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Mathematical Theory of Radiation

by

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Dedicated to Larry Horwitz on occasion of his 65th birthday.

Abstract

In this paper we present an informal review of our recent work [5-8] whose goal is to develop a mathematical theory of the physical phenomenon of emission and absorption of radiation by systems of non-relativistic matter such as atoms and molecules.

Introduction

In this paper we outline our recent work [5-8] on mathematical theory of emission and absorption of electromagnetic radiation by systems of non-relativistic matter such as atoms and molecules. Related results but for more restrictive models were obtained in [2-4, 16, 21-22, 26-31, 34, 41]. Moreover, these works, except for [2-4], which consider an explicitly solvable model of harmonic oscillator, use explicitly or effectively a fixed infrared cut-off and coupling constrants depending on this cut-off.

In our paper we use three distinct type of techniques: methods of constructive field theory, a method of positive commutators and a renormalization group technique. The first one is an adaption to the new situation of well known methods (see e.g. [18,19,16], the second one is an extention to the field-theoretical context of some ideas originating in

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the theory of Schrödinger operators (see [33,35,15]). While [6] was in preparation more advanced commutator techniques appeared in [22,9]. The third, renormalization group technique is new. Its central characteristic is that the renormalization group transformation is defined directly, without intermiditaries such as (euclidian) propagators, on the equation in question.

We work with the units in which $m_{\rm el} = 1$, $\hbar = 1$ and $c = 1$, so that the electron charge, e, is a dimensionless quantity.

Radiation

The quantized electro-magnetic field (in Coulomb gauge) is described by quantized transverse vector potential

$$A(x) = \int \big(e^{-ik\cdot x}a(k) + h.c.\big)g(k)d^3k,$$

where

$$g(k) = \big(2\omega(k)\big)^{-1/2} \quad \text{and } \omega(k) = |k|,$$

with the dynamics given by the Hamiltonian

$$H_{\rm rad} = \int \omega(k)a^\dagger(k)\cdot a(k)d^3k/(2\pi)^3 \ ,$$

acting on $\mathcal{H}_{\rm rad}$ = Fock space. Here $a(k)$ and $a^\dagger(k)$ are the *transverse* annihilation and creation operators, respectively (i.e. they are operator-valued vector fields on $\mathbf{R}^3$ satisfying $k\cdot a(k) = 0$ and $k \cdot a^\dagger(k) = 0$).

Matter

Quantum, non-relativistic matter is described by a Schrödinger operator of the form

$$H_{\rm part} \ = \ \sum_1^N \frac{1}{2m_j}p_j^2 + V(x)$$

on a Hilbert space $\mathcal{H}_{\rm part}$ (say, $L^2(\mathbf{R}^{3N})$).

Spectrum of $H_{\rm part}$

Typically, the spectrum of $H_{\rm part}$ is of the form (Hunziker-van Winter-Zhislin Theorem)

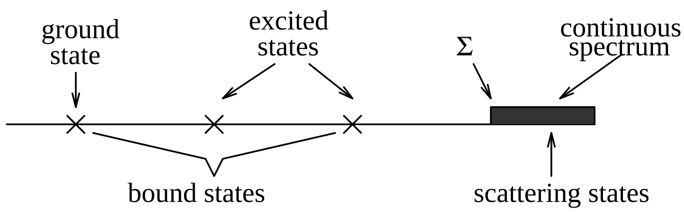

Figure 1.

Matter + Radiation

Think about A as a (quantum) *connection* on $\mathbf{R}^3$ and replace the particle momentum operator by the covariant one:

$$p \to p_A = p - eA(x) \ ,$$

where e is a *particle charge.* As a result one arrives at the *Hamiltonian for the system of matter and radiation*

$$H(e) \ = \ \sum \frac{1}{2m_j} p_{j,A}^2 + V(x) + H_{\rm rad}$$

acting on the Hilbert space $\mathcal{H}_{\rm part} \otimes \mathcal{H}_{\rm rad}$. If we assume ultraviolet cut-off in the form

$$\int \frac{|g|^2}{\omega} < \infty \ ,$$

then H is self-adjoint and is bounded from below:

$$H(e) \ \geq \ \inf {\rm spec}\, H_{\rm part} \ .$$

The *problem of radiation* is to find the fate of the bound states of $H_{\rm part}$ as the interaction between the particles and radiation is introduced, i.e. when the charges are "turned on".

