Integrals of Borcherds forms

by

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Introduction.

Let V be a non-degenerate inner product space over $\mathbb{Q}$ of signature $(n,2)$, and let D be the space of oriented negative 2-planes in $V(\mathbb{R})$. In [**2**], Borcherds constructed certain meromorphic modular forms $\Psi(F)$ on D with respect to arithmetic subgroups Γ_M of $G = O(V)$ by regularizing the theta integral of vector valued elliptic modular forms f of weight $1-\frac{n}{2}$ for $SL_2(\mathbb{Z})$ with poles at the cusp, cf. also [**1**], [**21**], [**7**], [**8**]. The Borcherds forms $\Psi(f)$ can be viewed as meromorphic sections of powers of a certain line bundle $\mathcal{L}$ on $X = \Gamma_M\backslash D$. Taking the standard Petersson metric $||\ ||$ on $\mathcal{L}$, it is of interest in Arakelov geometry to compute the integral:

$$\kappa(\Psi(f)) := -\mathrm{vol}(X)^{-1}\int_{\Gamma_M\backslash D}\log||\Psi(z,f)||^2\,d\mu(z), \tag{0.1}$$

where $d\mu(z)$ is a $G(\mathbb{R})$–invariant volume form on D.

In this paper, we give an explicit formula for $\kappa(\Psi(f))$ is almost all cases[2]. To describe it, suppose that M is a lattice in V such that the quadratic form $Q(x) = \frac{1}{2}(x,x)$ is $\mathbb{Z}$–valued and let $M^\sharp \supset M$ be the dual lattice. Recall that the modular form f is valued in the space $\mathbb{C}[M^\sharp/M]$, for a suitable choice of M, and has a Fourier expansion of the form

$$f(\tau) = \sum_{\mu\in M^\sharp/M}\sum_{m\in\mathbb{Q}} c_\mu(m)\,q^m\,\varphi_\mu \tag{0.2}$$

where $\tau\in\mathfrak{H}$, $q^m = e(m\tau)$, and where $c_\mu(m)$ is zero unless $m \in Q(\mu)+\mathbb{Z}$ and $m > -R$ for some positive integer R. In addition, if $m \le 0$, then $c_\mu(m)\in\mathbb{Z}$. Let

$$\Gamma_M = \{\gamma \in SO(V)(\mathbb{Q}) \mid \gamma M = M \text{ and } \gamma \text{ acts trivially in } M^\sharp/M\}, \tag{0.3}$$

and let $X_M = \Gamma_M\backslash D$, so that X is a quasi–projective variety. For each $m>0$ and $\mu\in M^\sharp/M$, there is a divisor $Z(m,\mu)$ on X, associated to the set of vectors $x\in\mu+M$ with $Q(x)=m$. These divisors, called rational quadratic divisors or Heegner divisors in [**2**], include the Heegner

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[2]In the *exceptional cases*, where $\dim(V)=3$ (resp. 4) and V has Witt index 1 (resp. 2), some additional regularization is required.

points, for $n = 1$, the Hirzebruch–Zagier curves on Hilbert modular surfaces, for $n = 2$, and the Humbert surfaces on Siegel threefolds, for $n = 3$. They are also special cases of the cycles considered in [**28**], [**29**], [**26**], etc.. Then, a key fact, due to Borcherds [**2**], is that the divisor of the form $\Psi(f)^2$, which has weight $c_0(0)$, is an explicit linear combination of these cycles:

$$\mathrm{div}(\Psi(f)^2) = \sum_{\mu} \sum_{m>0} c_\mu(-m)\, Z(m,\mu). \tag{0.4}$$

