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RETURN TO EQUILIBRIUM

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Abstract

We consider a system of quantum particles in a confining potential coupled to a thermal reservoir at a temperature $T \geq 0$. The reservoir contains black-body radiation of photons or phonons described by a massless free field Hamiltonian. We show that, starting at any normal state associated to the temperature T, the total system converges to equilibrium as $t \to +\infty$. The latter is described by a KMS state at the same temperature.

1. Introduction

In this paper we study the problem of *dissipation* in quantum mechanics. The problem is to analyze the behaviour of a small system, i.e., of a system with a finite number of degrees of freedom, called the *particle system,* coupled to a system of an infinitely many degrees of freedom, called the *reservoir.* One would like to show that, starting with an arbitrary state of the particle system and an equilibrium state of the reservoir, the coupled system converges to an equilibrium state of the total system. The latter state must have the same temperature as the reservoir. The system is then said to *return to* (or *approach*) *equilibrium.*

In this paper we prove return to equilibrium for all temperatures (in fact, uniformly in temperature), for arbitrary particle systems in confined potentials (see Eqn (2.2) for a technical definition) and for reservoirs consisting of either photons (quantized electromagnetic field) or free phonons (quantized oscillations of crystal lattices). Though our confinement condition, Eqn (2.2), can, probably, be weakened, it cannot be removed altogether: a particle system with a continuous spectrum (besides the discrete one) will be "ionized" by an interaction with the reservoir and would, presumably, be committed to a

Brownian motion rather than approach equilibrium.

The programme of mathematical understanding the phenomenon of dissipation, or friction, in classical and quantum systems was formulated by Tom Spencer and the last two authors, several years ago. It was successfully studied by V. Jakšič and C.-A. Pillet [JP1-5] who, in particular, have shown that systems with finite number of states coupled to a free phonon reservoir approach equilibrium at sufficiently high temperature, provided the interaction satisfies a certain rather rigid condition (see [Am, Ar1,2, D1,2, FNV1,2,3, HS1,2, Ma, Pu, Ro1,2, SDLL, SpD] for earlier results). Moreover, these authors for the first time have systematically treated the problem of return to equilibrium as a spectral problem for the Liouvillian operator (or equilibrium Hamiltonian, in the terminology of [Ro1]). They identified the latter problem with a problem of resonances, and applied spectral deformation techniques to tackle it. The spectral deformation used was a complex translation in the length of the reservoir wave vector. It is the choice of this transformation that caused the rigid requirement on the interaction term mentioned above.

In this paper we adopt the spectral approach of [JP3] and combine it with the methods of complex rotation and the decimation transformation and the renormalization group developed in our earlier work [BFS1-4] on a related subject. In conclusion of this review we mention especially one feature of our mathematical picture — it is continuous with regard to the temperature, T, even at $T = 0$.

The paper is organized as follows. In Sections 2 and 3 we describe the physical models we treat and in Section 4 we formulate the problem of return to equilibrium and present our main results. In Section 5 we reduce the problem of return to equilibrium to a spectral problem for the corresponding Liouvillian operator, L. In Sections 6–12 we review the construction of L. This construction as well as the result of Section 5 is due to [JP3]. In Section 13, we identify the domain of L and establish some relative bounds on the interaction part. In Section 14 we introduce our spectral deformation transformation and study the deformed Liouvillian operator. In Section 15 we review the definition and properties of the decimation map and in Section 16 we apply it to the deformed Liouvillian operator. Results of the latter section are used in Section 17 in order to obtain bounds on its resolvent. The latter bounds are used in Section 18 to study the evolution generated by the Liouvillian operator and to actually prove our main results in Section 19. Certain explicit computations related to the KMS states and Fermi Golden Rule are carried out in the appendices.

Units. All the quantities appearing in this paper are dimensionless. In the case of photons, the particle coordinates, x, are measured in units of the Bohr radius $r_{bohr} := \frac{\hbar^2}{me^2}$, the photon wave vector, k, in units of $\frac{e^2}{\hbar c}\frac{me^2}{\hbar^2}$, and the energy in units of $\frac{me^4}{\hbar^2} = \left(\frac{e^2}{\hbar c}\right)^2 mc^2$, which is twice the ionization energy of the ground state of the hydrogen, i.e., 2 Rydberg. Here m and e are the electron mass and charge, respectively. In these units the perturbation parameter is $\lambda = \sqrt{\alpha^3 K}$, where $\alpha = \frac{e^2}{\hbar c}$, the fine-structure constant, and K is the ultraviolet cut-off in the interaction term. The physical value of α is $\approx \frac{1}{137}$, but it will be treated as a small, dimensionless parameter.

In what follows $D_f(A)$ stands for the form domain of an operator A.

2. Hamiltonian

In this section we describe the Hamiltonian of a confined particle system coupled to either photons or phonons. The nature of the particle system is inessential for the considerations below, and therefore we do not specify it. All that matters is that it is described by a Schrödinger operator

$$H^p = \sum_{j=1}^{N} \frac{1}{2m_j} p_j^2 + V(x) \tag{2.1}$$

in a box $B = [0, R]^{3N}$ with Dirichlet or periodic boundary conditions. Thus H^p acts on the state space $\mathcal{H}^p_\infty := L^2(B)$. Here m_j, x_j and $p_j = -i\nabla_{x_j}$ are the mass, position and momentum of the j-th particle ($j = 1, \ldots, N$) and $x = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$. Moreover, the potential $V(x) : \mathbb{R}^{3N} \to \mathbb{R}$, is assumed to be of the Kato-type. Next we make a rather technical assumption:

$$\min_{i \neq j} |E_i - E_j| > 0 \ , \qquad \text{where} \qquad \{E_i\}_0^\infty = \sigma(H^p) \ . \tag{2.2}$$

Above, $p^2 = \sum_{j=1}^{N} \frac{1}{m_j} p_j^2$, and the superindex p stands the "particle" and the subindex $\beta = \infty$, for the inverse temperature (then ∞ reflects the fact that for the moment we consider the zero temperature case).

The reservoir is described by the massless free field Hamiltonian

$$H^r = \int \omega(k) a^*(k) a(k) d^3k \ ,$$

acting on the Boson Fock space $\mathcal{H}^r_\infty := \mathcal{F}$ describing photon/phonon states. Here $\omega(k) = |k|$, the photon/phonon dispersion law, and $a^*(k)$ and $a(k)$ are photon/phonon creation and annihilation operators acting on $\mathcal{H}^r_\infty$. Of course, for photons, we deal with transverse vector bosons, and, phonons, with scalar bosons. Hence the Fock space and the creation and annihilation operators should be interpreted accordingly in each of these cases. In the photon case the basic field is the quantized transverse vector potential,

$$A_0(y) = \int \frac{d^3k}{\sqrt{2\omega}} (e^{-ik\cdot y} a^*(k) + h.c.) \ , \tag{2.5}$$

with $k \cdot a^*(k) = 0$. In the phonon case we denote the basic field as

$$\varphi_0(y) = \int \frac{d^3k}{\sqrt{2\omega}} (e^{-ik\cdot y} a^*(k) + h.c.) \ . \tag{2.6}$$

We also introduce an ultraviolet cut-off and rescaling of size of A' as

$$A' = \check{\chi} * A_0 \quad \text{and} \quad \varphi = \check{\chi} * \varphi_0 \ , \tag{2.7}$$

where $\chi = \chi(k)$ is a C^∞ function of the size $K^{-1/2}$ and decaying sufficiently fast on the scale $K > 0$, i.e., $\chi(k)$ is of of the form $K^{-1/2}\chi_0(k/K)$ for some fixed function χ_0 with appropriate properties. We require χ to be dilation analytic in the sense that $\chi(e^\theta k)$ has an analytic continuation into a disc $\{z \in \mathbb{C} | |z| \leq \theta_0\}$, for some θ_0, as a vector function

$$\theta \to \chi(e^\theta \cdot) \in L^2\left(\frac{d^3k}{\omega^2}\right) \ .$$

Note that under our assumptions, the integral $\int \frac{|\chi(e^\theta(k)|^2}{\omega^2} d^3k$ is independent of K. This is the desired ultraviolet behaviour.

Now we are ready to introduce the Hamiltonian of the particle system coupled to the reservoir:

$$H'(\lambda) = \sum_{j=1}^N \frac{1}{2m_j}(p_{j,A'})^2 + V(x) + H^r \ , \tag{2.8}$$

where $p_{j,A} = -i\nabla_{x_j} - \lambda A'(x_j)$, for photons, and

$$H_\lambda = H^p + H^r + \sum_1^N \varphi(x_j) \tag{2.9}$$

for phonons. Both Hamiltonians act on the space

$$\mathcal{H}_\infty = \mathcal{H}^p_\infty \otimes \mathcal{H}^r_\infty \ , \tag{2.10}$$

but in each case $\mathcal{H}^r_\infty$ has a somewhat different interpretation as described above. In Eqns (2.8) and (2.9) we omitted various identity operators, e.g. the symbol H^r really stands for $\mathbf{1}^p_\infty \otimes H^r$, where $\mathbf{1}^p_\infty$ is the identity operator on $\mathcal{H}^p_\infty$, etc.

Under the conditions on χ and $V(x)$ mentioned above the operators $H'(\lambda)$ and H_λ are self-adjoint for all λ, H_λ on $D(H_\lambda) = D(H_0)$, while for $H'(\lambda)$ the domain is not identified. The relation $D(H'(\lambda)) = D(H'(0))$ is proven for sufficiently small $|\lambda|$ (see [BFS4]). In what follows we assume that $|\lambda|$ is sufficiently small so that $H'(\lambda)$ and H_λ are self-adjoint on $D(H'(0))$ and $D(H_0)$, respectively.

3. Pauli-Fierz transformation

For technical reasons Hamiltonian (2.8) is not suitable for analysis when one passes to positive temperatures. To pass to a more convenient Hamiltonian we use the Pauli-Fierz transform (as in [BFS2]):

$$H(\lambda) := e^{-i\lambda\overline{x}\cdot\overline{A}(0)} H'(\lambda) e^{i\lambda\overline{x}\cdot\overline{A}(0)} \ , \tag{3.1}$$

where $\overline{x} = \sum_1^N x_j$ and $\overline{A}(y) = \check{\chi}_1 * A'(y)$ with $\chi_1(k)$, a smooth fast decaying at ∞ function independent of K and satisfying $\chi_1(0) = 1$ and $\chi_1'(0) = 0$. We also assume that $\theta \to \chi_1(e^{-\theta}\cdot) \in L^\infty$ is analytic in $|\theta| < \theta_0$. Let $g := \frac{\chi\chi_1}{\sqrt{2\omega}}$. To simplify notation we consider only the one particle case, $N = 1$. The r.h.s. is explicitly evaluated in the following

Proposition 3.1. *The operator $H(\lambda)$ is of the form*

$$\begin{aligned} H(\lambda) \ &= \ \sum \frac{1}{2m_j}\big(p - \lambda A(x)\big)^2 + V(x) \otimes \mathbf{1}^r_\infty \\ &\quad + \mathbf{1}^p_\infty \otimes H_f - \lambda x \cdot E + \frac{2}{3}\lambda^2 \int |g|^2 \omega |x|^2 \ , \end{aligned} \tag{3.2}$$

where $A(y) = A'(y) - \overline{A}(0)$ and E is the electric field at $x = 0$:

$$E = \int (ia - ia^+)\omega g \ . \tag{3.3}$$

Proof. Due to the property $[A'(y), A'(z)] = 0$, we have

$$e^{-i\lambda\overline{x}\overline{A}(0)}\big(p_j - A'(x_j)\big)e^{i\lambda\overline{x}\overline{A}(0)} = p_j - \lambda A(x_j) \ . \tag{3.4}$$

Next, we compute

$$e^{-i\lambda\overline{x}\overline{A}(0)} a e^{i\lambda\overline{x}\overline{A}(0)} = a + i\lambda\overline{x}g \tag{3.5}$$

and therefore, using the definition of E,

$$\begin{aligned} & e^{-i\lambda\overline{x}\overline{A}(0)} H^r e^{i\lambda\overline{x}\overline{A}(0)} \\ &= \int (a + i\lambda\overline{x}^{\perp}g)^*\omega(a + i\overline{x}^{\perp}g) \\ &= H^r - \lambda\overline{x}E + K \ , \end{aligned} \tag{3.6}$$

where $\overline{x}^{\perp}$ is the projection of $\overline{x}$ onto the plane $k^{\perp}$ and

$$K = \int (\lambda\overline{x}^{\perp})^2 |g|^2 \omega \ .$$

Using that $|y^{\perp}|^2 = |y|^2 - (\hat{k}\cdot y)^2$, where $\hat{k} = k/|k|$, and that

$$\begin{aligned} \int (\hat{k}\cdot q)^2 d\Omega_k &= 2\pi |q|^2 \int_0^{\pi} \cos^2\theta \sin\theta d\theta \\ &= \frac{4\pi|q|^2}{3} \ , \end{aligned}$$

where $d\Omega_k$ is the surface measure on the unit sphere $|k| = 1$, we obtain

$$K = \frac{2}{3}\lambda^2|\overline{x}|^2 \int |g|^2\omega \ . \tag{3.7}$$

(Remember that the integrals without limits and/or measure of intergration stand for integrals over $\mathbb{R}^3$.) Collecting all the relations above, we arrive at (3.2). $\square$

The advantage of $H(\lambda)$ compared to $H'(\lambda)$ lies in the fact that the new coupling functions

$$G_y = G'_y - G'_0 \tag{3.8}$$

replacing the old one $G'_y(k) := e^{iky}\chi(k)/\sqrt{2\omega(k)}$, have much better infrared behaviour.

