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ASYMPTOTIC COMPLETENESS

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Dedicated to the memory of F.A. Berezin

Apology. Little more than 20 years ago my Ph.D. adviser, F.A. Berezin, suggested to me to look into Scattering Theory. In his own words I did not have time to do anything on my own but it would keep me off the streets. I would get myself an education. For better or worse it did keep me off the streets, but 20 years later I am still an amateur. In the first 10 years of my preoccupation with Scattering Theory, progress was very limited and extremely painful. In the last decade it was rather remarkable, but too fast too keep track of unless one devoted all one's time to the subject. I did not. So this loosely written review reflects my bumpy ride. It sketches some general notions which I find fundamental, not only for scattering but also outside of it. Unfortunately, I understand them only partially. I omit details, because I either find them boring or do not understand them well enough to find their proper place in the system of things. I dedicate this review to Felix Berezin. To his launching me on this tough subject, I owe many exhilarating moments of my scientific journey.

Acknowledgement. With the exception of the last section this review keeps close to parts of the talks on mathematical problems in Quantum Mechanics I have given at MIT, Princeton, ETH and Jerusalem. I am grateful to the audiences for remarks and encouragement. I was lucky to work on the scattering theory in collaboration with Avy Soffer. My understanding of that theory which I will try to convey below is formed by this collaboration.

Problem. Scattering theory studies asymptotic behaviour of the time-dependent Schrödinger equation

$$i\frac{\partial\psi_t}{\partial t} = H\psi_t \tag{1}$$

as $t \to \pm\infty$. Here ψ_t is a differentiable path or orbit in the state space $L^2(X)$, where X is the configuration space of the system in question. Usually, it is $\mathbb{R}^{3N}$ or a subspace thereof corresponding to the center-of-mass frame.

$$H = -\Delta + V(x) ,$$

the Schrödinger operator on $L^2(X)$, with Δ being a Laplace-Beltrami operator on X (the kinetic energy operator) and $V(x)$, a real function on X (the potential). We assume that $V(x)$ is not too singular so that H is self-adjoint (see Kato, 1953).

The self-adjointness of H is equivalent to the existence of unitary dynamics, i.e. the existence of global solutions to the Cauchy problem for (1) satisfying $\|\psi_t\| = \text{const}$ (see [Simon, 1976]). Once this is established the next problem is the classification of solutions (orbits) according to their asymptotic behaviour as $t \to \pm\infty$. It is called *asymptotic* completeness. It is the main mathematical problem of Scattering Theory. It was first attacked just a few years after the birth of Quantum Mechanics. While during the next 60 years there was a rather satisfactory progress on the one-two body problem, the many-body problem alluded researchers and only limited progress was made. However, in the last decades, a remarkable development took place in this area. In this article I sketch the main highlights of this process.

I concentrate on the many body asymptotic completeness. To this end we consider a system consisting of N ν-dimensional particles. The configuration space of such a system in the center-of-mass frame is

$$X \ = \ \{x \in \mathbb{R}^{\nu N} \mid \sum m_i x_i = 0\} \ .$$

Here $x = (x_1, \ldots, x_N)$. We equip X with the inner product $\langle x, y\rangle = \sum_{i=1}^{N} m_i x_i y_i$. The kinematics of many particles in $\mathbb{R}^\nu$ translates into geometry of X. This was understood by many people (see e.g. Zhislin, 1960, Enss, 1977, Simon, 1977, Sigal, 1982, 1987). Passing from kinematic to geometric language, which is crucial in the latest constructions in the many-body scattering theory was suggested by Agmon, 1982. A brief kinematic-geometric dictionary is given below.

Kinematics	**Geometry**
N particles	System, $\mathfrak{X}$, of subspaces of X
$V(x) = \sum V_{ij}(x_i - x_j)$	$V(x) \to 0$ as $\lvert x\rvert \to \infty$, except along the subspaces from $\mathfrak{X}$
$N = 2$	$\mathfrak{X} = \phi$
$N = 3$	Subspaces from $\mathfrak{X}$ intersect only at the origin

Fig. 1

$N \geq 4$	No restrictions (only on generations)

Break-up

Cone in X

subspaces
from $\mathfrak{X}$

Fig. 3. Break-up: physical picture

Fig. 2. Break-up: geometric representation

In the kinematic language asymptotic completeness states that as $|t| \to \infty$, the system in question breaks up into independent, freely moving, stable subsystems. In the geometrical language one says that the system's motion is a superposition of a free motion on a ray starting at the origin (= the free motion of the centers-of-mass of subsystems) and a bounded motion in the transversal direction (stable internal motion of the subsystems):