Our first result concerns the generating function for the degrees of the cycles $Z(m,\mu)$. Let Ω be the first Chern form of the metrized line bundle $\mathcal{L}^\vee$ on X, dual to $\mathcal{L}$, and let

$$\deg(Z(m,\mu)) = \int_{Z(m,\mu)} \Omega^{n-1} \tag{0.5}$$

be the volume of the cycle $Z(m,\mu)$ with respect to Ω. Similarly, let

$$\mathrm{vol}(X) = \int_X \Omega^n. \tag{0.6}$$

From now on, we exclude the two exceptional cases. In addition, for simplicity here in the introduction, we make the following 'class number 1' assumption[3]: Let

$$K_M = \{ g \in SO(V)(\mathbb{A}_f) \mid gM = M \;\; \text{and } g \text{ acts trivially on } M^\sharp/M \; \}$$

and suppose that

$$SO(V)(\mathbb{A}_f) = SO(V)(\mathbb{Q})^+ \, K_M,$$

where $SO(V)(\mathbb{Q})^+ = SO(V)(\mathbb{Q}) \cap SO(V)(\mathbb{R})^+$, for $SO(V)(\mathbb{R})^+$ the identity component of $SO(V)(\mathbb{R})$.

We then show the following, using the Siegel–Weil formula and results of [**28**], [**29**], [**30**].

Theorem I. *For each $\mu \in M^\sharp/M$, there is an Eisenstein series $E(\tau, s; \mu, \frac{n}{2}+1)$, for $\tau \in \mathfrak{H}$ and $s \in \mathbb{C}$, of weight $\frac{n}{2}+1$ such that*

$$E(\tau, s_0; \mu) = \delta_{\mu,0} + \mathrm{vol}(X)^{-1} \sum_{m>0} \deg(Z(m,\mu))\, q^m,$$

where $s_0 = \frac{n}{2}$.

Analogous results for generating functions for cycles of higher codimension for anisotropic V's are discussed in [**28**], section 3, and in [**26**].

The integral $\kappa(\Psi(F))$ can also be expressed using the second term in the Laurent expansion at $s_0 = \frac{n}{2}$ of these Eisenstein series.

[3] Without this assumption, one must either work adelically, as is done in the body of the paper, or introduce sums over a suitable collection of lattices determined by the decomposition (1.3)

Theorem II. *For each $\mu \in M^\sharp/M$, the Fourier coefficients in the expansion*

$$E(\tau, s; \mu) = \sum_m A_\mu(s, m, v)\, q^m$$

have Laurent expansion at $s = s_0 = \frac{n}{2}$

$$A_\mu(s, m, v) = a_\mu(m) + b_\mu(m, v)(s - s_0) + O((s - s_0)^2).$$

Let

$$\kappa_\mu(m) = \begin{cases} \lim\limits_{v\to\infty} b_\mu(m, v) & \text{if } m > 0, \\ \frac{1}{2}\left(\log(2\pi) - \gamma\right) & \text{if } m = 0, \\ 0 & \text{if } m < 0. \end{cases}$$

Then, for f with Fourier expansion (0.2),

$$\kappa(\Psi(f)) = \sum_\mu \sum_{m\ge 0} c_\mu(-m)\, \kappa_\mu(m).$$

In addition, we derive the useful relation

$$-\mathrm{vol}(X)\, c_0(0) = \sum_\mu \sum_{m>0} c_\mu(-m)\ \deg(Z(m, \mu)).$$

The quantities $\kappa_\mu(m)$ can be calculated quite readily in any particular case; this will be done in a sequel [**33**].

As an illustration, consider the case where $M = \mathbb{Z}^5$ with quadratic form of signature $(3, 2)$ defined by $Q(x) = \frac{1}{2}{}^t x Q x$ where

$$Q = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 2 & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}.$$

In this example, which is worked out in detail in section 5, $|M^\sharp/M| = 2$ and, labeling the cosets by $\mu = 0,\ 1$, we have

$$E(\tau, \frac{3}{2}; \mu) = \delta_{\mu,0} + \zeta(-3)^{-1} \sum_{\substack{m>0 \\ 4m\equiv\mu \mod (4)}} H(2, 4m)\, q^m,$$

where $H(2, N)$ is the N-th coefficient in Cohen's Eisenstein series of weight $\frac{5}{2}$, [**11**],

$$\mathcal{H}_2(\tau) = \zeta(-3) + \sum_{\substack{N>0 \\ N\equiv 0,1 \mod (4)}} H(2, N)\, q^N.$$