Observe that Eqn. (3.6) implies that $\pm\lambda x \cdot E \leq H_f + K$, which, in turn, yields that

$$\|(H_f + K)^{-1/2}\lambda x \cdot E(H_f + K)^{-1/2}\| \leq 1 \ . \tag{3.9}$$

In what follows H *stands for either* $H(\lambda)$ *or* H_λ whenever the expression in question is valid for both operators. Also whenever we deal with an abstract situation H stands for a general self-adjoint operator. Furthermore, $H^{(0)}$ will stand for either $H(0)$ or H_0 and v, for either

$$\lambda \sum_{i=1}^{N} \left[-\frac{1}{m_j} p_j \cdot A(x_j) + \frac{\lambda}{2m_j} A(x_j)^2 - x_j \cdot E + \frac{2}{3}\lambda |x_j|^2 \int |G_0|^2 \omega \right] \tag{3.10}$$

in the photon case, or

$$\lambda \sum_{1}^{N} \varphi(x_j) \tag{3.11}$$

in the phonon case.

In order to simplify notation we assume $N = 1$ and treat only the phonon case.

This completes the description of the Hamiltonians of a particle system coupled to a reservoir. It provides a mathematical framework for understanding dynamics of the total system at *zero temperature*, $T = 0$. The convergence to equilibrium in this case means that

$$e^{iE_0 t}\langle \psi, e^{-iHt}\psi \rangle \to \left|\langle \psi_0, \psi \rangle\right|^2 \text{ as } t \to \infty$$

$\forall \psi \in \mathcal{H}_\infty$. Here ψ_0 is the ground state of H and E_0, its energy.

4. Positive temperatures: Mathematical framework and results

A mathematical framework for treating quantum systems at positive densities is given by a pair $(\mathcal{A}, \alpha)$. Here $\mathcal{A}$ is a $*$ algebra of observables and $\alpha : t \to \alpha_t$ is a group of $*$ automorphisms of $\mathcal{A}$ (evolution). $\mathcal{A}$ is usually chosen to be a von Neumann algebra. (See [BR1, HHW].) In our case, $\mathcal{A}$ is generated by operators $B \otimes e^{i\varphi(f)}$ acting on $\mathcal{H}_\infty = \mathcal{H}^p_\infty \otimes \mathcal{H}^r_\infty$, where B is a bounded operator on $L^2(\mathbb{R}^{3N})$ and $\varphi(f) \equiv \int \varphi(x) f(x) d^3x$ with $f \in L^2(\mathbb{R}^3; \mathbb{R})$, in the case phonons and similar operators for the case of photons; and

$$\alpha_t(A) = e^{iHt} A e^{-iHt} \qquad \forall A \in \mathcal{A} \ . \tag{4.1}$$

States of infinite systems are given by positive functionals, ω', on $\mathcal{A}$, normalized as $\omega'(\mathbf{1}) = 1$. A state ω, is said to be an *equilibrium state* if it is invariant under α, i.e. $\omega \circ \alpha_t = \omega$. If a Hamiltonian $\tilde{H}$ is s.t. the operator $e^{-\beta \tilde{H}}$ is trace class, like in the case of a particle system in a confining potential, then the Gibbs states

$$\tilde{\omega}(A) = \mathrm{tr}(A\tilde{\rho}) \ ,$$

given by the density matrices $\tilde{\rho} = e^{-\beta \tilde{H}} / \mathrm{tr}(e^{-\beta \tilde{H}})$, yield examples of equilibrium states.

Two key problems about infinite systems are

Problem #1. Find all equilibrium states for a given system;

Problem #2. Determine stability properties of these states.

In particular, one would like to show that all equilibrium states are the KMS - states and that all KMS - states are stable in the sense that given a KMS state ω we have

$$\omega' \circ \alpha_t \to \omega \text{ as } t \to +\infty \tag{4.2}$$

for any state ω' normal w.r to ω. If the latter relation holds we say that a system in question *approaches the equilibrium* (at the temperature determined by ω).

Definitions of the KMS and normal states are given in Supplement I (see also Section 5, Eqn. (5.5)). Here we only mention that the KMS states for massless particles are one-parameter families of states satisfying certain periodicity condition. The parameter in question is called the *temperature*. Such states are usually constructed as limits of Gibbs states of suitable approximations of a given system. Normal states associated with a given state ω are in some sense are local perturbations of ω. In our case examples of normal states associated with the KMS state ω are

$$\omega^p \otimes \omega^r \ , \tag{4.3}$$

where ω^p is any state (i.e., a density matrix) of the particle system and ω^r is the KMS equilibrium state of the reservoir at the same temperature. From now on we fix a temperature $T > 0$ and do not display it in the notation. One expectes that for a large class of systems all equilibrium states are the KMS states.

As $T \to 0$, i.e., $\beta = 1/T \to \infty$,

$$\omega(A) \to \omega_\infty(A) := \langle \psi_0, A\psi_0 \rangle \ , \tag{4.4}$$

where, recall, ψ_0 is the ground state of H. The normal w.r.t. ω_∞ states are given by

$$\omega_\rho(A) := \mathrm{tr}(A\rho) \tag{4.5}$$

for some positive trace-class operator ρ on $\mathcal{H}_\infty$ s.t. $\mathrm{tr}\rho = 1$. Thus definition (4.2) incorporates also the notion of convergence to equilibrium for zero temperature, $T = 0$.

Existence of equilibrium states is established in the following

Theorem 4.1. *For $|\lambda|$ sufficiently small and for every temperature $T > 0$, there is a KMS state, ω, of the entire system at this temperature unique among normal states w.r. to $\omega^{(0)} := \omega^p \otimes \omega^r$, where ω^p is the Gibbs state of the particle system and ω^r is the KMS state of the reservoir at temperature T.*

The existence proof, for any λ, is given in Appendix I (see also Theorem 11.1). The uniqueness statement follows from Theorem 11.1 and Corollary 16.4.

The main result of this paper is

Theorem 4.2. *Under the assumptions given in Section 2, each of the two systems described in Section 2 converges to equilibrium for any given temperature.*

This theorem states in particular that starting at any state of the particle system and an equilibrium state of the reservoir the state of the total system converges to the equilibrium state at the temperature of the reservoir.

5. Reduction to a spectral problem

The goal of this section is to derive the property of approach to equilibrium from spectral properties of a certain operator – the Liouville operator – giving an alternative description of dynamics at positive temperatures. To this end we construct a Hilbert space, $\mathcal{H}$, a representation, π, of $\mathcal{A}$ in the von Neumann algebra of bounded operators on $\mathcal{H}$ and self-adjoint operator L on $\mathcal{H}$ s.t. $\pi(\mathcal{A})\psi$ is dense in $\mathcal{H}$ $\forall \psi \in \mathcal{H}$ and

$$\pi(\alpha_t(A)) = e^{iLt}\pi(A)e^{-iLt} \ , \tag{5.1}$$

$\forall A \in \mathcal{A}$. Moreover, we construct conjugate representation, π', of $\mathcal{A}$ on $\mathcal{H}$ s.t. $\pi'(\mathcal{A})$ commutes with $\pi(\mathcal{A})$:

$$[\pi'(A)\ ,\ \pi(B)] = 0 \quad \forall A\ ,\ B \in \mathcal{A}\ ,$$

s.t. $\pi'(\mathcal{A})\psi$ is dense in $\mathcal{H}$ $\forall \psi \in \mathcal{H}$ and s.t. for this representation we also have

$$\pi'(\alpha_t(A)) = e^{iLt}\pi'(A)e^{-iLt} \qquad \forall A \in \mathcal{A}\ . \tag{5.2}$$

Once such a construction is given, the normal (w.r. to π) states are those states which are given by density matrices, i.e.,

$$\omega'(A) = \mathrm{tr}(\rho\pi(A)) \tag{5.3}$$

for all $A \in \mathcal{A}$ and some trace class, positive operator ρ on $\mathcal{H}$ s.t. $\mathrm{tr}\rho = 1$

Finally, in what follows P_ψ stands for an orthogonal projection onto $\psi \in \mathcal{H}_\beta$ and $\overline{P}_\psi = \mathbf{1} - P_\psi$.

Using the standard spectral analysis we derive the following

Proposition 5.1. *(i) If L has only one eigenvector, Ω, and this eigenvector corresponds to the zero eigenvalue, then the system converges to equilibrium, in the sense of the ergodic mean convergence:*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \omega'(\alpha_t(A))dt = \omega(A)\ , \tag{5.4}$$

for any $A \in \mathcal{A}$ and any normal w.r. to π state ω'. Here $\omega(A) := \langle \Omega, \pi(A)\Omega\rangle$, an equilibrium state. (One also says that the system is ergodic.) (ii) If, moreover, the spectrum of L on $\Omega^\perp$ is absolutely continuous then the system has the convergence to equilibrium property.

Proof. Clearly it suffices to prove the proposition for normal states determined by density matrices which are rank one projections (pure states). Since $\pi'(\mathcal{A})\Omega$ is dense in $\mathcal{H}$, it therefore suffices to prove the proposition for states of the form

$$\begin{aligned}\omega'(A) &\equiv \langle B\Omega, \pi(A)B\Omega\rangle \\ &= \langle \Omega, \pi(A)B^*B\Omega\rangle \end{aligned}\tag{5.5}$$

where $B \in \pi'(\mathcal{A})$, s.t. $\|B\Omega\| = 1$. For such states we have due to (5.1) and the relation $L\Omega = 0$,

$$\omega'(\alpha_t(A)) = \langle \Omega, \pi(A) e^{-iLt} B^* B \Omega \rangle \ . \tag{5.6}$$

By the condition in (i), L has only one simple eigenvalue at 0. In the case (ii), the rest of its spectrum is, in addition, absolutely continuous. Hence

$$e^{-iLt} \to P_\Omega \quad \text{as} \quad t \to \infty \tag{5.7}$$

in the sense of ergodic mean convergence in case (i) and the usual sense in case (ii). The last two relations imply the statement of the proposition. $\square$

Remark 5.2. It is clear from the proof above that one can estimate the rate of convergence in (4.3) if one has information about smoothness of

$$(L - \lambda - i0)^{-1} \overline{P}_\Omega$$

in some average sense. Existence of derivatives in λ of those boundary values would imply a power law convergence in (4.3) while analyticity in a strip around $\mathbb{R}$ would yield exponential bounds. We do not develop the corresponding abstract theory here but leave these questions to other publications.

Our programme is to prove the approach to equilibrium constructing objects $\mathcal{H}$, π, π' and L described above and by demonstrating spectral characteristics of the operator L described in Proposition 5.1(ii). The construction of an operator L, which we call, following $[JP]$, the Liouville operator, is done in Section 6-11. The rest of the paper is devoted to investigation of the spectral properties of L.

We begin with GNS constructions for the particle system and reservoir. The corresponding objects are labeled by the supisinduces p and r, respectively.

A GNS construction associates with a von Neumann ($C*$?) algebra $\mathcal{A}$ and a cyclic separating state ω on it yields a Hilbert space $\mathcal{H}$, a representation, π, of $\mathcal{A}$ in the von Neumann algebra of bounded operators on $\mathcal{H}$ and a cyclic vector Ω in $\mathcal{H}$ s.t.

$$\Omega(A) = \langle \Omega, \pi(A) \Omega \rangle$$

and similarly for conjugate representation π'. The GNS constructions we use for the particle system and reservior are explicit. Once this is done we construct the corresponding objects for the total system by a perturbation theory.

6. Construction of $\mathcal{H}^p$, π^p, Ω^p and L^p

Since the particle system has a finite number of degrees of freedom a construction of a GNS representation for it is straightforward. The equilibrium states of the system are given by the Gibbs states

$$\omega^p(A) = tr(A\rho^p) \ , \tag{6.1}$$

where $\rho^p := e^{-\beta H}/tr(e^{-\beta H})$, $\beta = \frac{1}{T}$, the Gibbs density matrix. A GNS Hilbert space can be realized as the space of all Hilbert-Schmidt operators on $\mathcal{H}^p_\infty$ with the cyclic vectors, Ω^p, given by $\sqrt{\rho^p}$. After a standard identification arising from $|\phi\rangle\langle\psi| \leftrightarrow \phi \otimes C\psi$ we arrive at

$$\mathcal{H}^p = \mathcal{H}^p_\infty \otimes \mathcal{H}^p_\infty \ , \ \Omega^p = \sqrt{\rho^p}(x, y) \ , \tag{6.2}$$

$$\pi^p(A) = A \otimes \mathbf{1}^p_\infty \ , \quad \pi^{p\prime}(A) = \mathbf{1}^p_\infty \otimes CAC \ , \tag{6.3}$$

where $\sqrt{\rho^p}(x, y)$ is the integral kernel of the squares root of the Gibbs density matrix ρ^p. Here C is the anti-linear operator of complex conjugation. Moreover, the Liouvillian operator is given by $L^p = \pi(H^p) - \pi'(H^p)$ which becomes

$$L^p = H^p \otimes \mathbf{1}^p_\infty - \mathbf{1}^p_\infty \otimes H^p \ . \tag{6.4}$$

Clearly, we have $L^p\Omega^p = 0$.

7. Construction of $\mathcal{H}^r$, π^r, Ω^r and L^r

Since the reservoir has an infinite number of degrees of freedom, an explicit construction of a GNS representation for it is more involved. It was given by H. Araki and J. Woods [AW] and relies on the fact that the reservoir is given in terms of the free quantum field. It goes as follows:

$$\mathcal{H}^r = \mathcal{H}^r_\infty \otimes \mathcal{H}^r_\infty \ , \quad \Omega^r = \Omega_1 \otimes \Omega_2 \ , \tag{7.1}$$

$$\pi^r(a(k)) := \sqrt{1+\rho}a_1(k) + \sqrt{\rho}a_2^*(k) \ , \tag{7.2}$$

$$\pi^{r\prime}(a(k)) := \sqrt{\rho}a_1^*(k) + \sqrt{1+\rho}a_2(k) \ , \tag{7.3}$$

where, recall, $\mathcal{H}_\infty^r$ is a bosonic Fock space, Ω_i and $a_i(k)$ are the vacuum and annihilation operators on the i-th factor in the product $\mathcal{H}_\infty^r \otimes \mathcal{H}_\infty^r$, $i = 1, 2$, and

$$\rho(k) = (e^{\beta\omega(k)} - 1)^{-1} \ , \tag{7.4}$$

the Planck photon/phonon density.