Spectrum and motion

The main mathematical task of a quantum theory described by a Hamiltonian H is to classify the solutions of the Schrödinger equation,

$$i\frac{\partial\psi}{\partial t} = H\psi \ ,$$

according to their temporal behaviour. The key invariant for such a classification is the spectrum of H. In the case of Schrödinger opertors, for instance, the discrete spectrum is in correspondence with bounded (in configuration space) motion (bound states), while the continuous spectrum, with the unbounded one (scattering states) (the Ruelle theorem). However, this classification is not sufficient for our purposes. A central rôle in our analysis is played by states which are localized for long periods of time, but eventually escape to ∞, i.e. are, in fact, unbounded. These are the resonance states or resonances. In other words, the (quantum) *resonances* are states which belong to continuous spectrum subspace, but which behave for long time intervals as eigenfunctions of discrete spectrum. In the problem of radiation of interest here the notion of resonances plays a key role. We define this notion little more carefully in the next section.

Resonances

Extension of the notion of spectrum. For a self-adjoiont operator A on a Hilbert space $\mathcal{H}$ and for any $f, g \in \mathcal{H}$

$$\text{Point spec } = \{ \text{ poles of } \langle f, (z-A)^{-1} g \rangle \}$$

$$\text{Continuous spec } = \{ \text{ cuts of } \langle f, (z-A)^{-1} g \rangle \}$$

Now for a dense set $\mathcal{D} \subset \mathcal{H}$, we consider the Riemann surface, $\mathcal{R}$, of $\langle f, (z-A)^{-1} g \rangle$, $z \in \mathbb{C}^+$, for $f, g \in \mathcal{D}$.

Definition. *Resonances* of $(A, \mathcal{D})$ = *complex* poles of the Riemann surface $\mathcal{R}$.

The study of resonances has a long history which we are unable to reproduce here. The definition above, though the most straightforward, is one of many equivalent definitions. In the spirit of the definition above the resonances were introduced in [39] and studied in [24,25,37,40] among many other papers (see [37] and the papers above for some other references).

Spectrum of $H(0)$

Set the charge (coupling constant) e to zero:

$$H(0) \;=\; H_{\text{part}} \otimes 1_{\text{rad}} + 1_{\text{part}} \otimes H_{\text{rad}} \;.$$

$H(0)$ describes the non-interacting matter and radiation. Its spectrum is

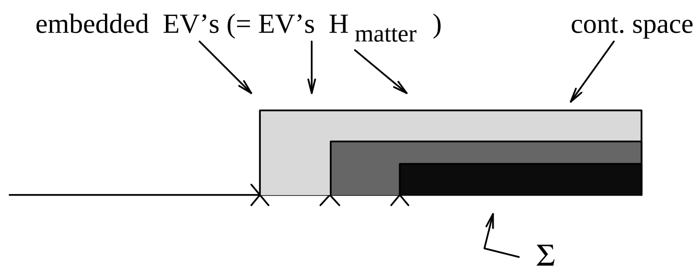

Figure 2.

Rotate this spectrum (take a different projection of the Riemann surface of $\langle f, (z - H(0))^{-1} g\rangle$, $z \in \mathbb{C}^+$, onto the spectral plane):

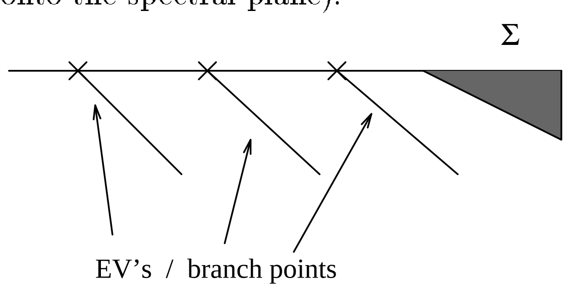

Figure 3.

Thus on the technical level the problem of resonances is the problem of perturbation of eigenvalues of $H(0)$, and of the corresponding eigenfunctions. The main mathematical difficulties are:

(a) even for bounded energy intervals the number of photons is infinite (soft photons), i.e., roughly speaking, we have to deal with partial differential operators in infinitely many variables

(b) the eigenvalues to be perturbed sit on the top of the branch points.

Both problems are features of a fundamental problem of Quantum Physics, the *infrared problem.*

Mathematical Results

Now we formulate in an informal way our main results.