In this case, as explained in [**47**] and [**19**], $\Gamma_M\backslash D \simeq \mathrm{Sp}_4(\mathbb{Z})\backslash\mathfrak{H}_2$ is the Siegel threefold of level 1, $\mathrm{vol}(X) = \zeta(-1)\zeta(-3)$, and $Z(m,\mu)$, for $4m \equiv \mu \mod (4)$, is the Humbert surface $\mathcal{G}_{4m}$, in the notation of [**47**]. Thus, Theorem I implies that

$$\deg(H_N) = -\frac{1}{12}\,H(2,N),$$

a relation due to van der Geer, [**47**]. Also, we find that, for $m > 0$ with $4m = n^2 d$ for a fundamental discriminant d, and with $4m \equiv \mu \mod (4)$,

$$\kappa_\mu(m) = \zeta(-3)^{-1}\,H(2,4m)\,\bigg[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - \frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} - C + \sum_{p|n}\bigg(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\bigg)\bigg].$$

If $4m \not\equiv \mu \mod (4)$, then $\kappa_\mu(m) = 0$. Here $L(s,\chi_d)$ is the L-series for the quadratic character χ_d and the other quantities are explained in section 5. It is shown by Gritesenko and Nikulin [**19**] that the Siegel cusp Δ_5 of weight 5 and quadratic character arises as a Borcherds form $\Psi(\mathbf{f}_5) = 2^{-6}\,\Delta_5(z)$, for a suitable meromorphic form $\mathbf{f}_5$ of weight $-\frac{1}{2}$ with expansion

$$\mathbf{f}_5(\tau) = \big(\,10 + 108\,q + 808\,q^2 + \dots\,\big)\,\varphi_0 + \big(\,q^{-\frac{1}{4}} - 64\,q^{\frac{3}{4}} - 513\,q^{\frac{7}{4}} + \dots\,\big)\,\varphi_1.$$

Thus, by Theorem II,

$$-\,\mathrm{vol}(X)^{-1}\int_X \log\big(|\Delta_5(z)|^2\,\det(y)^5\big)\cdot\Omega^3 = 10\,\bigg[-\frac{4}{3} - 2\,\frac{\zeta'(-3)}{\zeta(-3)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2}\,\log(2) + \log(\pi)\bigg] - 7\log(2).$$

The main idea in the proof of Theorem II is the following. Recall that, in Borcherds construction, it is essentially the quantity $\log||\Psi(f)||^2$, rather than the meromorphic form $\Psi(f)$ itself, which arises as a regularized theta integral. Therefore, after some justification, we can compute the integral of this quantity by first integrating the theta kernel over X and then taking the regularized integral against f. This procedure is valid provided the integral of the theta kernel is termwise abosolutely convergent, and it is for this reason that the exceptional cases must be excluded. The Siegel–Weil formula then identifies the integral of the theta kernel as a special value of an Eisenstein series of weight $\frac{n}{2} - 1$ at the point $s_0 = \frac{n}{2}$. The regularized integral of this series against f can then be evaluated by using Maass operators, which shifts the weight to $\frac{n}{2} + 1$, and a Stokes theorem argument from section 9 of [**2**].

In fact, the method used here should also be applicable to the calculation of the integrals of the functions arising via Borcherds's construction for more general signatures (p,q), and it would be interesting to investigate such cases. Note, in particular, that the remarkable product formulas for the $\Psi(F)$'s in the case of signature $(n,2)$ play no role.

Possible applications of the formula for $\kappa(\Psi(f))$ to arithmetic geometry are discussed in section 6. The main point is that there should be a close connection between the second term in the Laurent expansion of the Fourier coefficients of the Eisenstein series $E(\tau,s;\mu)$ at $s_0=\frac{n}{2}$, and the heights of the divisors $Z(m,\mu)$ on X, after extension to a suitable integral model. Such a connection is also suggested by the results of joint work [**32**] with Michael Rapoport and Tonghai Yang in which we compute the heights of Heegner type divisors on the arithmetic surfaces $\mathfrak{X}$ defined by Shimura curves, the case $n=1$ with V anisotropic. In fact, for suitablly defined classes $\widehat{\mathfrak{Z}}(m,v)\in\widehat{CH}^1(\mathfrak{X})$, the arithmetic Chow group[4] of $\mathfrak{X}$, and for a normalized version $\mathcal{E}(\tau,s;\varphi)$ of the Eisenstein series $E(\tau,s;\varphi)$ of weight $\frac{3}{2}$, we show that