$Y \in \mathfrak{X}$

(free motion of the center-of-mass)

Fig. 4. Free motion of stable clusters

In rigorous terms, asymptotic completeness states that for any orbit ψ_t (i.e. a solution to (1) with an L^2 initial condition) there are $\varphi_Y \in L^2(Y)$ for all $Y \in \mathfrak{X}$, s.t.

$$\|\psi_t - \sum_{(Y, \psi_{Y^\perp})} \psi_{Y^\perp} \otimes e^{-i(\varepsilon_{Y^\perp} - \Delta_Y)t} \varphi_Y\| \ \to \ 0$$

as $t \to +\infty$ (and similarly for $t \to -\infty$). Here Δ_Y is the Laplace-Beltrami operator on $Y \in \mathfrak{X}$, $\psi_{Y^\perp}$ is an eigenfunction (labeled by ψ!) of $H_{Y^\perp}$ (the "trace of H on the subspace $Y^\perp$", to be defined later) with an eigenvalue $\varepsilon_{Y^\perp}$ and the sum runs over all pairs $(Y, \psi_{Y^\perp})$, where $Y \in \mathfrak{X}$ and $\psi_{Y^\perp}$ is an eigenfunction of $H_{Y^\perp}$. The collection of such pairs can be called the *boundary*, ∂_∞, of X *at* ∞. Thus this boundary is determined by a certain type of geometry of X but also by dynamical data of our system. Now,

$$H_{Y^\perp} \;=\; -\Delta_{Y^\perp} + V_{Y^\perp}$$

on $L^2(Y^\perp)$, where, as before, $\Delta_{Y^\perp}$ is the Laplace-Beltrami operator on $Y^\perp$ and $V_{Y^\perp}(x_{Y^\perp})$ is the limit of $V(x)$ as $|x_Y| \to \infty$ while $x_{Y^\perp}$ is kept fixed. Here x_Y is the projection of x onto Y, etc.

Conditions on the potential. In the kinematic language one imposes restrictions on the pair potentials, say, that for $y \in \mathbb{R}^\nu$ in a neighbourhood of infinity

$$V_{ij}(y) \;=\; O(|y|^{-\mu})$$

for some $\mu > 0$. Behaviour of V_{ij} in a bounded part of $\mathbb{R}^\nu$ is not important as long as H is self-adjoint. In the geometrical language one way to formulate conditions on $V(x)$ is to assume that for each $Y \in \mathfrak{X}$

$$\nabla_Y V(x) \;=\; O(|x|^{-\mu}) \text{ on any closed cone in } Y \backslash \Big(\bigcup_{Z \subsetneq Y} Z \Big)$$

again for some $\mu > 0$.

Result and Brief History. The main result I would like to report is

Theorem. *Let $\mu > \sqrt{3} - 1$. Then asymptotic completeness holds.*

Careful investigation of scattering in quantum systems began with the birth of Quantum Mechanics. The mathematical methods used date back to works of Lord Rayleigh and A. Sommerfeld on wave equation in the 19th and the beginning of the 20th century. Since then Scattering Theory counted among the most active areas among physicists and mathematicians with early important works by W. Heisenberg, J. Schwinger, H. Eckstein, S. Weinberg, W. Hunziker, T. Kato, S. Birman, Ya. Povzner, T. Ikebe and K. Friedrichs, among others. A major breakthrough was made by L.D. Faddeev in 1963, who solved the three body problem. While the one-body problem has undergone a significant development, especially in works of S. Agmon and L. Hörmander, the events in the many-body case took a slow turn and concentrated, with the exception of two papers, K. Hepp, 1969 and R. Lavine, 1973, on the improvement of the three-body result.

The situation changed with the works of V. Enss in 1978, who introduced phase-space (or micro-local in the terminology of PDE's) methods into the scattering theory and of E. Mourre in 1981, who introduced the method of local positive commutators.

Merging these two sets of ideas and developing the method which can be loosely termed as of microlocally positive commutators, I.M. Sigal and A. Soffer, 1987, have proved asymptotic completeness for N-body *short-range* (i.e. with $\mu > 1$) systems. Finally, extending microlocal analysis to include also so-called asymptotic projections, J. Derezinski, 1993 and I.M. Sigal and A. Soffer, 1993b have proved asymptotic completeness for *long-range* systems with $\mu \geq 1$ and $\mu > \sqrt{3} - 1$, respectively (see also L. Zelinski, 1992). Powerful ideas of G.-M. Graf, 1990, and, in the latter case, of D. Yafaev, 1993, played important roles in the proofs.