In order to verify that the construction above indeed gives a GNS representation we compute, using (6.1)–(6.3) that

$$\langle \Omega^r, \pi^r(a^*(k))\pi^r(a(\ell))\Omega^r \rangle = \rho(k)\delta(k-\ell) \ . \tag{7.5}$$

Since the reservoir is described by a free field Hamiltonian, the correlation function ω^r $(a^*(k)a(\ell))$ determines entirely the state ω^r, i.e.,

$$\omega^r(A) = \langle \Omega^r, \pi^r(A)\Omega^r \rangle \qquad \forall A \in \mathcal{A}^r \ .$$

We think about the representation (7.2)-(7.3) as a Bogolubov tranform. However, $\overline{a}$ and $\overline{a}'$ do not yield a Fock representation of the CCR: they do not have a vacuum in $\mathcal{F} \otimes \mathcal{F}$. Thus this representation of the CCR is not equivalent to a tensor product of two Fock representations.

Lemma 7.1. *The Liouville operator L^r for the reservoir acts on the space $\mathcal{H}^r := \mathcal{H}_\infty^r \otimes \mathcal{H}_\infty^r \equiv \mathcal{F} \otimes \mathcal{F}$ according to the formula*

$$L^r = \int \omega(k)[a_1^*(k)a_1(k) - a_2^*(k)a_2(k)]d^3k \ . \tag{7.6}$$

Proof. Plug expressions (7.2)-(7.3) for $\pi^r(a(k))$ and $\pi^{r\prime}(a(k))$ into the formula

$$L^r = \int \omega(k)[\pi^r(a^*(k))\pi^r(a(k)) - \pi^{r\prime}(a^*(k))\pi^{r\prime}(a(k))]d^3k \ , \tag{7.7}$$

which follows from the definition of L^r, to obtain (7.6). $\square$

8. Composition of subsystems

GNS constructions for the particle system and the reservoir yield a GNS construction for the composite, not interacting system:

$$\mathcal{H}^{(0)} := \mathcal{H}^p \otimes \mathcal{H}^r \ , \ \Omega^{(0)} = \Omega^p \otimes \Omega^r \ , \tag{8.1}$$

$$\pi^{(0)} := \pi^p \otimes \pi^r \ , \ \pi^{(0)\prime} := \pi^{p\prime} \otimes \pi^{r\prime} \ , \tag{8.2}$$

$$L^{(0)} := L^p \otimes \mathbf{1}^r + \mathbf{1}^p \otimes L^r \ , \tag{8.3}$$

where $\mathbf{1}^p$ and $\mathbf{1}^r$ are the identity operators on the spaces $\mathcal{H}^p$ and $\mathcal{H}^r$, respectively.

The operator $L^{(0)}$ can also be represented as

$$L^{(0)} = H^{(0)} \otimes \mathbf{1}_\infty - \mathbf{1}_\infty \otimes H^{(0)} \ , \tag{8.4}$$

on $\mathcal{H}^{(0)} = \mathcal{H}_\infty \otimes \mathcal{H}_\infty$. Here $\mathbf{1}_\infty$ is the identity operators on $\mathcal{H}_\infty$.

9. Perturbation

Now we use perturbation theory in order to pass from the GNS construction of the uncoupled system to that of the coupled one, i.e., from group of automorphisms generated by the Hamiltonian

$$H^{(0)} = H^p \otimes \mathbf{1}^r_\infty + \mathbf{1}^p_\infty \otimes H^r \tag{9.1}$$

to the one generated by the Hamiltonian

$$H = H^{(0)} + v \ . \tag{9.2}$$

Here both Hamiltonians are acting on $\mathcal{H}^p_\infty \otimes \mathcal{H}^r_\infty$ and the perturbation v is given in (3.9) or (3.10). We set

$$\mathcal{H} = \mathcal{H}^{(0)} \ , \quad \pi = \pi^{(0)} \ , \tag{9.3}$$

and

$$L = L^{(0)} + I \ , \tag{9.4}$$

where the uncoupled Lionvillian operator, $L^{(0)}$, is given by (8.3) with L^p and L^r given in (6.4) and (7.6), respectively, while the interaction, I, is given by

$$I := \pi(v) - \pi'(v) \ . \tag{9.5}$$

A straightforward way to view the operators $\pi^{\#}(v)$ is by using the integral representation, (3.9) or (3.10), for v and the fact that the integrand is a sum of terms each of which is a product of a bounded operator on $\mathcal{H}^p$ and one of the operators, either $a(k)$ or $a^*(k)$. We write out the explicit form of the operator I in the phonon case:

$$
\begin{aligned}
I = \lambda \int d^3k \Big\{ &(g_x \otimes \mathbf{1}^p_\infty)(\sqrt{1+\rho}a_1^* \otimes \mathbf{1}^r_\infty + \sqrt{\rho}\mathbf{1}^r_\infty \otimes a_2) \\
&- (\mathbf{1}^p_\infty \otimes \overline{g}_x)(\sqrt{\rho}a_1 \otimes \mathbf{1}^r_\infty + \sqrt{1+\rho}\mathbf{1}^r_\infty \otimes a_2^* + h.c. \Big\} ,
\end{aligned}
\tag{9.6}
$$

where $g_x(k) = \frac{\chi(k)}{\sqrt{2\omega}} e^{-ix\cdot k}$.

The domain of definition and self-adjointness of the operator L are tackled in the next section.

10. Relative bounds and self-adjointness

The main result of this section is

Theorem 10.1. *Under the conditions stated in Section 2, the operator L is self-adjoint.*

This theorem follows from Proposition 10.2 below and the Nelson's commutator theorem (see [GJ, RSIV]).

We begin with some notation. We introduce the auxillary estimating operator

$$
\Lambda = \mathbf{1}^p \otimes L^r_{\text{aux}}, \quad \text{where} \quad L^r_{\text{aux}} = \int \omega[a_1^* a_1 + a_2^* a_2] d^3k \ . \tag{10.1}
$$

acting on $\mathcal{H}^p \otimes \mathcal{H}^r$. Clearly, the operators $L^{(0)}$ and Λ commute, are self-adjoint and obey

$$
D(L^{(0)}) \supseteq D(\Lambda) \ .
$$

Proposition 10.2. *The operators I and $[I, \Lambda]$ are Λ - form bounded with the relative form-bound* $\leq \text{const} \cdot |\lambda| \cdot (\int \frac{|\chi|^2}{\omega(\omega+1)})^{1/2}$ *with the constant depending on the size, R, of the particle box.*

Proof. A proof of this proposition in the photon case is somewhat lengthy. It follows from the similar statement for v and $H^{(0)}$ (see [BFS2,4]). We omit it here. We present the proof only for the phonon case and only for the operator I. The proof for the operator $[I,\Lambda]$ identical. We write $\pi^{\#}$ for either π or π'. We decompose $\varphi(x)$ as

$$\varphi(x)=\varphi_1(0)+\delta\varphi(x)\ , \tag{10.2}$$

where $\varphi_1(x)=\check{\chi}_1*\varphi(x)$, with $\chi_1\in C_0^\infty$ supported in $|k|\le 2$ and $\equiv 1$ in $|k|\le 1$, and the second term is defined by this relation, to obtain

$$I=I'+I''\ , \tag{10.3}$$

where

$$I'=\mathbf{1}^p\otimes(\pi^r(\varphi_1(0))-\pi^{r\prime}(\varphi_1(0)) \tag{10.4}$$

and

$$I''_\beta=\pi^p(\pi^r(\delta\varphi(x))-\pi^{p\prime}_\beta(\pi^{r\prime}(\delta\varphi(x))\ . \tag{10.5}$$

Now we compute, using (7.2)-(7.3) and omitting the identity operators

$$\pi^r(\varphi_1(0))-\pi^{r\prime}(\varphi_1(0))=a_1(f)+a_1^*(f)-a_2(f)-a_2^*(f)\ , \tag{10.6}$$

where $a_j(f)=\int a_j\overline{f}$, $a_j^*(f)=\int a_j^*f$ and

$$f:=\left(\sqrt{1+\rho}-\sqrt{\rho}\right)\frac{\chi\cdot\chi_1}{\sqrt{2\omega}}\ . \tag{10.7}$$

By the definition of ρ, Eqn (7.4), we have

$$f=\frac{e^{\beta\omega/2}-1}{\sqrt{e^{\beta\omega}-1}}\frac{\chi\cdot\chi_1}{\sqrt{\omega}}\ . \tag{10.8}$$

Hence,

$$\int\frac{f^2}{\omega}\le\int_{\omega\le 2}\frac{\chi^2}{\omega} \tag{10.9}$$

and therefore I' is Λ - form bounded with the form bound $\le$ const $\cdot\,|\lambda|\big(\int_{\omega\le 1}\frac{\chi^2}{\omega}\big)^{1/2}$, by standard estimates (see [BFS2,4]).

In order to estimate I'' we use the mean value theorem to represent

$$\delta\varphi(x)=x\cdot\nabla\varphi_1(y)+\check{\overline{\chi}}_1*\varphi(x)\ , \tag{10.10}$$

where $\overline{\chi}_1 := 1 - \chi_1$, for some point y on the line segment joining 0 and x. The coupling function for the field $\nabla\varphi_1(y)$ is $\frac{ik\chi\cdot\chi_1}{\sqrt{2\omega}} e^{iy\cdot k}$. By Eqns (7.2)-(7.3), the coupling functions for $\pi^{r\#}\big(\nabla\varphi(y)\big)$ are

$$F_y := \sqrt{1+\rho}\, ik \frac{\chi\cdot\chi_1}{\sqrt{2\omega}} e^{-ik\cdot y} \quad \text{and} \quad H_y := \sqrt{\rho}\, ik \frac{\chi\cdot\chi_1}{\sqrt{2\omega}} e^{-ik\cdot y} \ , \tag{10.11}$$

and similarly for $\check{\overline{\chi}}_1 * \varphi(x)$. They clearly satisfy

$$\int \frac{|F_y|^2}{\omega} \leq \int_{\omega\leq 2} \chi^2 \quad \text{and} \quad \int \frac{|H_y|^2}{\omega} \leq \int_{\omega\leq 2} \chi^2 \ . \tag{10.12}$$

Hence $\pi^{p\#}\big(x\cdot\pi^{r\#}(\nabla\varphi_1)(y)\big)$ are Λ-form bounded with the relative bound $\leq$ const$\big(\int\limits_{\omega\leq 2}\chi^2 + \int \frac{\chi^2}{\omega+1}\big)^{1/2}$ (see [BFS2]). Similarly, $\pi^{p\#}\Big(\pi^{r\#}\big(\check{\overline{\chi}}_1 * \varphi(x)\big)\Big)$ are Λ-form bounded with the relative bound $\leq$ const$\big(\int\limits_{\omega\geq 1}\frac{\chi^2}{\omega^2}\big)^{1/2}$. This completes the proof. □

11. Basic properties of L

In this section we establish key properties of the Liouville operator L mentioned in Section 5. The main result of this section is the following

Theorem 11.1. *Let α_t be defined in (4.1). The operator L has the following properties*

(i) $e^{iLt}\pi^{\#}(A)e^{-iLt} = \pi^{\#}(\alpha_t(A))$;

(ii) *for each β, there is a vector $\Omega = \Omega_\beta \in \mathcal{H}$ s.t. $L\Omega = 0$;*

(iii) *the functional $\omega(A) := \langle\Omega, \pi(A)\Omega\rangle$ is a KMS state for α;*

(iv) $\Omega = \Omega^{(0)} + O(\lambda)$.

Proof. (i) We consider only the representation π. To begin with, a simple computation gives

$$\begin{aligned} e^{iL_0t}\pi(a(k))e^{-iL_0t} &= e^{-i\omega t}\pi(a(k)) \\ &= \pi\big(e^{-i\omega t}a(k)\big) \ . \end{aligned}$$

These relations imply that

$$e^{iL_0t}\pi(A)e^{-iL_0t} = \pi\big(e^{iH_0t}Ae^{-iH_0t}\big) \ . \tag{11.1}$$

Next, by the definition, $I := \pi(v) - \pi'(v)$, and the fact that the representations π and π' commute, we have

$$e^{iIt}\pi(A)e^{-iIt} = \pi(e^{ivt}Ae^{-ivt}) \ . \tag{11.2}$$

The last two relations and the Trotter product formula

$$e^{-iLt} = s - \lim_{n\to\infty} (e^{-iL_0 t/n} e^{-It/n})^n \tag{11.3}$$

imply (i) for $\pi^{\#} = \pi$.