I. **Binding.** $H(e)$ has a ground state (i.e. there is an eigenvalue at the bottom of its spectrum). This ground state, ψ, is exponentially localized in the particle space: for some $\alpha > 0$

$$\|e^{\alpha|x|}\psi\| < \infty \ ,$$

where x stands for the particle coordinates.

II. **Instability of Excited States.** For e sufficiently small, $H(e)$ has no other eigenvalues, besides the ground state one. Unstable states have *life-times* predicted by the Fermi Golden Rule.

III. **Resonances**. Assume e is sufficientlly small. Then while the ground state of H_{part} yields the ground state of $H(e)$, the excited states of H_{part} bifurcate into resonances of $H(e)$. The Riemann surface of $\langle f, (z - H(e))^{-1} g\rangle$ has the following structure

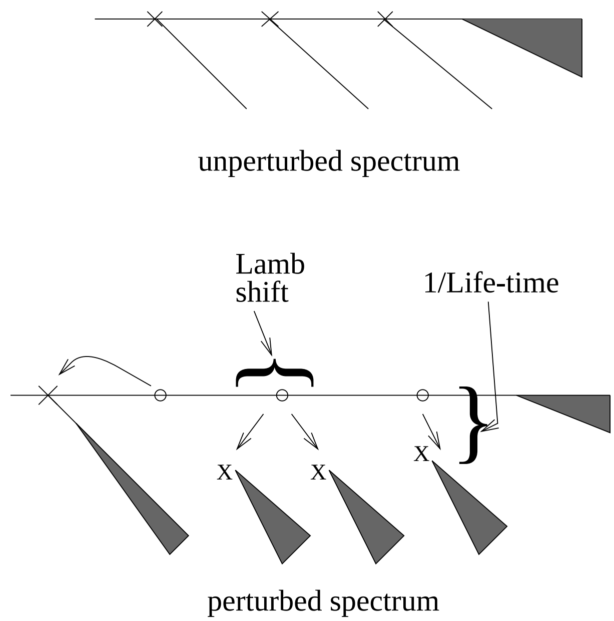

Figure 4.

Physical Picture

Since the photon mass is zero, even tiny energy fluctuations allowed by the uncertainty principle ($\Delta E \Delta t \geq \hbar$) produce an infinite number of photons (soft photons). Thus even in the vacuum an atom is surrounded by a cloud of photons born out of this vacuum, interacting with the atom and vanishing back into the vacuum. This cloud changes (renormalizes) the properties the atom would have if there were no radiation in Nature at all. This change of the atomic characteristics was first demonstrated by Lamb and Retherford in 1951 and is known under the name of the *Lamb shift*.

In the remaining sections we outline some of the ideas entering into our analysis.

Riemann Surface of $H(e)$

Fact: There are $\epsilon > 0$ and a family $H(e, \theta)$, analytic in $|\theta| \leq \epsilon$, s.t. the Riemann surface of $H(e)$ is determined by the spectrum of $H(e, \theta)$, $\text{Im}\theta > 0$. In particular, the complex eigenvalues of $H(e, \theta)$ are independent of θ and occur at the poles of this Riemann surface, i.e.

$$\{\text{resonances of } H(e)\}$$
$$= \{\text{complex EV}'\text{s of } H(e,\theta)\}$$

($H(e,\theta)$ is a complex deformation of $H(e)$.)

Thus the problem of finding the resonances of $H(e)$, or more generally its Riemann surface, is reduced to the one of understanding the spectrum of $H(e,\theta)$ for $\operatorname{Im}\theta > 0$.

Renormalization Group Analysis

Let $H(e,\theta)$ be a complex deformation of $H(e)$, determining the Riemann surface of $H(e)$. Assume we are interested in the fate of an eigenvalue E_j of H_{part} and in the spectrum of $H(e,\theta)$ near E_j. Let P_{part} by the projection on the eigenspace corresponding to an eigenvalue E_j of the operator $H_{\text{part}}(\theta)$, which is a complex deformation of H_{part}. We define $e(\rho)$, the *effective charge* at the photon energy scale ρ, by

$$e(\rho) = e \cdot \left(\rho^{-1} \int_{\omega \leq \rho} \frac{|g|^2}{\omega}\right)^{\frac{1}{2}} .$$

For any ρ, $1 \geq \rho > 0$, we define a transformation $\mathcal{R}_\rho$ on operators on $\mathcal{H}_{\text{part}} \otimes \mathcal{H}_{\text{rad}}$. This transformation has the following properties:

(i) $\mathcal{R}_\rho$ is isospectral in the sense that

$$0 \in \sigma_\#(H) \leftrightarrow 0 \in \sigma_\#(\mathcal{R}_\rho(H)) ,$$

where $\sigma_\#(A)$ stands for either point or continuous spectrum of A.

