| = (1-\alpha)\left\||A|(1+|A|)^{-\alpha}(1+\varepsilon|A|)^{-2+\alpha}\right\| \le \varepsilon^{-(1-\alpha)}(1-\alpha),$$

we conclude that

$$\left\|\frac{dD_\varepsilon}{d\varepsilon}G_\varepsilon D_\varepsilon + D_\varepsilon G_\varepsilon \frac{dD_\varepsilon}{d\varepsilon}\right\|$$

is bounded by the right side of (7.15).

We can estimate $D_\varepsilon dG_\varepsilon/d\varepsilon\, D_\varepsilon$ as in Lemma 7.7. The only place that the form of D entered was in (7.13) which gets replaced by

$$\|D_\varepsilon A G_\varepsilon D\| \le \|D_\varepsilon A\|\,\|G_\varepsilon D_\varepsilon\| \le c\varepsilon^{-(1-\alpha)}\left(1 + \varepsilon^{-1/2}\|\tilde{F}_\varepsilon\|\right).$$

We have thus proved (7.15).

Given (7.15), suppose that we know that

$$\|\tilde{F}_\varepsilon\| \le C\varepsilon^{-\gamma}$$

with $\gamma \le 1$. By (7.15),

$$\left\|\frac{d\tilde{F}_\varepsilon}{d\varepsilon}\right\| \le C\varepsilon^{-(1-\alpha)}\left(1 + \varepsilon^{-1/2-\gamma/2}\right)$$

which upon integration yields

$$\|\tilde{F}_\varepsilon\| \le C\left(1 + \varepsilon^{-\gamma/2-1/2+\alpha}\right).$$

Since $\alpha > \frac{1}{2}$, this is an improvement and so after a finite number of steps beginning with $\|\tilde{F}_\varepsilon\| \leq C\varepsilon^{-1}$ (by (7.9)) we find $\|\tilde{F}_\varepsilon\|$ is bounded, so (7.14) follows by taking $\varepsilon = 0$. □

8. Weighted L^2 estimates

Let $L^2_\alpha(R^n)$ be the weighted L^2-space:

$$L^2_\alpha = \left\{ f \,|\, (1 + x^2)^{+\alpha/2} f \in L^2 \right\}.$$

For two body Schrödinger operators within a general class of potentials, Agmon [1] and Kuroda [19] showed that for any $\alpha > \frac{1}{2}$

$$\sup_{0<\mu<1} \|(H - \lambda - i\mu)^{-1}\|_{\alpha, -\alpha} < \infty \tag{8.1}$$

where $\|\cdot\|_{\alpha,\beta}$ is the norm as an operator from L^2_α to L^2_β. λ must avoid the threshold zero and any eigenvalues. $\alpha > \frac{1}{2}$ is optimal in the sense that (8.1) fails for $H = -\Delta$ if $\alpha \leq \frac{1}{2}$ (but (8.1) can be "logarithmically" improved using Besov spaces [2]). In this section we want to show how to obtain (8.1) for general N-body Schrödinger operators. We will also prove sufficient Hölder continuity to ensure that boundary values exist. In Mourre's original paper [24], he remarked without proof that for the three body operator where he obtained (7.1), $(|A| + 1)^{-1}$ could be replaced by $(|x| + 1)^{-1}$. We found an interpolation argument allowing improvement to $\alpha > \frac{2}{3}$ and then learned of Mourre's unpublished improvement leading to (7.14) which allows $\alpha > \frac{1}{2}$. We will prove:

THEOREM 8.1. *Let H be an N-body Schrödinger operator with potentials obeying hypothesis* M. *Let $\mathfrak{B}$ be the (closed) set of thresholds and eigenvalues and fix $\frac{1}{2} < \alpha \leq 1$. Then*:

(a)
$$\sup_{0<\mu<1} \|(H - \lambda - i\mu)^{-1}\|_{\alpha, -\alpha} < \infty$$

for all $\lambda \notin \mathfrak{B}$, uniformly in compact subsets of $R \setminus \mathfrak{B}$.

(b) *For such λ, λ',*

$$\|(H - \lambda - i\mu)^{-1} - (H - \lambda' - i\mu')^{-1}\|_{\alpha, -\alpha} \leq C_\theta [|\lambda - \lambda'| + |\mu - \mu'|]^\theta$$

for any θ, $\theta < \frac{2}{3}(\alpha - \frac{1}{2})$; for all $0 \leq \mu, \mu' \leq 1$, with C_θ uniform as λ, λ' run through a fixed compact subset of $R \setminus \mathfrak{B}$.