(ii)–(iv) There are two ways to prove the statements – one way through the finite volume approximation and thermodynamic limit and the other by a perturbation theory. The first proof is worked out in Appendix I while the second proof is based on an explicit construction of Ω given in

Proposition 11.2. *The vector $\Omega^{(0)}$ is in the domain of the operator $e^{-\beta L_\lambda}$ for any $\beta \in \mathbb{R}$ and the vector*

$$\Omega := e^{-\beta L_\lambda/2}\Omega^{(0)}/\|e^{-\beta L_\lambda/2}\Omega\| \ , \tag{11.4}$$

where $L_\lambda = L_0 + \pi(v)$, has all the properties mentioned in statements (ii)– (iii) of Theorem 11.1. Moreover, in the formula above the operator L_λ can be replaced by

$$L_\rho := L_0 - \pi'(v) \ . \tag{11.5}$$

A proof of this proposition is given also in Appendix I. □

12. Spectral deformation

In this section we apply the method of spectral deformation to the operator L in order to untangle various branches of its continuous spectrum. To keep things explicit we use the simplest realization of this method by means the dilatation group. Namely, we define

$$U(\theta) = U_\infty(\theta) \otimes U_\infty(-\theta) \tag{12.1}$$

on $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_\infty$, where

$$U_\infty(\theta) = \mathbf{1}^p_\infty \otimes U^r(\theta) \tag{12.2}$$

on $\mathcal{H}_\infty = \mathcal{H}^p_\infty \otimes \mathcal{H}^r_\infty$, with $U^r(\theta)$ being the lifting of the dilation group

$$U_\theta : f(k) \to e^{\frac{-3\theta}{2}} f(e^{-\theta}k) \ , \tag{12.3}$$

$f \in L^2(\mathbb{R}^3)$, from the one photon/phonon space to the Fock space $\mathcal{H}^r_\infty$ (e.g.

$$U^r(\theta) : \prod_{j=1}^n a_j^*(f_j)\Omega_\infty \to \prod_{j=1}^n a_j^*(U_\theta f_j)\Omega_i) \ .$$

The twist of taking the dilation parameters, in the two factors in (12.1), of opposite signs is crucial.

Note that, the cyclic vector $\Omega = \Omega_\beta$ is not invariant under $U(\theta)$: $U(\theta)\Omega \neq \Omega$ for $\theta \neq 0$.

Now we define on the form domain of Λ

$$L_\theta := U(\theta)LU(\theta)^{-1} \tag{12.4}$$

and similarly for $L^{(0)}$ and I. Thus we have

$$L_\theta = L_\theta^{(0)} + I_\theta \ . \tag{12.5}$$

We compute

$$L_\theta^{(0)} = L^p \otimes \mathbf{1}^r + \mathbf{1}^p \otimes L_\theta^r \ , \tag{12.6}$$

where

$$L_\theta^r = \int \omega[e^{-\theta} a_1^* a_1 - e^\theta a_2^* a_2] d^3k \ . \tag{12.7}$$

Moreover we have in the phonon case (θ is real)

$$\begin{aligned} I_\theta = \lambda e^{-\theta} \int \frac{d^3k}{\sqrt{2\omega}} \chi_\theta \Big\{ & (e^{-ie^{-\theta} x\cdot k} \otimes \mathbf{1}^p_\infty)(\sqrt{1+\rho_\theta} a_1^* \otimes \mathbf{1}^r_\infty + \sqrt{\rho_\theta} \mathbf{1}^r_\infty \otimes a_2) \\ & - (\mathbf{1}^p_\infty \otimes e^{-ie^{-\theta} x\cdot k})(\sqrt{\rho_\theta} a_1 \otimes \mathbf{1}^r_\infty + \sqrt{1+\rho_\theta} \mathbf{1}^r_\infty \otimes a_2^*) + h.c. \Big\} \ , \end{aligned} \tag{12.8}$$

where $\chi_\theta(k) = \chi(e^{-\theta}k)$ and $\rho_\theta(k) = \rho(e^{-\theta}k)$, and similarly in the photon case.

The family $L_\theta^{(0)}$ is analytic in the strip $\{\theta \in \mathbb{C} \big| |\theta| < \frac{\pi}{2}\}$.

Due to the twist in the definition of $U^r(\theta)$ we have the following formula

$$L_{i\varphi}^r = \cos\varphi \cdot L^r - i \sin\varphi \cdot L^r_{\text{aux}} \ , \tag{12.9}$$

where, recall, L^r_{aux} is given in (10.1).

Eqns (7.2)-(7.4) and (12.8) show that the coupling functions for the operator I_θ are of the form

$$\frac{f(e^{-\theta}k)}{\sqrt{e^{e^\theta\beta\omega}-1}} \ , \tag{12.10}$$

where the function $f(e^{-\theta}k)$ is analytic in θ in the strip

$$\{\theta\in\mathbb{C}\big|\,|\mathrm{Im}\theta|<\frac{\pi}{2}\} \ , \tag{12.11}$$

while, for every fixed θ, $|\mathrm{Im}\theta|<\frac{\pi}{2}$, $\frac{|f(e^{-\theta}k)|}{|\sqrt{e^{e^\theta\beta\omega}-1}|}$ is exponentially decaying in $|k|$ on the scale K uniformly in β.

On the other hand, if $|\mathrm{Im}\theta|<\frac{\pi}{2}$, then the denominator in (12.10) has only one zero in k: at $k=0$. This zero produces an additional infrared singularity $\frac{1}{\sqrt{\beta\omega}}$, which we dealt with in Section 10. Using Eqn (12.9) and the estimates of that section but applied to the operator $L_\theta=L_\theta^{(0)}+I_\theta$, we arive at

Theorem 12.1. *Let $\chi_\theta(k):=\chi(e^{-\theta}k)$. For $|\mathrm{Im}\theta|<\frac{\pi}{2}$ and for $|\lambda|$ sufficiently small, (i) L_θ are defined as closed operators with the form domain $D_f(L_\theta)\supset D_f(\Lambda)$, (ii) I_θ is Λ-form bounded with the relative bound $\le \mathrm{const}|\lambda|\big(\int\frac{|\chi_\theta|^2}{\omega(\omega+1)}\big)^{1/2}\frac{1}{\sin(\mathrm{Im}\theta)}$, (iii) $L_\theta u$ is an analytic in θ, $|\mathrm{Im}\,\theta|<\frac{\pi}{2}$, for any $u\in D(\Lambda)$.*

We cannot prove that the family L_θ is analytic of type A, but the following partial result suffices for us.

Theorem 12.2. *(i) If $0<\pm\mathrm{Im}\theta<\frac{\pi}{2}$, then $\{z\in\mathbb{C}|\pm\mathrm{Im}z\ge 1\}\subset\rho(L_\theta)$;*
(ii) The family L_θ is analytic of type A (in the sense of Kato) in the strips $\Big\{\theta\in\mathbb{C}\Big|0<\pm\mathrm{Im}\theta<\frac{\pi}{2}\Big\}$;
(iii) For any u and v which are $U(\theta)$-analytic in a strip $\Big\{\theta\in\mathbb{C}\Big|\,|\mathrm{Im}\theta|<\delta_1\Big\}$ for some $\frac{\pi}{3}\ge\delta_1>0$,

$$\langle u,(L-z)^{-1}v\rangle=\langle u_{\overline{\theta}},(L_\theta-z)^{-1}v_\theta\rangle \tag{12.12}$$

for $\mathrm{Im}z\gg 1$ and $0<\mathrm{Im}\theta<\delta_1/2$.

Proof. (i) Consider only the $+$ case and let z be s.t. $\mathrm{Im}z>K$. Without a loss of generality we can assume that $\theta=i\varphi$. Then $\varphi>0$. By the Schwarz inequality

$$\begin{aligned}\|u\|\cdot\|(L_{i\varphi}-z)u\|&\ge-\mathrm{Im}\langle L_{i\varphi}-z\rangle_u\\&=\langle\sin\varphi\cdot L^r_{\mathrm{aux}}-\mathrm{Im}I_{i\varphi}+\mathrm{Im}z\rangle_u \ ,\end{aligned} \tag{12.13}$$

Proceeding as in the proof of Proposition 10.2 we obtain

$$\mathrm{Im} I_{i\varphi} \leq \sin\varphi(\varepsilon L^r_{\mathrm{aux}} + C\lambda^2\varepsilon^{-1}) \tag{12.14}$$

for any $\varepsilon > 0$. Take $\varepsilon = \frac{1}{2}$. The last three estimates and the inequality $\mathrm{Im} z \geq 1$ give

$$\|u\| \cdot \|(L_{i\varphi} - z)u\| \geq \gamma\langle\Lambda + 1\rangle_u \ , \tag{12.15}$$

where $\gamma = \frac{1}{2}\sin\varphi$. The last estimate implies that

$$\|(L_{i\varphi} - z)u\| \geq \gamma\|u\| \ ,$$

provided $\mathrm{Im} z \gg 1$. Similarly, we show that $\|(L_{-i\varphi} - \overline{z})u\| \geq \gamma\|u\|$ under the same conditions. Since $L_{-i\varphi} = L^*_{i\varphi}$, the last two statements imply that $z \in \rho(L_{i\varphi})$.

(ii) In fact, estimate (12.15) implies more: for $\gamma = \frac{1}{2}\sin(\mathrm{Im}\theta)$

$$\|(L_\theta - z)u\| \geq \gamma\|\Lambda^{1/2}u\| \ . \tag{12.16}$$

The last estimate can be rewritten as

$$\|\Lambda^{1/2}(L_\theta - z)^{-1}\| \leq \gamma^{-1} \ . \tag{12.17}$$

Now, using conditions on $V(x)$ and proceeding similarly as in the proof of Proposition 10.2, we obtain

$$\|\Lambda^{1/2}\partial_\theta L_\theta \Lambda^{-1/2}\| \leq C \ . \tag{12.18}$$

The last two estimates and the computation

$$\partial_\theta(L_\theta - z)^{-1} = -(L_\theta - z)^{-1}\partial_\theta L_\theta(L_\theta - z)^{-1}$$

imply that $(L_\theta - z)^{-1}$ is analytic in $\theta \in S^{\pm}$, provided $\pm\mathrm{Im} z$ is sufficiently large. Here $S^{\pm} := \{\theta \in \mathbb{C} | 0 < \pm\mathrm{Im}\theta < \frac{\pi}{2}\}$.

(iii) Now to fix ideas we assume that $\mathrm{Im}\theta > 0$ and $\mathrm{Im} z > 0$. Let u and v be $U(\theta)$-analytic for $|\mathrm{Im}\theta| < \delta_1$ for some $\frac{\pi}{3} > \delta_1 > 0$. Then in a standard way

$$\langle u, (L_\theta - z)^{-1}v\rangle = \langle u_{-is}, (L_{\theta+is} - z)^{-1}v_{is}\rangle \tag{12.19}$$

for $0 \leq s \leq \delta_1$ and for θ as above. The r.h.s. of this equation is analytic in θ as long as $\theta + is \in S^+$ and $\mathrm{Im}z \gg 1$. In particular, this shows that $\langle u, (L_\theta - z)^{-1} v\rangle$ has an analytic continuation on a dense set of u's and v's into the strip $\left\{\theta \in \mathbb{C} | -\frac{\delta_1}{2} < \mathrm{Im}\theta < \frac{\pi}{2}\right\}$, provided $\mathrm{Im}z \gg 1$.

It remains to identify the analytic continuation, $f_\theta(u, v)$, of $\langle u, (L_\theta - z)^{-1} v\rangle$ constructed above with $\langle u, (L_\theta - z)^{-1} v\rangle$ for $\theta \in \mathbb{R}$. To this end using the definition of f_θ we find

$$
\begin{aligned}
f_\theta(u, (L_\theta - z)w) &= \langle u_{-is}, (L_{\theta+is} - z)^{-1}(L_{\theta+is} - z)w_{is}\rangle \\
&= \langle u_{-is}, w_{is}\rangle = \langle u, w\rangle
\end{aligned}
$$

If $L_\theta - z$ is invertible, then we have

$$
f_\theta(u, v) = \langle u, (L_\theta - z)^{-1} v\rangle \ .
$$

For $\theta \in \mathbb{R}$ and $\mathrm{Im}z > 0$, $L_\theta - z$ is invertible and therefore this equality holds. On the other hand we know that $f_\theta(u, v) = \langle u_{\overline{\theta}'}, (L_{\theta+\theta'} - z)^{-1} v_{\theta'}\rangle$ for any θ' with $\mathrm{Im}\theta' > 0$. The last two relations with $\theta = 0$ yield (12.12). $\square$

Finally, using perturbative expressions developed in the proof of Propositionn 11.2 we demonstrate in Appendix I the following

Theorem 12.3. *The eigenvector Ω described in Proposition 11.2 (see Eqn. (11.4)) is dilatation analytic in the strip $\{\theta \in \mathbb{C} || \mathrm{Im}\theta| < \frac{\pi}{2}\}$ and consequently $\Omega_\theta := U(\theta)\Omega$ is a zero eigenvector of the operator L_θ.*

13. Spectrum of $L_\theta^{(0)}$

In this section we determine the spectrum of the unperturbed but rotated Liouville operator $L_\theta^{(0)}$.

It is easy to determine the spectrum of L_θ^r:

$$
\sigma_{pp}(L_\theta^r) = \{0\} \ , \tag{13.1}
$$

$$
\sigma_{\mathrm{ess}}(L_\theta^r) = e^{-i\mathrm{Im}\theta}\overline{\mathbb{R}^+} + e^{i(\mathrm{Im}\theta+\pi)}\overline{\mathbb{R}^+} \ , \tag{13.2}
$$

see Fig. 13.1 below.

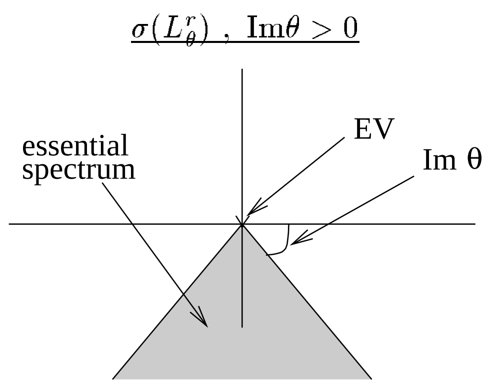

Fig. 13.1

Since the spectrum of H^p is purely discrete,

$$\sigma(H^p) = \sigma_d(H^p) = \{E_j\}_0^\infty \ , \tag{13.3}$$

we have that

$$\sigma(L^p_\theta) = \sigma_d(H^p) - \sigma_d(H^p) = \{E_i - E_j | i, j = 0, \ldots\} \ . \tag{13.4}$$

Using this, we obtain

$$\sigma_p\big(L^{(0)}_\theta\big) = \sigma_d(H^p) - \sigma_d(H^p) \ , \tag{13.5}$$

$$\sigma_{\mathrm{ess}}\big(L^{(0)}_\theta\big) = \big(\sigma_d(H^p) - \sigma_d(H^p)\big) + \big(e^{-i\mathrm{Im}\,\theta}\overline{\mathbb{R}+} + e^{i(\mathrm{Im}\,\theta+\pi)}\overline{\mathbb{R}+}\big) \ ; \tag{13.6}$$

see Fig. 13.2 below.