Propagation Set. Now I will describe the key notion which guides the proof of asymptotic completeness and which I believe can be useful outside the problem at hand. The propagation set plays the role for the Schrödinger equation similar to the role wave front set for the wave equation.

Consider a class of symbols on the space-time $X \times \mathbb{R}$ (endowed into the usual symplectic form $dx \wedge dp - dE \wedge dt$) satisfying the estimates

$$|\partial^{\alpha}_{(x,t)} \partial^{\beta}_{(p,E)} \varphi| \ \leq \ C_{\alpha,\beta} \langle (x,t) \rangle^{-|\alpha|}$$

on any set which is compact in p and E. These are classical symbols in which role of the coordinate and momentum are interchanged. Now for any $\psi \in L^{\infty}\big(\mathbb{R}, L^2(X)\big)$ we define its propagation set by

$$PS(\psi) \ = \bigcap_{\|\phi\psi\|_{L^2(\frac{dx\,dt}{\langle t \rangle})} < \infty} \operatorname{char} \phi \ .$$

Here the intersection is taken over pseudodifferential operators ϕ with symbols described above and satisfying the estimate indicated and $\operatorname{char}\phi$, the characteristic set of ϕ, is the null set of its symbol. The inequality under the intersection sign means that $\|\phi\psi\|^2_{L^2(dx)}$, the probability that ψ is localized in $\operatorname{supp}\varphi$, vanishes in some probabilistic sense as $|t| \to \infty$. In other words, as $|t| \to \infty$, the state ψ concentrates with rarer and rarer exceptions on $PS(\psi)$.

For each point at ∞, $\omega = (Y, \psi_{Y^\perp})$, we introduce the classical Hamiltonian function

$$h_\omega(z) \ = \ \lambda_{Y^\perp} + h_Y(z) \ ,$$

where $z \in T^*Y$, $\lambda_{Y^\perp}$ is the eigenvalue of $H_{Y^\perp}$ corresponding to the eigenfunction $\psi_{Y^\perp}$ and h_Y, the "trace of h along Y", is given by

$$h_Y(z) \ = \ \frac{1}{2}|p_Y|^2 + V_Y(x_Y) \ ,$$

where p_Y and x_Y are projections of p on Y' and x on Y, respectively, and $V_Y(x) = V(x) - V_{Y^\perp}(x)$. Recall that the bicharacteristics of h_ω are just its classical trajectories, i.e. the solutions of the Hamiltonian equations

$$\dot{x} = \frac{\partial h_\omega}{\partial p}, \qquad \dot{p} = -\frac{\partial h_\omega}{\partial x}, \qquad h_\omega = E .$$

Theorem. *Let $\mu > \sqrt{3} - 1$ and let ψ solve (1). Then $PS(\psi)$ consists of asymptotics to the bicharacteristics of h_ω for $\omega \in \partial_\infty$.*

Discussion. The Hamiltonians h_ω, $\omega \in \partial_\infty$, play the role of the principal symbol. These Hamiltonians as well as their bicharacteristics are partially quantized. A bicharacteristic for h_ω with $\omega = (Y, \psi_{Y^\perp})$ describes the free (classical) motion along Y and bounded (quantum) motion in $Y^\perp$ (see Fig. 1).

The motion along partially quantized bicharacteristics is unstable. In a random moments of time system jump from one bicharacteristic onto another. That is why the convergence of $\|\phi\psi\|_{L^2(dx)}$ to zero as $|t| \to \infty$, for the symbol, φ, of ϕ supported outside $PS(\psi)$, is in certain average. We expect that for φ supported on parts of the phase space outside of $PS(\psi)$, through which tunnelling between different bicharacteristics is not ruled out by the energy or similar considerations, there is not uniform decay of $\|\phi\psi\|_{L^2(dx)}$ as $|t| \to \infty$, i.e. the estimate $\|\phi\psi\|_{L^2(\frac{dx\,dt}{\langle t \rangle})} < \infty$ cannot be sharpened.

The theorem above was first proven in I.M. Sigal and A. Soffer, 1987 for $\mu > 0$ but only for the part of the phase-space away from the critical set

$$\bigcup_{\substack{\omega \in \partial_\infty \\ \omega > Y}} \{z \mid \nabla h^Y_\omega(z) = 0\} , \tag{2}$$

where, for $\omega = (Z, \psi_{Z^\perp})$, $\omega > Y$ means $Z \supset Y$ and h^Y_ω stands for $\lambda_{Z^\perp} + \frac{1}{2}|\xi^Y_Z|^2 + I_Z(x^Y_Z)$ with ξ^Y_Z and x^Y_Z standing for the projections of ξ and x on $Z' \ominus Y'$ and on $Z \ominus Y$, respectively. This result sufficed to prove asymptotic completeness for the short-range forces, i.e. for $\mu > 1$. In the long-range case, i.e. for $\mu \leq 1$, one has to control $PS(\psi)$ near the critical set (which corresponds to the case of multiple characteristics in the problem of propagation of singularities). Moreover, near the critical set not only the bicharacteristic motion becomes rather complicated (in particular, not ballistic), but the microlocal nature of problem breaks down, the quantum phenomena begins playing the crucial role.