(ii) $\mathcal{R}_\rho(P_{\text{part}} \otimes wH_{\text{rad}}) = P_{\text{part}} \otimes wH_{\text{rad}}$ for all $w \in \mathbb{C}$, i.e. $\mathcal{R}_\rho$ has a complex line $P_{\text{part}} \otimes \mathbb{C}H_{\text{rad}}$ of fixed points.

(iii) $\mathcal{R}_\rho(P_{\text{part}} \otimes E \cdot \mathbf{1}_{\text{rad}}) = P_{\text{part}} \otimes \frac{1}{\rho}E \cdot \mathbf{1}_{\text{rad}}$ for any $E \in \mathbb{C}$, i.e. $P_{\text{part}} \otimes \mathbb{C} \cdot id$ is an unstable manifold for the manifold $P_{\text{part}} \otimes \mathbb{C}H_{\text{rad}}$ of fixed points w.r. to $\mathcal{R}_\rho$

(iv) $\exists$ a Banach space, $\mathcal{B}$, of operators on $\mathcal{H}_{\rm rad}$ s.t. $\forall H \in \mathcal{B}$ and sufficiently close to $\mathbb{C}H_{\rm rad}$ $\exists\ E \in \mathbb{C}$ s.t. $\mathcal{R}_\rho\ (P_{\rm part} \otimes (H - E \cdot \mathbf{1}_{\rm rad})) \to P_{\rm part} \otimes wH_{\rm rad}$ for some $w \in \mathbb{C}$, i.e. a neighbourhood of $P_{\rm part} \otimes \mathbb{C}H_{\rm rad}$ is spanned by the unstable manifold $P_{\rm part} \otimes \mathbb{C}\cdot \mathbf{1}_{\rm rad}$ and a stable manifold.

(v) For $\rho \gg e^2$, $\mathcal{R}_\rho(H(e,\theta) - z) \in P_{\rm part} \otimes \mathcal{B}$ for any $|z| \le \Delta E$, where $\Delta E = \frac{1}{2}{\rm dist}\big(E_j, \sigma(H_{\rm part} \backslash \{E_j\})\big)$.

(vi) $\mathcal{R}_{\rho_1} \circ \mathcal{R}_{\rho_2} = \mathcal{R}_{\rho_1\rho_2}$ (i.e. $\mathcal{R}_\rho$ is a semigroup).

Note that properties (iv)-(v) imply that for $H(e,\theta)$ there are complex numbers $E_j(e)$ and $w(e,\theta)$ s.t.

$$\mathcal{R}_\rho(H(e,\theta) - E_j(e)) \to P_{\rm part} \otimes w(e,\theta)\ H_{\rm rad}$$

as $\rho \to 0$. In fact, we can prove more:

(a) $w(e,\theta) = w(e)e^{-\theta}$, $E_j(e)$ and $w(e)$ are independent of θ and

$$\mathcal{R}_\rho(H(e,\theta) - z) = w(e)e^{-\theta}H_{\rm rad} + O(\rho e(\rho))$$

for every $z \in \mathcal{D}_\rho$ where $\mathcal{D}_\rho := \{z \in \mathbb{C} \big| |\langle \mathcal{R}_\rho(H(e,\theta) - z)\rangle| \le \frac{1}{2}\rho\}$, with the expectation taken with a state $\psi_{\rm part} \otimes \Omega$, $\psi_{\rm part}$ = any normalized vector from ${\rm Ran}P_{\rm part}$ and Ω = the vacuum vector in $\mathcal{H}_{\rm rad}$,

(b) besides the obvious relation $\mathcal{D}_{\rho'} \subset \mathcal{D}_\rho$ if $\rho' < \rho$, the domains $\mathcal{D}_\rho$ satisfy

$$\mathcal{D}_\rho \supset \mathcal{D}(E_j(e), \frac{1}{5}\rho)$$

and

$$\bigcap_{\rho>0} \mathcal{D}_\rho = \{E_j(e)\}$$

where $\mathcal{D}(E,\rho) = \{z \in \mathbb{C} \big| |z - E| \le \rho\}$,

(c) $E(e)$ is an eigenvalue of $H(e,\theta)$,

(d) In D_ρ, $H(e,\theta)$ is isospectral to

$$P_{\rm part} \otimes w(e)e^{-\theta}H_{\rm rad} + E_j(e)$$

modulo $O(\rho e(\rho))$ (infrared asymptotic freedom). See Fig. 5.