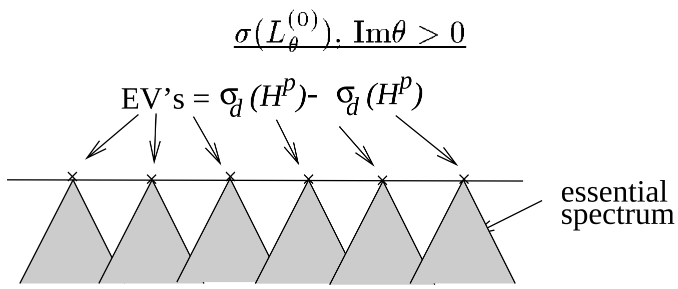

Fig. 13.2

Remark 13.1 If we consider χ not just as an ultraviolet cut-off but as an adjustable coupling function and if we set χ to be of a certain form in the infrared region, namely

$$\chi(k) = |k|\chi_1(|k|^2) \ , \tag{13.6}$$

where $\chi_1(s^2)$, is analytic in s in the strip $\{z \in \mathbb{C} \big| |\mathrm{Im} z| < \frac{2\pi}{\beta_0}\}$, for some $0 < \frac{2\pi}{\beta_0} \leq \frac{2\pi}{\beta}$, as well as dilation analytic, then one can use a complex shift transformation, besides the dilation one, to achieve the picture for the spectrum of $L^r(\theta)$ given in Fig. 13.3 (cf. [JP 1-3]).

$\underline{\sigma\big(L^r(\theta)\big)\ ,\ \mathrm{Im}\theta > 0}$

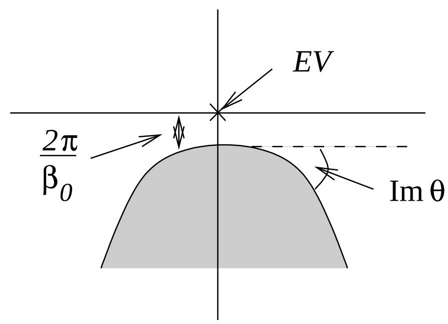

Fig. 13.3

This option would simplify slightly our approach.

In the remainder of this paper we develop perturbation theory for the operator L_θ. The goal of this theory is to give a sufficiently detailed description of spectral characteristics of the operator L_θ which will imply the main results of this paper – Theorems 4.1 and 4.2. The problem here is that the eigenvalues of $L_\theta^{(0)}$, $\mathrm{Im}\theta > 0$ (which are also the eigenvalues of $L^{(0)}$) which are of the main interest for us, are not isolated; they lie on top of thresholds (= branch points) of its continuous spectrum. To deal with this problem we adapt and use the decimation map - or, more generally, the renormalization group - techniques developed in [BFS1-4]. This is done in the next three sections.

14. <u>The decimation map</u>

In this section we recall the definition and main properties of the decimation map introduced in [BFS1] and developed in [BFS2].

Let projection operators P and $\overline{P}$ on a separable Banach space X form a partition of unity in the sense that $P + \overline{P} = \mathbf{1}$. Denote by C_P the set of all closed operators, H, on X whose domain have dense intersections with $\mathrm{Ran}P$ and $\mathrm{Ran}\overline{P}$ and which satisfy

$$\|R_{\overline{P}}\| < \infty \ , \tag{14.1}$$

$$\|PHR_{\overline{P}}\| < \infty \quad \text{and} \quad \|R_{\overline{P}}HP\| < \infty \; . \tag{14.2}$$

Here $H_P = PHP \restriction \mathrm{Ran}P$, etc. and $R_{\overline{P}} = \overline{P}H_{\overline{P}}^{-1}\overline{P}$. We define the map

$$D_P \; : \; C_P \to (\text{Closed operators on Ran}\, P) \tag{14.3}$$

by

$$D_P(H) = P(H - HR_{\overline{P}}H)P \restriction_{\mathrm{Ran}\, P} \; . \tag{14.4}$$

We call D_P the *decimation map.*

To describe properties of the map D_P we need the following

Definition 14.1.

(i) Given two closed operators, H and H', on Banach spaces X and X' we say that H and H' are isospectral at 0 iff

(a) $0 \in \sigma(H) \leftrightarrow 0 \in \sigma(H')$

(b) there are maps $P : D(H) \to D(H')$, $Q : D(H') \to D(H)$, $Q^{\#} : X \to X'$, and $P^{\#} : X' \to X$ s.t.

$$\mathrm{Null}Q \cap \mathrm{Null}H' = \{0\} \quad \text{and} \quad \mathrm{Null}P \cap \mathrm{Null}H = \{0\} \tag{14.5}$$

and

$$HQ = P^{\#}H' \quad \text{and} \quad Q^{\#}H = H'P \; . \tag{14.6}$$

(ii) Given two families, $H(z)$, $z \in \Omega_1$, and $H(z)$, $z \in \Omega_2$, of closed operators on Banach spaces X and X' we say that $H(z)$ and $H'(z)$ are isospectral in $\Omega \subset \Omega_1 \cap \Omega_2$ iff $H(z)$ and $H'(z)$ are isospectral for each $z \in \Omega$.

Observe that property (ib) implies that

$$H\psi = 0 \Rightarrow H'\psi' = 0 \quad \text{with} \quad \psi' = P\psi \neq 0$$

and

$$H'\psi' = 0 \Rightarrow H\psi = 0 \quad \text{with} \quad \psi = Q\psi' \neq 0 \; .$$

The latter relations imply, in particular, that

$$\sigma_p(H) = \sigma_p(H') \; ,$$

but also relate generalized eigenfunctions and the corresponding spectrum if those can be identified for one of the operators H or H'.

Furthermore we have

Proposition 14.2. *Assume (14.1) and (14.2) hold. Then Property (ib) of Definition 14.1 implies that $0 \in \sigma(H) \Rightarrow 0 \in \sigma(H')$ (the part of Property (ia) crucial for us), and we have always that* $\mathrm{Null}P \cap \mathrm{Null}H = \{0\}$.

Proof. Let $0 \in \rho(H')$. Then we can solve the equation $H'P = Q^{\#}H$ for P to obtain

$$P = H'^{-1}Q^{\#}H \ . \tag{14.7}$$

On the other hand the definition $H_{\overline{P}} = \overline{P}H_1\overline{P}$ implies

$$\begin{aligned} \overline{P} &= H_{\overline{P}}^{-1}\overline{P}H\overline{P} \cdot \overline{P} \\ &= H_{\overline{P}}^{-1}(\overline{P}H - \overline{P}HP) \ . \end{aligned} \tag{14.8}$$

Substituting expression (14.7) for P into the r.h.s., we find

$$\overline{P} = H_{\overline{P}}^{-1}(\overline{P} - \overline{P}HPH'^{-1}Q^{\#})H \ .$$

Adding this to (14.7) yields

$$\mathbf{1} = \left[H_{\overline{P}}^{-1}\overline{P} - H_{\overline{P}}^{-1}\overline{P}HPH'^{-1}Q^{\#} + H'^{-1}Q^{\#}\right]H$$

Since by our conditions $H_{\overline{P}}^{-1}\overline{P}HP$ is bounded, the expression in the square brackets represents a bounded operator. Hence H has a bounded inverse. So $0 \in \rho(H)$.

The second statement follows from Eqn (14.8) and the relation $P + \overline{P} = \mathbf{1}$. □

The main result of this section is the following

Theorem 14.3. *The operators H and $D_P(H)$ are isospectral at 0, provided (14.1)–(14.2) hold, i.e., H is in the domain of the map D_P.*

Proof. Clearly, to identify our situation with Definition 14.1 we take $X' = \mathrm{Ran}P$ and $H' = D_P(H)$. We also let P be the corresponding operator entering (ib) of Definition 14.1 and define three other operators entering (ib) as $P^{\#} = P$,

$$Q \equiv Q(H) := P - \overline{P}(H_{\overline{P}})^{-1}\overline{P}HP \tag{14.9}$$

and

$$Q^{\#} \equiv Q^{\#}(H) := P - PH\overline{P}(H_{\overline{P}})^{-1}\overline{P} \tag{14.10}$$

In our case $Q^{\#}(H) = Q(H^*)^*$, but we do not use this property.

The second relation in (14.4) is shown in Proposition 14.2, while the first one follows from the inequality $\|QP\varphi\| \geq \|P\varphi\| - \|\overline{R}HP\varphi\|$ and the property $P^2 = P$ and (14.7). Now we prove relations (14.5). Using the definition of $Q(H)$, we transform

$$\begin{aligned} HQ &\equiv HP - H\overline{P}H_{\overline{P}}^{-1}HP \\ &= PHP + \overline{P}HP - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}HP \\ &\quad - \overline{P}H\overline{P}H_{\overline{P}}^{-1}\overline{P}HP \\ &= PHP - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}HP \equiv P \cdot D_P(H) \ . \end{aligned}$$

Next, we have

$$\begin{aligned} Q^{\#}H &\equiv PH - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}H \\ &= PHP + PH\overline{P} - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}HP - PH\overline{P}H_{\overline{P}}^{-1}H\overline{P} \\ &= PHP - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}HP \equiv D_P(H)P \ . \end{aligned}$$

Finally, we demonstrate property (a) of Definition 14.1. Denote $H' \equiv D_P(H)$. Proposition 14.2 implies that $0 \in \rho(H)$ if $0 \in \rho(H')$.

Now let, conversely, $0 \in \rho(H)$ and show that $0 \in \rho(H')$. This statement follows from the relation

$$H'^{-1} = PH^{-1}P \tag{14.11}$$

which we set out to prove now. We have by the definition

$$\begin{aligned} H'PH^{-1}P &= PHPH^{-1} - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}HPH^{-1}P \\ &= P - PH\overline{P}H^{-1}P - PH\overline{P}H_{\overline{P}}^{-1}\overline{P}H(\mathbf{1} - \overline{P})H^{-1}P \\ &= P \ . \end{aligned}$$

Similarly one shows that $PH^{-1}PH' = P$. Hence H' has the bounded inverse $PH^{-1}P$.

Thus we have shown that $0 \in \rho(H) \leftrightarrow 0 \in \rho(D_P(H))$, which is equivalent to $0 \in \sigma(H) \leftrightarrow 0 \in \sigma(D_P(H))$. $\square$

We call $D_P(H)$ the *decimation* (or *Feshbach*) map: it maps operators on X into operators on $\operatorname{Ran} P$ in such a way that $H - z \cdot \mathbf{1}$ and $D_P(H - z)$ are isospectral at 0 on the set $\rho(H_{\overline{P}})$.

15. Elimination of particle and high photon/phonon energy degrees of freedom

In this section we deal with the first application of the decimation map and as a result we face some of the typical problems that arise. We pick an eigenvalue $\varepsilon = E_i - E_j$ of $L_\theta^{(0)}$ (which is an eigenvalue of L^p) and from now on we consider this fixed eigenvalue only. An important rôle in our analysis is played by the auxiliary Liouville operator

$$L_{\rm aux}^r = \int \omega(a_1^* a_1 + a_2^* a_2) \ , \tag{15.1}$$

introduced in Section 10, Eqn (10.1), to handle the problem of self-adjointness. With this operator we associate a partition of unity

$$\chi_{L_{\rm aux}^r \leq \rho} + \chi_{L_{\rm aux}^r > \rho} = \mathbf{1}^r \ , \tag{15.2}$$

where $\chi_{\lambda \leq \rho}$ and $\chi_{\lambda > \rho}$ are the characteristic functions of the intervals $(-\infty, \rho]$ and (ρ, ∞), respectively.

Let $P_{\varepsilon'}^p$ denote the projection onto the eigenspace of the operator L_θ^p, $\mathrm{Im}\theta > 0$, corresponding to an eigenvalue ε' and $\overline{P}_{\varepsilon'}^p = \mathbf{1} - P_{\varepsilon'}^p$. We define the partition of unity

$$P_\rho + \overline{P}_\rho = \mathbf{1} \ , \tag{15.3}$$

where

$$P_\rho = P_\varepsilon^p \otimes \chi_{L_{\rm aux}^r \leq \rho} \tag{15.4}$$

and

$$\overline{P}_\rho = P_\varepsilon^p \otimes \chi_{L_{\rm aux}^r \geq \rho} + \overline{P}_\varepsilon^p \ . \tag{15.5}$$

Finally with definition (14.4) in mind, we let

$$D_{\rho,\varepsilon}^{(0)} := D_{P_\rho} \ . \tag{15.6}$$

Let ε_- and ε_+ be the eigenvalues of L^p immediately on the left and on the right of ε, respectively. Define the domain (see Fig. 15.1)

$$S_{\varepsilon,\rho} = \left\{ z \in \mathbb{C} \middle| \varepsilon - \frac{2}{3}(\varepsilon - \varepsilon_-) \leq \mathrm{Re} z \leq \varepsilon + \frac{2}{3}(\varepsilon_+ - \varepsilon) \ , \mathrm{Im} z \geq -\frac{1}{2}\rho \right\} \ . \tag{15.7}$$

Fig. 15.1. Domain $S_{\varepsilon,\rho}$.

The main result of this section is

Proposition 15.1. *Let* $\frac{\pi}{2} > \mathrm{Im}\theta > \delta$, *for some* $0 < \delta < \frac{\pi}{2}$ *and* $\sqrt{\rho} \gg |\lambda|$ *(depending on* δ*). Then the operators* $L_\theta - z \cdot \mathbf{1}$, $z \in S_{\varepsilon,\rho}$, *are in the domain of definition of* $D^{(0)}_{\rho,\varepsilon}$.