$$\mathcal{E}'(\tau,\frac{1}{2};\varphi)=\sum_m\langle\,\widehat{\mathfrak{Z}}(m,v),\hat{\omega}\,\rangle\,q^m,$$

where $\tau=u+iv$, $\hat{\omega}\in\widehat{CH}^1(\mathfrak{X})$ is an extension of the metrized line bundle $\mathcal{L}^\vee$, dual to $\mathcal{L}$ to $\mathfrak{X}$, and $\langle\ ,\ \rangle$ is the Gillet–Soulé height pairing. Thus, the second term in the Eisenstein series gives a generating functions for the 'arithmetic volumes', at least in this example.

Here is a summary of the contents of the present paper. In section 1, we review the construction of the Borcherds forms $\Psi(F)$. An adelic formulation of this construction is given, which allows us to work more easily for general lattices and to make use of the adelic formulation of the Siegel–Weil formula and representation theory. Some explanation is given about how to pass back and forth between the adelic and classical version. In section 2, derive the formula for $\Psi(f)$, assuming certain facts about Eisenstein series, the Siegel–Weil formula, and about convergence. In section 3, we consider convergence questions and, in particular, justify the interchange of the integration of the theta kernel with the Borcherds regularized integral. In section 4, we first review the case of the Siegel–Weil formula which we need, including a refinement, already described by Weil, which is crucial in relating the integral over the orthogonal group occuring in this formula with the geometric integral we actually encounter. We then describe a general matching principle and apply it, together with the theory of [**28**], [**29**], [**30**], to prove that the degree generating function is given by the value of our Eisenstein series of weight $\frac{n}{2}+1$, as in Theorem I. This matching principle implies the coincidence of theta integrals for different quadratic spaces. For example, it shows that the degrees of the cycles $Z(m,\mu)$ occurring for spaces of signature $(n,2)$ always coincide with certain weighted representation numbers for spaces of signature $(n+2,0)$. This principle should have many other interesting applications. In section 5, we discuss the example of signature $(3,2)$ described above. In section 6, we give some speculations about the applications of the formulas for $\kappa(\Psi(f))$'s in arithmetic geometry.

[4]The definition of the class $\widehat{\mathfrak{Z}}(m,v)$ is still somewhat provisional.

This section also contains a brief discussion of the relation between our formula and work of Rohrlich [**45**] on analogous integrals of elliptic modular forms.

Work on the possibility of using Borcherds' forms $\Psi(F)$ in Arakelov theory began at the program on Arithmetic Geometry the Isaac Newton Institute during May–June, 1998. The main steps in computing $\kappa(\Psi(F))$ were done during a stay at Orsay in June of 1999. The examples in section 5 were worked out during a visit to Humbolt University in Berlin in June of 2001. The speculations in section 6 profited from discussions with Ulf Kühn at that time. I would like to thank these institutions and my hosts (J. Nekovar and C. Soulé in Cambridge, J.-B. Bost and G. Henniart in Orsay, and J. Kramer in Berlin) for providing a wonderful working environment.

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Contents.

§1. Borcherds' forms.

In this section we give an adelic formulation of a result of Borcherds on the construction of meromorphic modular forms. This formulation is convenient from the point of view of Hecke operators and Shimura varieties. Moreover, it is essential if we want to make use of the adelic version of the Siegel–Weil formula.

Let V be a vector space over $\mathbb{Q}$ with a non-degenerate quadratic form of signature $(n, 2)$, and let $H = \mathrm{GSpin}(V)$. We write $(x, y) = Q(x + y) - Q(x) - Q(y)$ for the associated bilinear form.