(c) *As maps from L^2_α to $L^2_{-\alpha}$, the norm limit*

$$(H - \lambda - i0)^{-1} = \lim_{\mu \downarrow 0} (H - \lambda - i\mu)^{-1}$$

exists if $\lambda \notin B$ and with this definition, $(H - z)^{-1}$ is Hölder continuous (as maps of L^2_α to $L^2_{-\alpha}$) on $\{z \,|\, \operatorname{Im} z > 0 \text{ or } z \in R \setminus \mathfrak{B}\}$.

Remark. The value of θ in (b) is presumably *not* optimal.

As a preliminary, we note that:

LEMMA 8.2. *For* $0 \le \alpha \le 1$:

$$(|A| + 1)^{\alpha}(H + i)^{-1}(x^2 + 1)^{-\alpha/2} \equiv J_{\alpha}$$

is bounded.

Proof. We need only prove the case $\alpha = 1$ and then use complex interpolation. Thus, we need only bound

$$(p \cdot x)(H + i)^{-1}(x^2 + 1)^{-1/2} = S_1 + S_2$$

where

$$S_1 = \left[p(H + i)^{-1/}\right] \cdot \left[x(x^2 + 1)^{-1/2}\right],$$

which is bounded, since $D(H) = D(H_0)$ and

$$S_2 = p\left[x, (H + i)^{-1}\right](x^2 + 1)^{-1/2},$$

which is bounded since

$$p\left[x, (H + i)^{-1}\right] = -2ip(H + i)^{-1}p(H + i)^{-1}$$

is bounded for the same reason. □

Proof of Theorem 8.1. (a) Writing

$$\begin{aligned}(x^2 + 1)^{-\alpha/2}&(H + i)^{-1}(H - z)^{-1}(H + i)^{-1}(x^2 + 1)^{-\alpha/2} \\ &= J_{\alpha}^{*}(|A| + 1)^{-\alpha}(H - z)^{-1}(|A| + 1)^{-\alpha}J_{\alpha}\end{aligned}$$

and using Theorem 7.8 and Lemma 8.2, we see that

$$(H + i)^{-1}(H - z)^{-1}(H + i)^{-1}$$

is bounded from L^2_{α} to $L^2_{-\alpha}$ uniformly in z in the required sets. Since

$$\begin{aligned}(H - z)^{-1} \equiv (H + i)^{-1} &+ (z + i)(H + i)^{-2} \\ &+ (z + i)^2(H + i)^{-1}(H - z)^{-1}(H + i)^{-1}\end{aligned}$$

we have the required result.

(b) By interpolation and part (a), we need only consider the case $\alpha = 1$. Using the operator $F_{\varepsilon}(z)$ of Section 7, and the device used in part (a), we need only prove

$$\|F_0(z) - F_0(z')\| \le C|z - z'|^{1/3} \tag{8.2}$$

for $|z-z'|$ small. Since $F_\varepsilon(z)$ is bounded, (7.7) implies that

$$\|F_\varepsilon(z)-F_0(z)\| \le C\varepsilon^{1/2} \tag{8.3}$$

uniformly in z in the required regions.

Now, for $\varepsilon < \varepsilon_0$:

$$\begin{aligned}\left\|\frac{d}{dz}F_\varepsilon(z)\right\| &= \|DG_\varepsilon^2(z)D\| \\ &= \|G_\varepsilon D\|^2 \le C\big(1+\varepsilon^{-1}\|F_\varepsilon\|\big) \le C\varepsilon^{-1}\end{aligned}$$

by Lemma 7.6. Thus

$$\|F_\varepsilon(z)-F_\varepsilon(z')\| \le C\varepsilon^{-1}|z-z'|. \tag{8.4}$$

With $\varepsilon = |z-z'|^{2/3}$, (8.3) and (8.4) yield (8.2).

(c) follows from (b). □

Appendix: Expanding in subsystems: The combinatorial methods

In this appendix, we want to describe an expansion of the form (3.1) with the c's combinatorial coefficients. Related ideas have appeared in a paper of Polyzou [26].

Definition. $f_T(H(a))$ (truncated f) is defined inductively in $N-\#(a)$ ($\#(a)$ = number of clusters in a) by:

$$f(H(a)) = \sum_{b\subseteq \boxed{a}} f_T(H(b)). \tag{A.1}$$

Thus $f_T(H(a_N)) = f(H(a_N))$ ($H(a_N)=H_0$) and (A.1) determines $f_T(H(a))$ given $f_T(H(b))$ for all $b \subsetneq a$. Clearly, by induction:

LEMMA A.1.

$$f_T(H(a)) = \sum_{b\subseteq \boxed{a}} d(b,a) f(H(b)) \qquad \text{with } d(a,a)=1.$$

Remark. The exact formula for d is irrelevant, but it can be obtained combinatorially. In fact, if

$$a = \{A_i\}_{i=1}^k \text{ and if } b_i = \{B\in b \mid B\subset A_i\},$$

then

$$d(b,a) = \prod_{i=1}^k d(b_i, A_i)$$

and

$$d(b_i, A_i) = \begin{cases} 1 & \text{if } \#(b_i) = 1 \\ (-1)^{(\#(b_i)-1)}[\#(b_i) - 1] & \text{if } \#(b_i) \geq 2. \end{cases}$$