Asymptotic observables. To deal with these problems one introduces fractional time scales and uses, in addition, asymptotic projection operators (or asymptotic observables) which allow one to localize in the space $T^*X \otimes L^2(X)$, rather than in T^*X or in $T^*(X \times \mathbb{R})$ alone.

I explain briefly what is going on here. To begin with in order for induction in the number of particles stay in the same class of Hamiltonians we consider more general Hamiltonians of the form $H(t) = H + W(x,t)$, where H is an N-particle Schrödinger operator described above and $W(x,t)$ is a real smooth function satisfying

$$|\partial^{\alpha}_{(x,t)} W(x,t)| \ \leq \ C_{\alpha} \langle t \rangle^{-\mu-|\alpha|} \ ,$$

where μ is the same as in the conditions on $V(x)$, though this is not necessary. Let $U(t)$ be the evolution group (from time 0 to time t) generated by $H(t)$. For a family, B, of self-adjoint operators, B_t (= observables), we introduce the asymptotic cut-off functions

$$f^{\pm}(B) \ = \ \lim_{t\to\pm\infty} U(t) f(B_t) U(t)^* \ ,$$

if the limits exist. In particular, we will be interested in the asymptotic cut-off functions for the velocity operator $v = \{|x|/t\}$. Originally, such limits are defined for $f \in C_0^{\infty}$ and then extended by continuity for more general f's which include in particular the characteristic functions, E_{Δ}, of intervals. In this way one can show that $E^{\pm}_{\Delta}(v)$ exist for any interval whose boundary does not contain the point zero. The point now is that the critical set of $H(t)$ (see (2)) can be characterized, adequately for our purposes, by the spectral projection $E^{\pm}_{\{0\}}(v)$. More precisely one shows that if $\psi \in (\mathrm{Ran}\, E^{\pm}_{\{0\}}(v))^{\perp}$, then for any $\varepsilon > 0$ there is ψ_{ε} s.t. $\|\psi - \psi_{\varepsilon}\| \leq \varepsilon$ and $PS(\psi_{\varepsilon})$ is a closed subset of the complement of set (2) in the set of asymptotics of bicharacteristics of h_{ω} for all ω. Moreover, if $\psi \in \mathrm{Ran}\, E^{\pm}_{\{0\}}(v)$, then $U(t)\psi$ is localized, in a rather strong sense in the ball $\{|x| \leq t^{\alpha}\}$ with $\alpha > 2(2+\mu)^{-1}$. This information is sufficient to prove asymptotic completeness of the decay rate, μ, of potentials greater than $\sqrt{3}-1$. Asymptotic observables were used for the quantum mechanical scattering theory in V. Enss, 1983 and R. Lavine, 1971. The characterization of the critical set in terms of asymptotic projections was introduced in I.M. Sigal and A. Soffer, 1989, 1993a for the observable of energy, i.e. H (in this case the relevant asymptotic projection is associated with the threshold set of H) and implicitly for A/t, where A is the dilation generator, and in J. Derezinski, 1993 for the velocity observable x/t. A review of the results up to June 1991 can be found in A. Soffer, 1992.

Recent results. In some parts of the phase-space the notion of the propagation set can be considerably sharpened as was shown in E. Skibstead, 1991 and Ch. Gérard, 1992. This leads to a sharper notion of the propagation set in the tree-body case. Ch. Gérard and I. Łaba, 1993, 1994 have proven asymptotic completeness for quantum many-particle systems placed in homogeneous magnetic fields whose potential decay at the rate $\mu > \sqrt{3}-1$. Asymptotic completeness for particles interacting via hard core potentials was proven by A. Iftimovici, 1993. The latter result is based on a generalization of the

Mourre theory to rough systems due to W. Amrein, A.M. Boutet de Monvel and G. Georgescu, 1988. Some of the ideas of the many-body scattering theory were used and further developed in the existence problem for non-linear Schrödinger equation in A. Soffer, 1993, J. Ginibre, A. Soffer and G. Velo, 1992. For other results and more detailed discussion see A. Soffer 1992 and also I.M. Sigal, 1991.

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