Let D be the space of oriented negative 2–planes in $V(\mathbb{R})$. Recall that D is isomorphic to the open subset Q_- of the quadric $Q \subset \mathbb{P}(V(\mathbb{C}))$ defined by

$$Q_- = \{w \in V(\mathbb{C}) \mid (w,w) = 0,\ (w,\bar{w}) < 0\}/\mathbb{C}^\times. \tag{1.1}$$

The isomorphism is given by $z \mapsto v_1 - iv_2 = w$, where v_1, v_2 is a properly oriented basis for $z \in D$ with $(v_1,v_1) = (v_2,v_2) = -1$ and $(v_1,v_2) = 0$. For a compact open subgroup $K \subset H(\mathbb{A}_f)$, the space

$$X_K = H(\mathbb{Q})\backslash\Big(D \times H(\mathbb{A}_f)/K\Big) \tag{1.2}$$

is the set of complex points of a quasi-projective variety rational over $\mathbb{Q}$ (via canonical models). This variety is projective if and only if V is anisotropic over $\mathbb{Q}$. It is smooth if the image of K in $SO(V)(\mathbb{A}_f)$ is neat. Fix a component D^+ of D, and write

$$H(\mathbb{A}) = \coprod_j H(\mathbb{Q})H(\mathbb{R})^+ h_j K,$$

where $H(\mathbb{R})^+$ is the identity component of $H(\mathbb{R}) \simeq \mathrm{GSpin}(n,2)$. Then

$$X_K \simeq \coprod_j \Gamma_j\backslash D^+, \tag{1.3}$$

where $\Gamma_j = H(\mathbb{Q}) \cap \big(\ H(\mathbb{R})^+ h_j K h_j^{-1}\ \big)$.

Let $\mathcal{L}_D$ be the restriction to $D \simeq Q_-$ of the tautological bundle on $\mathbb{P}(V(\mathbb{C}))$. The action of $O(V)(\mathbb{R})$ on $V(\mathbb{C})$ induces an action of $H(\mathbb{R})^+$ on $\mathcal{L}_D$, and hence there is a holomorphic line bundle

$$\mathcal{L} = H(\mathbb{Q})\backslash\Big(\mathcal{L}_D \times H(\mathbb{A}_f)/K\Big) \longrightarrow X_K. \tag{1.4}$$

This line bundle is also algebraic and has a canonical model over $\mathbb{Q}$, [**20**]. On the component $\Gamma_j\backslash D^+$, $\mathcal{L}$ has the form $\Gamma_j\backslash\mathcal{L}_D$. Define a Hermitian metric $h_{\mathcal{L}}$ on $\mathcal{L}_D$ by taking

$$h_{\mathcal{L}}(w_1,w_2) = -\frac{1}{2}(w_1,\bar{w}_2). \tag{1.5}$$

This metric is clearly invariant under the natural action of $O(V)(\mathbb{R})$ and hence descends to $\mathcal{L}$.

For a Witt decomposition

$$V(\mathbb{R}) = V_0 + \mathbb{R}e + \mathbb{R}f, \tag{1.6}$$

where e and f, with $(e,f)=1$ and $(e,e)=(f,f)=0$, span a hyperbolic plane with orthogonal complement V_0, note that $\mathrm{sig}(V_0)=(n-1,1)$ and let

$$C=\{y\in V_0\mid (y,y)<0\} \tag{1.7}$$

be the negative cone. Then $D\simeq Q_-$ is isomorphic to the tube domain

$$\mathbb{D}=\{z\in V_0(\mathbb{C})\mid y=\mathrm{Im}(z)\in C\}, \tag{1.8}$$

via the map

$$\mathbb{D}\longrightarrow V(\mathbb{C}),\qquad z\mapsto w(z):=z+e-Q(z)f. \tag{1.9}$$

composed with the projection to Q_-. The map $z\mapsto w(z)$ can be viewed as a nowhere vanishing holomorphic section of $\mathcal{L}_D$. Note that this section has norm