Proof. For simplicity we conduct the proof only for particle systems with finite number of states and for phonons. In the general case the proof of bound (15.9) is more involved. As before, $\theta = i\varphi \in \mathbb{C}^+$ is fixed. Let $\overline{L}_\theta := L^{(0)}_\theta \overline{P}_\rho + \overline{P}_\rho I_\theta \overline{P}_\rho$. We want to show that $\overline{L}_\theta - z \cdot \mathbf{1}$ is invertible on $\mathrm{Ran}\overline{P}_\rho$ as long as $z \in S_{\varepsilon,\rho}$. First, we prove that $L^{(0)}_\theta$ is invertible on $\mathrm{Ran}\overline{P}_\rho$. One can reduce the problem at hand as follows. Write the eigenprojection $\overline{P}^p_\varepsilon$ as

$$\overline{P}^p_\varepsilon = \sum_{\substack{\varepsilon' \in \sigma(L^p) \\ \varepsilon' \neq \varepsilon}} P^p_{\varepsilon'} \ . \tag{15.8}$$

Due to this formula and expression (15.5) for $\overline{P}_\rho$ it suffices to show that

$$\mathbf{1}^p \otimes (L^r_\theta - z + \varepsilon') \text{ is invertible on } \mathrm{Ran}(P^p_{\varepsilon'} \otimes \mathbf{1}^r)$$

$$\text{for } \varepsilon' \neq \varepsilon \text{ and}$$

$$\mathbf{1}^p \otimes (L^r_\theta - z + \varepsilon) \text{ is invertible on } \mathrm{Ran}(P^p_\varepsilon \otimes \chi_{L^r_{\mathrm{aux}} \geq \rho}) \ .$$

The operator $L^r_\theta = \cos\varphi L^r - i \sin\varphi L^r_{\mathrm{aux}}$ is normal and both statements follow readily from an examination of its spectrum on an appropriate subspace. Thus the operators $L^{(0)}_\theta - z \cdot \mathbf{1}$ are invertible of $\mathrm{Ran}\overline{P}_\rho$. With this in mind we use the notation $(L^{(0)}_\theta - z \cdot \mathbf{1})^{-1/2}$ on $\mathrm{Ran}\overline{P}_\rho$ in an informal but clearly understandable way.

The same analysis as above shows moreover that

$$\sqrt{\sin\varphi\rho + \mathrm{Im}z}\|(L^{(0)}_\theta - z)^{-1/2}\overline{P}_\rho\| + \sqrt{\sin\varphi}\|\Lambda^{1/2}(L^{(0)}_\theta - z)^{-1/2}\overline{P}_\rho\| \leq C \ . \tag{15.9}$$

Now we remember that $\frac{\pi}{2} > \varphi > \delta$ and therefore $\sin\varphi > \sin\delta$. We absorb $\sin\delta$ into the constants below and do not display it. We transform

$$\overline{L}_\theta - z \cdot \mathbf{1} = (L^{(0)}_\theta - z)^{1/2}(\mathbf{1} + M)(L^{(0)}_\theta - z)^{1/2} \ , \tag{15.10}$$

where $M = (L^{(0)}_\theta - z)^{-1/2}\overline{P}_\rho I_\theta \overline{P}_\rho (L^{(0)}_\theta - z)^{-1/2}$. Following the proof of Proposition 10.2 (Λ-form boundedness of I) one shows that the operator M is well defined and, in the phonon case, satisfies the estimate

$$\|M\| \leq C \left(\int \frac{|\chi_\theta|^2}{\omega(\omega+1)} \right)^{1/2} \frac{\lambda}{\sqrt{\rho}} \ . \tag{15.11}$$

Thus picking $|\lambda|/\sqrt{\rho}$ sufficiently small we can achieve that $\|M\| \leq \frac{1}{2}$. Hence $\overline{L}_\theta - z \cdot \mathbf{1}$ is invertible on Ran$\overline{P}_\rho$. Moreover, due to (15.9), we have the estimate

$$\|\overline{P}_\rho (\overline{L}_\theta - z)^{-1} \overline{P}_\rho\| \leq \frac{\text{const}}{\rho + \operatorname{Im} z} \ . \tag{15.12}$$

Similar estimates as above (see also [BFS] for a detailed analysis of the zero temperature case) show also that

$$\begin{aligned} &\|P_\rho I_\theta \overline{P}_\rho (\overline{L}_\theta - z)^{-1} \overline{P}_\rho\| + \|\overline{P}_\rho (\overline{L}_\theta - z)^{-1} \overline{P}_\rho I_\theta P_\rho\| \\ &\qquad \leq \text{const} \left(\int \frac{|\chi_\theta|^2}{\omega(\omega+1)} \right) \frac{\lambda}{\rho} \ . \end{aligned} \tag{15.13}$$

This completes the proof of Proposition 15.1. □

The above proposition allows us to pass isospectrally from $L_\theta - z \cdot \mathbf{1}$, $z \in S_{\varepsilon,\rho}$, $0 < \operatorname{Im}\theta < \frac{\pi}{2}$, to $D^{(0)}_{\rho,\varepsilon}(L_\theta - z \cdot \mathbf{1})$. The spectrum of the latter operator on the invariant subspace RanP_ρ determines the spectrum of L_θ, $0 < \operatorname{Im}\theta < \frac{\pi}{2}$, in the set $S_{\varepsilon,\rho}$.

16. Instability of eigenvalues and the Fermi Golden Rule

In this section we derive an important consequence of Proposition 15.1 - the instability of the eigenvalues of $L^{(0)}_\theta$, $\frac{\pi}{2} > \operatorname{Im}\theta > 0$, under the perturbation I_θ. Our main result is the following

Theorem 16.1. *There is a number $\nu > 0$ independent of λ, β and θ s.t. for $|\lambda|$ sufficiently small, the operator L_θ, $\frac{\pi}{2} > \operatorname{Im}\theta > 0$, has no spectrum in the domain*

$$\{z \in \mathbb{C} \big| \operatorname{Im} z \geq -\nu\lambda^2\} \backslash K \ , \tag{16.1}$$

but a simple eigenvalue at 0. Here the domain K is given by (see Fig. 16.1)

$$K = \{z \in \overline{\mathbb{C}^-} \big| |\operatorname{Re} z| \leq -\operatorname{Im} z \tan\varphi + C(-\operatorname{Im} z)^{3/2}\} \tag{16.2}$$

for some constant C.

Fig. 16.1. Domain K.

Proof. In this proof θ, $0 < \mathrm{Im}\theta < \frac{\pi}{2}$, is fixed, and sometimes is not displayed. We also fix $\varepsilon \in \sigma(L^p)$ and consider two cases (a) $\varepsilon \neq 0$ and (b) $\varepsilon = 0$, separately.

Take $\rho = |\lambda|^{4/3}$ in the photon case and $\rho = 1$ in the phonon case. Due to Proposition 14.1 and the remark after its proof it suffices to examine the spectrum of the operator

$$L_\varepsilon(z) := D^{(0)}_{\rho,\varepsilon}(L_\theta - z) \ , \tag{16.3}$$

acting on the subspace $\mathrm{Ran}P_\rho$, for $z \in S_{\varepsilon,\nu}$. Using the definition of $D^{(0)}_{\rho,\varepsilon}$, see Eqns (13.4) and (14.6), we write it as

$$L_\varepsilon(z) = L^{(0)}_\theta + W - z \cdot \mathbf{1} \ , \tag{16.4}$$

where

$$W = P_\rho\big(I_\theta - I_\theta \overline{R}(z,\theta) I_\theta\big) P_\rho \tag{16.5}$$

with

$$\overline{R}(z,\theta) = \overline{P}_\rho(\overline{P}_\rho L_\theta \overline{P}_\rho - z)^{-1}\overline{P}_\rho \ . \tag{16.6}$$

Estimates of [BFS2], in the phonon case, and of [BFS4], in the photon case, can be easily adapted to the present situation to yield that

$$W = (\Lambda_\varepsilon \otimes \mathbf{1}^r)P_\rho + O(\lambda^{7/3}) \ , \tag{16.7}$$

where Λ_ε is a rank $P^p_\varepsilon \times$ rank P^p_ε matrix (or a bounded operator in the case of rank $P^p_\varepsilon = \infty$ as for $\varepsilon = 0$). A key fact about Λ_ε is that $\Gamma_\varepsilon := -\mathrm{Im}\Lambda_\varepsilon$ is non-negative definite and is given by

$$\Gamma_\varepsilon := 2\pi P_\varepsilon I \overline{P}_\varepsilon \delta(\overline{L}^{(0)} - \varepsilon)\overline{P}_\varepsilon I P_\varepsilon \restriction \mathrm{Ran}P_\varepsilon \ , \tag{16.8}$$

where $P_\varepsilon = P^p_\varepsilon \otimes P^r_{\Omega^r}$, $P^r_{\Omega^r}$ is the rank-one projection onto $\Omega^r := \Omega_1 \otimes \Omega_2 \in \mathcal{H}^r$, $\overline{P}_\varepsilon = \mathbf{1} - P_\varepsilon$ and $\overline{L}^{(0)} = L^{(0)}\overline{P}_\varepsilon$. We consider Γ_ε as an operator on $\mathrm{Ran}P^p_\varepsilon$ ($\approx \mathrm{Ran}P_\varepsilon$).

Remark 16.2. Λ_ε is the matrix of the second order perturbation theory for the imaginary parts of resonances of L branching out the eigenvalue ε of $L^{(0)}$ under the perturbation I (expression (16.8) is referred to his literature as the Fermi Golden Rule (see [RSIV])).

A crucial property of the operator Λ_0 is proven in the following

Theorem 16.3. *Ω^p is a zero eigenvector of the operator $\Lambda_0 : \Lambda_0\Omega^p = 0$.*

Proof. First we observe that

$$L_\theta^{(0)} = \mathbf{1}^p \otimes L_\theta^r = O(\rho) = O(\lambda^{4/3}) \quad \text{on} \quad \mathrm{Ran}P_\rho \ .$$

Next, since Ω_θ is a zero eigenvector of the operator L_θ (see Theorem 12.3), we have by Proposition 14.1 that the vector

$$\varphi_\lambda := P_\rho\Omega_\theta$$

is a zero eigenvector of the operator $L_0(0)$. By Theorem 11.1 and the relation $U(\theta)\Omega^{(0)} = \Omega^{(0)}$, we have that

$$\varphi_\lambda = \Omega^{(0)} + O(\lambda) \ .$$

Hence due to Eqns (16.4) and (16.7),

$$0 = L_0(0)\varphi_\lambda = (\Lambda_0 \otimes \mathbf{1}^r)\Omega^{(0)} + O(\lambda^{7/3}) \ .$$

Since the operator Λ_0 is of the order λ^2, we have $(\Lambda_0 \otimes \mathbf{1}^r)\Omega^{(0)} = 0$ and consequently $\Lambda_0\Omega^p = 0$. □

Since Ω^p is a real vector ($C\Omega^p = \Omega^p$), it is also a zero eigenvector for $\mathrm{Re}\Lambda_0$ and $\mathrm{Im}\Lambda_0 = -\Gamma_0$, separately.

Next, we concentrate on the operator Γ_ε We have

Proposition 16.4.

(i) Γ_ε *is non-negative,* $\Gamma_\varepsilon \geq 0$*, and real,* $C\Gamma_\varepsilon = \Gamma_\varepsilon C$*;*

(ii) $\Gamma_\varepsilon > 0$ *if* $\varepsilon \neq 0$*;*

(iii) Γ_ε *has a simple eigenvalue at* 0 *with the eigenvector* $\Omega^p \in \mathrm{Ran}P_\varepsilon^p$ *if* $\varepsilon = 0$*;*

(iv) $\inf_\beta \inf_\varepsilon \gamma_\varepsilon > 0$*, where* $\gamma_\varepsilon = \inf\{\sigma(\Gamma_\varepsilon)\backslash\{0\}\}$.

This proposition is proven in Appendix II. (Note that statement (i) follows directly from the definition of Γ_ε and a simple computation.)

Eqns (16.4) and (16.7) imply that the operator $L_\varepsilon(z)$ is of the form

$$L_\varepsilon(z) = \Lambda_\varepsilon \otimes \mathbf{1}^r + \mathbf{1}^p \otimes L_\theta^r + O(\lambda^{7/3}) \quad \text{on} \quad \mathrm{Ran}P_\rho \ . \tag{16.9}$$

(a) $\varepsilon \neq 0$. Eqn (16.9) and statement (i) show that the spectrum of $L_\varepsilon(z) + z \cdot \mathbf{1}$, $z \in S_{\varepsilon,\rho}$, lies in the half space

$$\{z \in \mathbb{C} \big| \mathrm{Im} z \leq -\gamma_\varepsilon + O(\lambda^{7/3})\} .$$

Take $\sigma := \frac{1}{2\delta^2}\gamma$ and $\nu = \sigma\lambda^2 = \frac{1}{2}\gamma$, where $\gamma = \min_\varepsilon \gamma_\varepsilon$. Then for $|\lambda|$ sufficiently small, L_θ has no spectrum in the strip $S_{\varepsilon,\nu}$, provided $\varepsilon \neq 0$.

(b) $\varepsilon = 0$. The problem here is that Γ_ε has a zero eigenvalue. To see how this effects the spectrum of $L_\varepsilon(z)$ consider the sum of the first two terms on the r.h.s. of (16.9), i.e. ignore for the moment the remainder $O(\lambda^{7/3})$. Besides of the spectrum coming from non-zero eigenvalues of Λ_ε and the spectrum of L_θ^r, similar to the spectrum in the case $\varepsilon \neq 0$, this sum has the spectrum $0 + \sigma(L_\theta^r) = \sigma(L_\theta^r)$ coming from the zero eigenvalue of $\Lambda_{\varepsilon=0}$ (see Fig. 16.2).