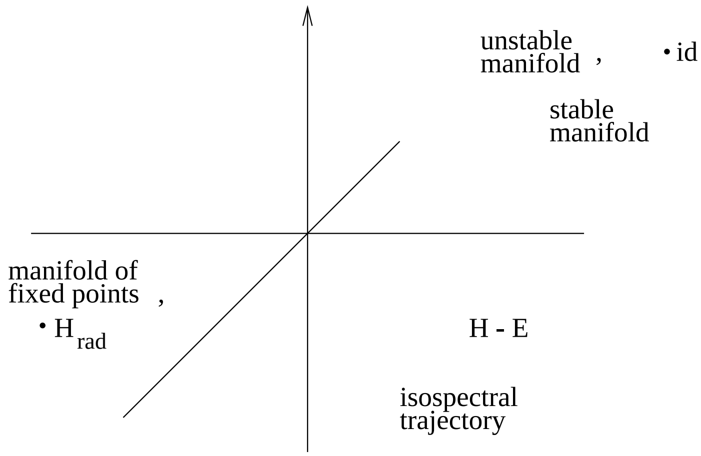

Figure 5.

We do not specify here the space, $\mathcal{B}$, of quantum Hamiltonians on which $\mathcal{R}_\rho$ acts. It can be found in [5-7]. Instead, we write out the explicit definition of $\mathcal{R}_\rho$. Let $\chi_{H_f \leq \rho}$ be the spectral projection for H_f, corresponding to the interval $[0, \rho]$, and let $P_\rho = P_{\mathrm{part}} \otimes \chi_{H_f \leq \rho}$ and $\overline{P}_\rho = \mathbf{1} - P_\rho$. We define

$$\mathcal{R}_\rho = E_\rho \circ S_\rho \circ F_\rho \ ,$$

where $E_\rho(H) = \frac{1}{\rho} H$,

$$F_\rho(H) = P_\rho(H - HR_{\geq\rho}H)P_\rho \ ,$$

$$R_{\geq\rho} = \overline{P}_\rho(\overline{P}_\rho H \overline{P}_\rho)^{-1}\overline{P}_\rho$$

(cf. [20]) and $S_\rho(H) = e^{-A\ln\rho} H e^{iA\ln\rho}$ with

$$A = \frac{i}{2}\int a^*(k)(k \cdot \nabla_k + \nabla_k \cdot k)a(k) d^3k \ .$$

For the procedure above to work one needs that $e(\rho) \to 0$ as $\rho \to 0$ or, more precisely, that, $\int \frac{|g|^2}{\omega^2} < \infty$. This fails in the infrared region if $g = \mathrm{const.}\omega^{-1/2}$, as $\omega \to 0$, as in the case of interest. There are several ways to relax this condition. One is to compose $\mathcal{R}_\rho$ with a Bogolubov transform. The latter is chosen to remove the terms in the effective potential which are linear in $a^\#$ (the marginal terms leading to the restriction $\int \frac{|g|^2}{\omega^2} < \infty$). The other is to apply $\mathcal{R}_\rho$ not to $H(e,\theta)$ but to $H^{PF}(e,\theta)$, the complex deformation of $e^{-ieA(0)\cdot\sum x_j} H(e) e^{ieA(0)\cdot\sum x_j}$ (the result of applying the Pauli-Fierz transformation to $H(e)$). The Pauli-Fierz transform improves the infrared behaviour of coupling functions, $g(k)$, at the expense of a large x behaviour. The latter, however, is easier to control (see [6-8] for details).

Conclusion

The results presented above describe the states of non-relativistic matter interacting with quantized electro-magnestic field. We show that such a system has only one *stable state* – at the bottom of its energy spectrum. This state is exponentially localized in the particle coordinates. The system has a series of unstable but long-living states-*resonances*. These unstable states are the ones that control the process of emission and absorption of radiation.

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