$$||w(z)||^2=-\frac{1}{2}\,(w(z),\bar{w}(z))=-(y,y)=:|y|^2. \tag{1.10}$$

For $h\in O(V(\mathbb{R}))$ or $H(\mathbb{R})$, we have

$$h\cdot w(z)=w(hz)\,j(h,z) \tag{1.11}$$

for a holomorphic automorphy factor

$$j:H(\mathbb{R})\times D\longrightarrow\mathbb{C}^\times. \tag{1.12}$$

For $k\in\mathbb{Z}$, holomorphic sections of $\mathcal{L}^{\otimes k}$ can be identified with holomorphic functions

$$\Psi:D\times H(\mathbb{A}_f)\longrightarrow\mathbb{C} \tag{1.13}$$

such that $\Psi(z,hk)=\Psi(z,h)$ for all $k\in K$ and

$$\Psi(\gamma z,\gamma h)=j(\gamma,z)^k\,\Psi(z,h) \tag{1.14}$$

for all $\gamma\in H(\mathbb{Q})$. The norm of the section $(z,h)\to\Psi(z,h)\cdot w(z)^{\otimes k}$ associated to Ψ is then

$$||\Psi(z,h)||^2=|\Psi(z,h)|^2\,|y|^{2k}. \tag{1.15}$$

We will refer to this as the Petersson norm of Ψ. Note that, under the isomorphism (1.3), Ψ corresponds to the collection $(\Psi(\cdot,h_j))_{\{j\}}$ of holomorphic functions on D^+ automorphic of weight k with respect to the Γ_j's.

Remark: In the case $n=1$, so that $\mathbb{D}=\mathfrak{H}^+\cup\mathfrak{H}^-$, the automorphy factor is

$$j(g,z)=\det(g)^{-1}(cz+d)^2,$$

so that the 'classical weight' of a section of $\mathcal{L}^{\otimes k}$ is $2k$.

We now give a version of Borcherds' construction [**2**] of meromorphic sections of (a certain twist of) $\mathcal{L}^{\otimes k}$. These are obtained by a regularized theta lift for the dual pair $(SL_2, O(V))$.

The basic theta kernel is constructed as follows. Let $S(V(\mathbb{A}))$, $S(V(\mathbb{A}_f))$, and $S(V(\mathbb{R}))$ be the Schwartz spaces of $V(\mathbb{A})$, $V(\mathbb{A}_f)$, and $V(\mathbb{R})$ respectively. For $z \in D$, let $\mathrm{pr}_z : V(\mathbb{R}) \to z$ be the projection with kernel $z^{\perp}$, and, for $x \in V(\mathbb{R})$, let

$$R(x,z) = -(\mathrm{pr}_z(x), \mathrm{pr}_z(x)) = |(x, w(z))|^2 |y|^{-2}. \tag{1.16}$$

Then the majorant associated to z is

$$(x,x)_z = (x,x) + 2R(x,z), \tag{1.17}$$

and the Gaussian is the function

$$\varphi_\infty \in S(V(\mathbb{R})) \otimes A^0(D), \qquad \varphi_\infty(x,z) = e^{-\pi(x,x)_z}. \tag{1.18}$$

Here $A^0(D)$ is the space of smooth functions on D. Note that, for $h \in O(V(\mathbb{R}))$,

$$\varphi_\infty(hx, hz) = \varphi_\infty(x,z). \tag{1.19}$$

Let $G = SL_2$ and let $G'_{\mathbb{A}}$ be the 2-fold metaplectic cover of $G(\mathbb{A})$. Let $G'_{\mathbb{Q}} \subset G'_{\mathbb{A}}$ be the image of $G(\mathbb{Q})$ under the canonical splitting homomorphism. The group $G'_{\mathbb{A}}$ acts in $S(V(\mathbb{A}))$ via the Weil representation ω (determined by the standard additive character ψ of $\mathbb{A}/\mathbb{Q}$ such that $\psi_\infty(x) = e(x) = e^{2\pi i x}$) and this action commutes with the linear action of $O(V)(\mathbb{A})$. It will sometimes be convenient to write this linear action as $\omega(h)\varphi(x) = \varphi(h^{-1}x)$. For $z \in D$, $h \in O(V)(\mathbb{A}_f)$ and $g' \in G'_{\mathbb{A}}$, we let $\theta(g', z, h)$ be the linear functional on $S(V(\mathbb{A}_f))$ defined by

$$\varphi \mapsto \theta(g', z, h; \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g')\Big(\varphi_\infty(\cdot, z) \otimes \omega(h)\varphi\Big)(x). \tag{1.20}$$