Fig. 16.2. Spectrum of $\Lambda \otimes \mathbf{1}^r + \mathbf{1}^p \otimes L_\theta^r$ for $\varepsilon = 0$.

This part touches the real axis at the point $z = 0$. Thus, because of the remainder $O(\lambda^{7/3})$ we can control the spectrum of the operator $L_\varepsilon(z)$ only outside an $O(\lambda^{7/3})$-neighbourhood of the point $z = 0$ - the spectral point of our interest. To get at the point $z = 0$ we have to perform a renormalization group analysis on $L_\varepsilon(z)$ in the spirit of [BFS 2,3]. This analysis is based on the family of projection operators P_ρ and yields

Theorem 16.5. *Let the conditions of Section 2 hold. Then for all θ, $0 < \mathrm{Im}\theta < \frac{\pi}{2}$,*

(α) the part of the spectrum of L_θ in $S_{\varepsilon,\rho}$ lies in the set K,

(β) 0 is a simple eigenvalue of L_θ.

A proof of this theorem is sketched in Appendix III.

Combining the results for $\varepsilon \neq 0$ and $\varepsilon = 0$ obtained above and using that

$$\bigcup_{\varepsilon \in \sigma(L^p)} S_{\varepsilon,\rho} = \{z \in \mathbb{C} \big| \mathrm{Im} z \geq -\frac{1}{2}\rho\}$$

and that $\rho = \frac{1}{2}\gamma$, we arrive at the statement of the Theorem 16.1. □

Corollary 16.6. *Under the conditions of Section 2, the spectrum of $L \cdot P_\Omega^\perp$ is absolutely continuous and consequently the statements of Theorems 4.1 and 4.2 are true.*

Proof. Let ψ and φ be $U(\theta)$-analytic vector in the strip $\left\{\theta \in \mathbb{C} \big| |\mathrm{Im}\theta| \leq \frac{\pi}{2}\right\}$ and let $\psi_\theta = U(\theta)\psi$ and $\varphi_\theta = U(\theta)\varphi$. By Theorem 12.2 we have that

$$\langle \varphi, (L-z)^{-1}\psi \rangle = \langle \varphi_{\overline{\theta}}, (L_\theta - z)^{-1}\psi_\theta \rangle \tag{16.10}$$

for $0 < \mathrm{Im}\theta < \frac{\pi}{2}$ and $\mathrm{Im}z \geq K$, where K is given in that theorem. Now fix θ with $0 < \mathrm{Im}\theta < \frac{\pi}{2}$. Then Theorem 16.1 implies that the r.h.s. of (16.10) is analytic in z in domain (16.1) with a simple pole at $z = 0$. Since for a given strip $U(\theta)$-analytic vectors form a dense set in $\mathcal{H}$, the result follows. □

Appendix I. Existence of the KMS states (incomplete)

In this appendix we present two proofs of statements (ii) and (iii) of Theorem 11.1. The first proof follows a standard general argument (see e.g. [BRII, Sections 6.2.2 and 6.3.4]). We just sketch it here. Let Λ be a box in $\mathbb{R}^3$. With this box we associate, in a standard way, the Fock space $\mathcal{H}^r_{\infty,\Lambda} = \mathcal{F}_\Lambda$ and the Hamiltonian H_Λ on $\mathcal{H}_{\infty,\Lambda} := \mathcal{H}^p_\infty \otimes \mathcal{H}^r_{\infty,\Lambda}$ (see e.g. [GJ,H, BR II]). Then the operator $e^{-\beta H_\Lambda}$ is trace class for $\beta > 0$, so that we can define the Gibbs state

$$\omega_\Lambda(A) = \mathrm{tr}(A\rho_\Lambda) \ ,$$

where $\rho_\Lambda = e^{-\beta H_\Lambda}/\mathrm{tr}e^{-\beta H_\Lambda}$, for all observables "localized" in $\Lambda_0 \subseteq \Lambda$ (see [H]). Now, the states ω_Λ are uniformly bounded:

$$\|\omega_\Lambda\| \leq \mathrm{tr}\rho_\Lambda = 1 \ .$$

Hence by the Alaoglu theorem the set $\{\omega_\Lambda\}$ is weak*-compact, i.e., there is a sequence $\{\omega_{\Lambda'}\}$, $\Lambda' \to \mathbb{R}^3$, and a state ω on $\mathcal{A}$ s.t.

$$\omega_{\Lambda'}(A) \to \omega(A) \quad \forall\, A \in \mathcal{A}$$

as $\Lambda' \to \mathbb{R}^3$. A standard argument (see [BRII, Proposition 5.3.25]) shows that ω is a KMS state at the temperature $\frac{1}{\beta}$. Note that we did not require $|\lambda|$ to be small.

Next, using the Araki-Woods representations π_Λ and π'_Λ of the local C^* algebras $\mathcal{A}_\Lambda$ (for the box Λ) by bounded operators on $\mathcal{H}_\Lambda := \mathcal{H}_{\infty,\Lambda} \otimes \mathcal{H}_{\infty,\Lambda}$ (see supplement I) we construct cyclic vectors Ω_Λ and Liouville operators L_Λ s.t.

$$\omega_\Lambda(A) = \langle \Omega_\Lambda, \pi_\Lambda(A)\Omega_\Lambda \rangle$$

$\forall A \in \mathcal{A}_\Lambda$, and

$$L_\Lambda \Omega_\Lambda = 0 \ .$$

(0bserve that, as it is shown in Supplement I, in the finite volume case

$$L_\Lambda = \pi_\Lambda(H_\Lambda) - \pi'_\Lambda(H_\Lambda) \ .$$

This representation does not survive the infinite volume limit since $\pi(H)$ and $\pi'(H)$ are not defined separately in the infinite volume case.)

Next, one can show that as $\Lambda \uparrow \mathbb{R}^3$,

$$\Omega_\Lambda \to \Omega \in D(L)$$

and

$$(L_\Lambda - z)^{-1}\psi \to (L - z)^{-1}\psi$$

for any $\psi \in \mathcal{H}_{\Lambda'}$ and any Λ' (one can embed $\mathcal{H}_{\Lambda'} \subset \mathcal{H}_\Lambda \subset \mathcal{H}$ for $\Lambda' \subset \Lambda$). The last two relations imply

$$L\Omega = 0 \ . \qquad \square$$

Now we proceed to the second proof of the existence of KMS states for our systems. The key argument here – the Dyson expansion - is also a standard one (see [BRII, Theorem 5.4.4]). However, the case treated in the literature is that of bounded perturbations, while the perturbations in our case are only relatively bounded. To accomodate this complication requires some extra work.

Observe that the state constructed below (see Proposition 11.2) is a thermodynamic limit of finite-volume states discussed above.

Proof of Proposition 11.2. We introduce the operator $\Gamma_{i\beta} := e^{-\beta L_\lambda} e^{\beta L_0}$ and show that for any β, the cyclic vector $\Omega^{(0)}$, defined in (8.1), is in the domain of $\Gamma_{i\beta}$ and that

$$\|\Gamma_{i\beta}\Omega^{(0)} - \Omega^{(0)}\| \le C\lambda \ ,$$

uniformly, in β. On the other hand, since $L^{(0)}\Omega^{(0)}=0$, we have $\Gamma_{i\beta}\Omega^{(0)}=e^{-\beta L_\lambda/2}\Omega^{(0)}=\Omega\cdot\|e^{-\beta L_\lambda/2}\Omega^{(0)}\|$ as in Proposition 11.2. After that an abstract argument (see [BRII, Proof of Theorem 5.4.4] implies that $\omega(A):=\langle\Omega,\pi(A)\Omega\rangle$, where π is given in (9.3), is a KMS state for the total system at the temperature $T=\frac{1}{\beta}$. This is exactly the state constructed at the beginning of this appendix.

Our analysis of $\Gamma_{i\beta}$ relies of the following Dyson formula:

$$\Gamma_{i\beta}=\sum_{r=0}^{\infty}(-1)^n\Gamma^{(n)} ,$$

where $\Gamma^{(0)}=\mathbf{1}$ and, with $I':=\pi(v)$,

$$\Gamma^{(n)}=\int_{\Delta_\beta}d^n s\tau_{is_n}(I)\dots\tau_{is_1}(I) ,$$

with $\Delta_\beta=\{0\le s_n\le\dots\le s_1\le\beta\}$ and $\tau_{is}(A)=e^{-sL_0}Ae^{sL_0}$.

The appendix is not finished.

Appendix II. The operator Γ_ε (Fermi Golden Rule)

In this appendix we prove Proposition 16.4 of Section 16 concerning the operator Γ_ε defined in Eqn (16.8). We do this only in the phonon case. In the photon case the computations are similar but lengthier. In what follows we use the following notation: E_ℓ denote the eigenvalues of H^p and P_ℓ the corresponding eigenprojections so that $P^p_\varepsilon=\sum\limits_{E_i-E_j=\varepsilon}P_i\otimes P_j$. Moreover, we set $E_{k\ell}=E_k-E_\ell$ and $\overline{P}_i=\mathbf{1}-P_i$.

For simplicity we assume the following non-degeneracy conditions:

(a) $E_i-E_j=E_{i'}-E_{j'}$ only if $i=i'$ and $j=j'$

and

(b) the eigenvalues E_j's of H^p are simple.

Proof of Proposition 16.4. Statement (i) follows from the definition of Γ_ε and a simple computation.

Next, let $P^p_{ij}=(P_i\otimes P_j)$ and write Γ_ε as

$$\Gamma_\varepsilon=\Gamma_\varepsilon^{\mathrm{diag}}+\Gamma_\varepsilon^{\mathrm{off-diag}} ,$$

where

$$\Gamma_\varepsilon^{\rm diag} = (P_{ij}^p \Gamma_\varepsilon P_{ij}^p \big| E_{ij} = \varepsilon)$$

and

$$\Gamma_\varepsilon^{\rm off-diag} = (P_{ij}^p \Gamma_\varepsilon P_{ij}^p \big| (ij) \neq (i'j') \quad \text{and} \quad E_{ij} = E_{i'j'} = \varepsilon) \ .$$

By the non-degeneracy conditions, (a) and (b), above $\Gamma_\varepsilon^{\rm diag}$ has positive matrix elements and

$$\Gamma_\varepsilon^{\rm off-diag} = 0 \quad \text{if} \quad \varepsilon \neq 0 \ .$$

These two properties imply (ii).

To prove (iii) we use that as a computation below (see Eqn (II.9)) shows the matrix $\Gamma_0^{\rm off-diag}$ has strictly negative entries. Hence for a sufficiently large a, $(\Gamma_0 + a)^{-1} = \sum_{n=0}^{\infty} (\Gamma' + a)^{-1} \left[\Gamma''(\Gamma' + a)^{-1}\right]^n$, where $\Gamma' = \Gamma_0^{\rm diag}$ and $\Gamma'' = \Gamma_0^{\rm off-diag}$ are the diagonal and off-diagonal parts of Γ_0 respectively, is positivity improving (a vector is positive iff all its entries are positive). Since $\Gamma_0 \Omega^p = 0$ and since $\Omega^p \in {\rm Ran} P_0^p$ can be identified with the vector

$$\Omega^p = (e^{-\beta E_0/2}, e^{-\beta E_1/2}, \dots)$$

whose entries are positive, it follows by the Perron-Frobenius argument that 0 is the lowest and nondegenerate eigenvalue of Γ_0.

Statement (iv) is proven by a direct computation. □

In the next statement, the operator Γ_ε is computed explicitly.

Proposition II.1. *In the phonon case, the operator Γ_ε defined in Eqn (16.8) has the following representation*

$$\begin{aligned}\Gamma_\varepsilon = \sum_{E_i - E_j = \varepsilon} \int \Big\{ & P_i \big[\overline{g}_x \overline{P}_i \delta(H^p - E_i + \omega) g_x (1+\rho) + g_x \delta(H^p - E_i - \omega) \overline{g}_x \rho\big] P_i \otimes P_j \\ & + P_i \otimes P_j \big[g_x \overline{P}_j \delta(H^p - E_j + \omega) \overline{g}_x (1+\rho) + \overline{g}_x \delta(H^p - E_j - \omega) g_x \rho\big] P_j \Big\} \\ - 2 \sum_{\substack{E_i - E_j = \varepsilon \\ E_{i'} - E_{j'} = \varepsilon \\ (ij) \neq (i'j')}} \int \Big\{ & P_i \overline{g}_x P_{i'} \otimes P_j g_x P_{j'} \delta(\omega - E_{ii'}) \\ & + P_i g_x P_{i'} \otimes P_j \overline{g}_x P_{j'} \delta(\omega + E_{ii'}) \Big\} \sqrt{\rho(1+\rho)} \ , \end{aligned} \tag{II.1}$$

restricted to $\mathrm{Ran}P^p_\varepsilon$. *Here, recall,* $g_x(k) = \frac{\chi(k)}{\sqrt{2\omega}} e^{-ikx}$.