Also clearly:

LEMMA A.2.

$$f(H(a)) = f_T(H(a)) + \sum_{b \subsetneq a} c(b,a) f(H(b))$$

where

$$c(b,a) = \sum_{\boxed{b} \subseteq b' \subsetneq \boxed{a}} d(b,b').$$

The final aspect of the basic expansion requirements is contained in

THEOREM A.3. *$f_T(H(a))$ is a-compact.*

Proof. By the same argument used in the proof of Lemma 3.2, it suffices to consider the case where $f(x) = (x - z)^{-1}$ with $\operatorname{Re} z$ large in absolute value and negative. By choosing $\operatorname{Re} z$ large in absolute value and negative, one can be sure that the standard diagrammatic expansion ([31]) for $(H(a) - z)^{-1}$ converges for all a. Define the cluster, $a(D)$, of a diagram, D, to be the partition obtained by looking at the connected components of D. If the "value" of a diagram D is the operator $O(D)$, then clearly:

$$(H(a)) - z)^{-1} = \sum_{a(D) \subseteq a} O(D),$$

so, by induction and definition of f_T:

$$f_T(H(a)) = \sum_{a(D) = a} O(D).$$

That each $O(D)$ is $a(D)$-compact is the standard diagrammatic argument ([31]). □

Remark. As in Section 3, one can show that $f_T(H(a))(H + 1)$ is a-compact.

This completes the proof of existence of an expansion of the form (3.1) with the c's numbers. We want to demonstrate its uniqueness. As a preliminary, we pick for any $b \subsetneq a$, a one parameter family $U_b^a(t)$ of unitary translation operators with the following properties: U_b^a translates differences of centers of mass of clusters of b (so it is b-fibered) and only translates those intercluster c.m.'s for distinct clusters in b which are subclusters of the same cluster of a, so U_b^a acts as

$\tilde{U}_b^a \otimes 1$ under the decomposition $\mathcal{H}^a \otimes \mathcal{H}_a$. Finally, all such distinct clusters are translated so that

$$\text{(A.2)} \qquad s - \lim_{t\to\infty} V_\gamma (H_0 + 1)^{-1} U_b^a(t) = 0$$

for all γ with $\gamma \subset a$, $\gamma \not\subset b$.

LEMMA A.4. *Let $b \subsetneq a$ and let $c \subset a$. Then*

$$s - \lim_{t\to\infty} U_b^a(t)^{-1} f_T(H(c)) U_b^a(t)$$

exists and equals $f_T(H(c))$ if $c \subset b$ and equals 0 if $c \not\subset b$.

Proof. If $c \subset b$, $U_b^a(t)$ commutes with $f_T(H(c))$ since U_b^a is b-fibered. If $c \subset b$, then by looking at diagrams and using (A.2), we obtain the required $s - \lim$ result if $f(x) = (x - z)^{-1}$ with $\operatorname{Re} z$ large in absolute value and negative. Using the standard argument of [39] (see Lemma 3.2) any f can be accommodated. □

THEOREM A.5. *Let K be a linear combination of $f_T(H(b))$'s with $b \subsetneq a$. Suppose that K is a-compact; then $K = 0$.*

Proof. Write $K = \sum_{b \subsetneq a} f_T(H(b))$. Since $s - \lim U_c^a(t)^{-1} K U_c^a(t) = 0$ for any $c \subsetneq a$ on account of the a-compactness, we have by Lemma A.4, that

$$\sum_{b \subset c} f_T^b(H(b)) = 0$$

for any $c \subsetneq a$. By induction, $f_T^c(H(c)) = 0$. □

This theorem implies that any expansion of the form (3.1) with K compact and real number c's must have $K_f(a) = f_T(H(a))$; (see also Polyzou [26], Thm. 1).

Added Notes

1. As far as the Virial theorem alone is concerned, an approach very close to Mourre's and our kinds of extensions was discovered independently by H. Leinfelder in "On the Virial Theorem in Quantum Mechanics," to appear in Int. Eqn. and Op. Th.

2. E. Mourre and P. Perry have noted that $H - i\varepsilon M^2$ is not as mysterious as we indicated in Section 7. If we ignore the regularizing factor, we have $H + \varepsilon[H, A]$ which is up to terms of order ε^2 formally $e^{-\varepsilon A} A^{\varepsilon A}$, the complex dilated Hamiltonian. This links this method of Mourre to his earlier paper Commun. Math Phys. **68** (1979), 91–94.

3. The proof we gave of Theorem 6.1 in the preprint of this paper was more involved than the one given here. Independently of us, T. Kato found the simple proof we give here.

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