Then, for $\gamma \in O(V)(\mathbb{Q})$, we have

$$\theta(g', \gamma z, \gamma h; \varphi) = \theta(g', z, h; \varphi). \tag{1.21}$$

Also, by Poisson summation, [**50**], for $\gamma \in G'_{\mathbb{Q}}$,

$$\theta(\gamma g', z, h; \varphi) = \theta(g', z, h; \varphi). \tag{1.22}$$

Finally, for $g'_1 \in G'_{\mathbb{A}_f}$ and $h_1 \in O(V)(\mathbb{A}_f)$, we have

$$\theta(g' g'_1, z, h h_1; \varphi) = \theta(g', z, h; \omega(g'_1, h_1)\varphi). \tag{1.23}$$

In particular, if $K \subset H(\mathbb{A}_f)$ is as above, and if $\varphi \in S(V(\mathbb{A}_f))^K$, then the function

$$(z, h) \mapsto \theta(g', z, h; \varphi) \tag{1.24}$$

on $D \times H(\mathbb{A}_f)$ descends to a function on X_K. We may view it as a linear functional on the space $S(V(\mathbb{A}_f))^K$ and hence we obtain:

$$\begin{aligned} \theta : G'_{\mathbb{Q}} \backslash G'_{\mathbb{A}} \times X_K &\longrightarrow \Big(S(V(\mathbb{A}_f))^K\Big)^{\vee}. \\ (g', z, h) &\mapsto \theta(g', z, h; \cdot). \end{aligned} \tag{1.25}$$

Note that this function is *not* holomorphic in z.

Let K'_∞ be the full inverse image of $SO(2) \subset SL_2(\mathbb{R}) = G(\mathbb{R})$ in $G'_{\mathbb{R}}$, for each $r \in \frac{1}{2}\mathbb{Z}$ let χ_r be the character of K'_∞ such that

$$\chi_\ell(k')^2 = e^{2ir\theta}, \qquad \text{if} \quad k' \mapsto k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in SO(2) \tag{1.26}$$

under the covering projection. Let $K' \subset G'_{\mathbb{A}}$ be the full inverse image of $SL_2(\hat{\mathbb{Z}}) \subset G(\mathbb{A}_f)$, and note that

$$G'_{\mathbb{A}} = G'_{\mathbb{Q}} G'_{\mathbb{R}} K'. \tag{1.27}$$

The Gaussian (1.18) is an eigenfunction of K'_∞ with

$$\omega(k'_\infty)\varphi_\infty(x, z) = \chi_\ell(k'_\infty)\varphi_\infty(x, z), \tag{1.28}$$

for $\ell = \frac{n}{2} - 1$. It then follows from (1.23) that

$$\theta(g' k'_\infty k', z, h) = \chi_\ell(k'_\infty)\,(\omega(k')^{\vee})^{-1}\theta(g', z, h) \tag{1.29}$$

for all $k'_\infty \in K'_\infty$ and $k' \in K'$. In particular, the theta function has weight $\ell = \frac{n}{2} - 1$. Here $\omega(k')^{\vee}$ denotes the action of K' on the space $S(V(\mathbb{A}_f))^{\vee}$ dual to its action on $S(V(\mathbb{A}_f))$.

Now suppose that $F : G'_{\mathbb{Q}} \backslash G'_{\mathbb{A}} \to S(V(\mathbb{A}_f))^K$ is a function such that

$$F(g' k'_\infty k') = \chi_{-\ell}(k'_\infty)\,\omega(k')^{-1} F(g') \tag{1.30}$$

for all $k'_\infty \in K'_\infty$ and $k' \in K'$. Then, as a function of g', the $\mathbb{C}$–bilinear pairing

$$\langle\!\langle\, F(g'), \theta(g', z, h) \,\rangle\!\rangle = \theta(g', z, h; F(g')) \tag{1.31}$$

is left $G'_{\mathbb{Q}}$–invariant and right $K'_\infty K'$–invariant. Its integral over $G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}$, defined in general by a suitable regularization, is a function

$$\Phi(z,h;F) = \int^{\bullet}_{G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}} (\!(\; F(g'), \theta(g',z,h) \;)\!) \, dg' \tag{1.32}$$

on X_K.