Proof. In the phonon case the operator I entering Γ_ε is given in (9.6). We rewrite it here as

$$\begin{aligned} I &= \pi^p\big(a_1(\sqrt{1+\rho} g_x)\big) + \pi^p\big(a_2^*(\sqrt{\rho}\overline{g}_x)\big) \\ &\quad - \pi^{p\prime}\big(a_1^*(\sqrt{\rho}\overline{g}_x)\big) - \pi^{p\prime}\big(a_2(\sqrt{1+\rho} g_x)\big) + \text{ h.c.} \end{aligned}$$

where π^p and $\pi^{p\prime}$ are defined in (6.3). Substituting this expression into Eqn (12.8) and using that a_j annihilate Ω_j, $j = 1, 2$, we obtain

$$\begin{aligned} \Gamma_\varepsilon &= P^p_\varepsilon \Big\langle \big[\pi^p(a_1(\sqrt{1+\rho} g_x) + \pi^p(a_2(\sqrt{\rho}\overline{g}_x)) \\ &\quad - \pi^{p\prime}(a_1(\sqrt{\rho}\overline{g}_x)) - \pi^{p\prime}(a_2(\sqrt{1+\rho} g_x)\big] \delta(L^{(0)} - \varepsilon) \\ &\quad \times \big[\pi^p(a_1^*(\sqrt{1+\rho} g_x) + \pi^p(a_2^*(\sqrt{\rho}\overline{g}_x)) \\ &\quad - \pi^{p\prime}(a_1^*(\sqrt{\rho}\overline{g}_x)) - \pi^{p\prime}(a_2^*(\sqrt{1+\rho} g_x))\big] \Big\rangle_{\Omega_1 \otimes \Omega_2} P^p_\varepsilon \ , \end{aligned} \tag{II.2}$$

restricted to $\mathrm{Ran}P^p_\varepsilon$.

Now we pull the a_j's to the right, till they either contract with a_j^*'s or hit the vacuum Ω_j. To pull them through $\delta(L^{(0)} - \varepsilon)$ we use the following pull-through formulae

$$a_{1,2}(k)\delta(L^{(0)} - \varepsilon) = \delta(L^{(0)} \pm \omega(k) - \varepsilon) a_{1,2}(k) \ , \tag{II.3}$$

which is derived in a standard way using Eqn (7.6) (see [BFS2]). As a result we obtain

$$\begin{aligned} \Gamma_\varepsilon &= P^p_\varepsilon \int \Big\{ \pi^p(\overline{g}_x)\delta(L^p + \omega - \varepsilon)\pi^p(g_x)(1+\rho) \\ &\quad - \pi^p(\overline{g}_x)\delta(L^p + \omega - \varepsilon)\pi^{p\prime}(\overline{g}_x)\sqrt{(1+\rho)\rho} \\ &\quad + \pi^p(g_x)\delta(L^p - \omega - \varepsilon)\pi^p(\overline{g}_x)\rho \\ &\quad - \pi^p(g_x)\delta(L^p - \omega - \varepsilon)\pi^{p\prime}(g_x)\sqrt{(1+\rho)\rho} \\ &\quad - \pi^{p\prime}(g_x)\delta(L^p + \omega - \varepsilon)\pi^p(g_x)\sqrt{(1+\rho)\rho} \Big\} \\ &\quad + \pi^{p\prime}(g_x)\delta(L^p + \omega - \varepsilon)\pi^{p\prime}(\overline{g}_x)\rho \\ &\quad - \pi^{p\prime}(\overline{g}_x)\delta(L^p - \omega - \varepsilon)\pi^p(\overline{g}_x)\sqrt{(1+\rho)\rho} \\ &\quad + \pi^{p\prime}(\overline{g}_x)\delta(L^p - \omega - \varepsilon)\pi^{p\prime}(g_x)(1+\rho) \end{aligned} \tag{II.4}$$

restricted to $\mathrm{Ran}P^p_\varepsilon$. Now using that $L^p = H^p \otimes \mathbf{1}^p_\infty - \mathbf{1}^p_\infty \otimes H^p$, that

$$P^p_\varepsilon = \sum_{E_i - E_j = \varepsilon} P_i \otimes P_j \ ,$$

and that $\pi^p(A) = A \otimes \mathbf{1}$ and $\pi^{p\prime}(A) = \mathbf{1} \otimes CAC$, on $\mathcal{H}^p = \mathcal{H}^p_\infty \otimes \mathcal{H}^p_\infty$, we obtain

$$\begin{aligned}
\Gamma_\varepsilon = \sum_{\substack{E_i - E_j = \varepsilon \\ E_{i'} - E_{j'} = \varepsilon}} \int \Big\{ & P_i \overline{g}_x \delta(H^p + \omega - E_i) g_x P_i \otimes P_j \delta_{i,i'} \delta_{j,j'} (1+\rho) \\
& - P_i \overline{g}_x P_{i'} \otimes P_j g_x P_{j'} \delta(\omega - E_{ii'}) \sqrt{(1+\rho)\rho} \\
& + P_i g_x \delta(H^p - \omega - E_i) \overline{g}_x P_i \otimes P_j \delta_{i,i'}, \delta_{j,j'} \rho \\
& - P_i g_x P_{i'} \otimes P_j \overline{g}_x P_{j'} \delta(\omega + E_{ii'}) \sqrt{(1+\rho)\rho} \\
& - P_i \overline{g}_x P_{i'} \otimes P_j g_x P_{j'} \delta(\omega + E_{ii'}) \sqrt{(1+\rho)\rho} \Big\} \\
& + P_i \otimes P_j \overline{g}_x \delta(-H^p + \omega + E_j) g_x P_j \delta_{i,i'}, \delta_{j,j'} \rho \\
& - P_i g_x P_{i'} \otimes P_j \overline{g}_x P_{j'} \delta(\omega - E_{ii'}) \sqrt{(1+\rho)\rho} \\
& + P_i \otimes P_j g_x \delta(-H^p - \omega + E_j) \overline{g}_x P_j \delta_{i,i'} \delta_{j,j'} (1+\rho)
\end{aligned} \tag{II.5}$$

restricted to $\mathrm{Ran}(P^p_\varepsilon)$. Finally, using that

$$\delta(H^p \pm \omega - E_i) P_i = \delta(\omega) P_i \tag{II.6}$$

and similarly for $\delta(H^p \pm \omega - E_j)$, that

$$P_i e^{ikx} P_i \otimes P_j e^{-ikx} P_j \delta(\omega) = P_i \otimes P_j \delta(\omega) \tag{II.7}$$

and similarly for the other terms in (II.5), and finally that

$$\int 2\delta(\omega) \big[(1+2\rho) - 2\sqrt{(1+\rho)\rho} \big] g^2 = 0 \ , \tag{II.8}$$

we arrive at (II.1). □

Assuming for simplicity that all the eigenvalues E_j of H^p are simple we write out Γ_ε more explicitly as a (possibly, infinite) matrix:

$$\begin{aligned}
\Gamma_\varepsilon = \Bigg(\int \Big\{ & \sum_{\ell \neq i} \big| \langle \psi_\ell, g_x \psi_i \rangle \big|^2 \big[\delta(\omega - E_{i\ell})(1+\rho) + \delta(\omega + E_{i\ell})\rho \big] \\
& + \sum_{\ell \neq j} \big| \langle \psi_\ell, g_x \psi_j \rangle \big|^2 \big[\delta(\omega - E_{j\ell})(1+\rho) + \delta(\omega + E_{j\ell})\rho \big] \Big\} \delta_{i,i'}, \delta_{j,j'} \\
& - 2 \int \Big[\langle \psi_i, \overline{g}_x \psi_{i'} \rangle \langle \psi_j, g_x \psi_{j'} \rangle \delta(\omega - E_{jj'}) + \langle \psi_i, g_x \psi_{i'} \rangle \langle \psi_j, \overline{g}_x \psi_{j'} \rangle \delta(\omega + E_{jj'}) \Big] \\
& \sqrt{\rho(1+\rho)} \delta_{(ij) \neq (i'j')} \ ,
\end{aligned} \tag{II.9}$$

where the ψ_ℓ's are the eigenfunctions of H^p corresponding to the eigenvalues E_ℓ and (i,j) and (i',j') run through pairs satisfying $E_i - E_j = \varepsilon$ and $E_{i'} - E_{j'} = \varepsilon$. Here we have used that $E_{ii'} = E_{jj'}$ and assumed without loss of generality that the eigenfunctions ψ_i's are real.

References

[Am] Amann, A. Ground states of a spin-boson model. *Ann. Phys.* **208**, 414 (1991).

[Ar1] Arai, A. On a model of a harmonic oscillator coupled to a quantized, mass-less, scalar field, I. *J. Math. Phys.* **22**, 2539 (1981).

[Ar2] Arai, A. On a model of a harmonic oscillator coupled to a quantized, mass-less, scalar field, II. *J. Math. Phys.* **22**, 2549 (1981).

[AW1] Araki, H., Woods, E.J. Representation of the canonical commutation relations describing a nonrelativistic infinite free Bose gas. *J. Math. Phys.* **4**, 637 (1963).

[BFS1] Bach, V., Fröhlich, J., Sigal, I.M. Mathematical Theory of Nonrelativisitic Matter and Radiation. *Lett. Math. Phys.* **34**, 183 (1995).

[BFS2] Bach, V., Fröhlich, J., Sigal, I.M. Quantum Electrodynamics of Confined Non-Relativistic Particles. *Adv. in Math.* **137**, 205-298 (1998).

[BFS3] Bach, V., Fröhlich, J., Sigal, I.M. Renormalization Group Analysis of Spectral Problems in Quantum Field Theory. *Adv. in Math.*.**137**, 299-395 (1998).

[BFS4] Bach, V., Fröhlich, J., Sigal, I.M. Spectral Analysis for Systems of Atoms and Molecules coupled to the Quantized Radiation Field, preprint, Berlin, 1998.

[BR1] Bratteli, O., Robinson, D. Operator Algebras and Quantum Statistical Mechanics I. Berlin, Heidelberg, New York: Springer, 1979.

[BR2] Bratteli, O., Robinson, D. Operator Algebras and Quantum Statistical Mechanics II. Berlin, Heidelberg, New York: Springer, 1981.

[CDG] Cohen-Tannoudji, C., Dupont-Roc, J., Grynberg, G. Photons and Atoms – Introduction to Quantum Electrodynamics, Berlin, Heidelberg, New York: Springer, 1991.

[D1] Davies, E.B. Markovian master equations. *Commun. Math. Phys.* **39**, 91 (1974).

[D2] Davies, E.B. Markovian master equations II. *Math. Ann.* **219**, 147 (1976).

[D3] Davies, E.B. Quantum Theory of Open Systems. New York: Academic Press, 1976.

[DI] Dirac, P.A.M. The quantum theory of the emission and absorption of radiation. *Proc. Royal Soc. London Series A* **114**, 243 (1927).

[FNV1] Fannes, M. Nachtergaele, B., Verbeur, A. The equilibrium states of a spin-boson model. *Commun. Math. Phys.* **114**, 537 (1988).

[FNV2] Fannes, M. Nachtergaele, B., Verbeur, A. Quantum tunneling in the spin-boson model. *Europhysics Lett.* **4**, 963 (1987).

[FNV3] Fannes, M. Nachtergaele, B., Verbeur, A. Tunneling in the equilibrium state of a spin-boson model. *J. Phys. A* **21**, 1759 (1988).

[Ha] Haag, R. Local Quantum Physics. Berlin, Heidelberg, New York: Springer, 1993.

[HHW] Haag, R., Hugenholtz, N.M., Winnink, M. On the equilibrium states in quantum statistical mechanics. *Commun. Math. Phys.* **5**, 215 (1967).

[HS1] Hübner, M., Spohn, H. Radiative decay: Non perturbation approaches. Preprint.

[HS2] Hübner, M., Spohn, H. Spectral properties of the spin-boson Hamiltonian. Preprint.

[JP1] Jakšić, V. and C.-A. Pillet. On a Model for Quantum Friction I. Fermi's Golden Rule and Dynamics at Zero Temperature. *Ann. Inst. H. Poincaré* **62**, 47 (1995).

[JP2] Jakšić, V. and C.-A. Pillet. On a Model for Quantum Friction II. Fermi's Golden Rule and Dynamics at Positive Temperature. *Commun. Math. Phys.* **176**, 619 (1996).

[JP3] Jakšić, V. and C.-A. Pillet. On a Model for Quantum Friction III. Ergodic Properties of the Spin-Boson System. *Commun. Math. Phys.* **178**, 627 (1996).

[JP4] Jakšić, V. and C.-A. Pillet. Ergodic Properties of the non-Markovian Langevin Equation, (to appear in *Lett. Math. Phys.*).

[JP5] Jakšić, V. and C.-A. Pillet. Ergodic Properties of Classical Dissipative Systems I, (preprint 1996).

[LCD] Legget, A.J. et al. Dynamics of the dissipative two-state system. *Rev. Mod. Phys.* **59**, 1 (1987).

[LP] Lewis, J.T., Pulè, J.V. (1974). The equilibrium states of the free boson gass. *Commun. Math. Phys.* **36**, 1.

[Ma] Maassen, H. Return to thermal equilibrium by the solution of a quantum Langevin equation. *J. Stat. Phys.* **34**, 239 (1984).

[P] Pauli, W. Pauli Lectures on Physics: Volume 4. Statistical Mechanics. Edited by C.P. Enz. Cambridge, MA: The MIT Press, 1973.

[Pu] Pulè, J.V. The Bloch equations. *Commun. Math. Phys.* **38**, 241 (1984).

[Ro1] Robinson, D.W. C*-algebras and quantum statistical mechanics. In C*-Algebras and their Applications to Statistical Mechanics and Quantum Field Theory. Kastler, D., (ed.) Amsterdam: North Holland, 1976.

[Ro2] Robinson, D.W. Return to equilibrium. *Commun. Math. Phys.* **31**, 171 (1973).

[RS] Reed, M., Simon, B. Methods of Modern Mathematical Physics III. Scattering Theory. London: Academic Press, 1978.

[SDLL] Smedt, P., Dürr, D., Lebowitz, J.L. Liverani, C. Quantum system in contact with a thermal environment: Rigorous treatment of a simple model. *Commun. Math. Phys.* **120**, 120 (1988).

[Sp] Spohn, H. (1989). Ground state(s) of the spin-boson Hamiltonian. *Commun. Math. Phys.* **123**, 277 (1989).

[SpD] Spohn, S., Dümcke, R. Quantum tunneling with dissipation and the Ising model over $\mathbb{R}$. *J. Stat. Phys.* **41**, 381 (1985).