The Borcherds forms [**2**] arise when F comes from a certain type of vector valued automorphic form *with possible poles at the cusps*. To describe these, it is convenient to pass to a point of view intermediate between that just explained and the classical formulation.

Observe that $G'_{\mathbb{Q}} \cap (G'_{\mathbb{R}}K') \simeq SL_2(\mathbb{Z})$. Let Γ' be the full inverse image of $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R}) = G(\mathbb{R})$ in the metaplectic cover $G'_{\mathbb{R}}$. Thus Γ' is an extension of $SL_2(\mathbb{Z})$ by $\{\pm 1\}$. For each $\gamma' \in \Gamma'$, with image γ in $SL_2(\mathbb{Z})$, there is a unique element γ'' such that $\gamma'\gamma'' = \gamma \in G'_{\mathbb{Q}} \cap (G'_{\mathbb{R}}K')$. For $\tau = u + iv \in \mathfrak{H}$, the upper halfplane, let

$$g_\tau = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{\frac12} & \\ & v^{-\frac12} \end{pmatrix}, \tag{1.33}$$

and let $g'_\tau = [g_\tau, 1] \in G'_{\mathbb{R}}$. We then have

$$\gamma' g'_\tau = g'_{\gamma(\tau)} k'_\infty(\gamma',\tau) \tag{1.34}$$

for a unique $k'_\infty(\gamma',\tau) \in K'_\infty$. For $r \in \frac12\mathbb{Z}$, define an automorphy factor by

$$j_r : \Gamma' \times \mathfrak{H} \to \mathbb{C}^\times, \qquad j_r(\gamma',\tau) = \chi_{-r}(k'_\infty(\gamma',\tau))\, |c\tau + d|^r, \tag{1.35}$$

if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For example, if $r \in \mathbb{Z}$, $j_r(\gamma',\tau) = (c\tau+d)^r$.

Lemma 1.1. *Suppose that $(\rho, \mathcal{V})$ is a representation of K' and that*

$$\phi : G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}} \longrightarrow \mathcal{V}$$

is a function such that

$$\phi(g'k'_\infty k') = \chi_r(k'_\infty)\, \rho(k')^{-1}\, \phi(g').$$

Let

$$f(\tau) = v^{-r/2}\, \phi(g'_\tau).$$

Then, for all $\gamma = \gamma'\gamma'' \in SL_2(\mathbb{Z})$,

$$f(\gamma(\tau)) = j_r(\gamma',\tau)\, \rho(\gamma'') f(\tau).$$

Proof. We have

$$
\begin{aligned}
f(\gamma(\tau)) &= v(\gamma(\tau))^{-r/2}\,\phi(g'_{\gamma(\tau)})\\
&= |c\tau+d|^r\, v^{-r/2}\,\phi(\gamma g'_\tau k'_\infty(\gamma',\tau)^{-1}\,(\gamma'')^{-1})\\
&= |c\tau+d|^r\,\chi_{-r}(k'_\infty(\gamma',\tau))\,v^{-r/2}\,\rho(\gamma'')\,\phi(g'_\tau)\\
&= j_r(\gamma',\tau)\,\rho(\gamma'')f(\tau),
\end{aligned}
\tag{1.36}
$$

as claimed.

Note that we can view $\mathcal{V}$ as a representation of Γ' by setting $\rho(\gamma') = \rho(\gamma'')$.

Applying Lemma 1.1, via (1.29) and (1.30), we obtain automorphic forms

$$\vartheta(\tau, z, h) = v^{-\ell/2}\theta(g'_\tau, z, h), \tag{1.37}$$

of weight ℓ, and

$$f(\tau) = v^{\ell/2}F(g'_\tau), \tag{1.38}$$

of weight $-\ell$, valued in $S(V(\mathbb{A}_f))^\vee$ and $S(V(\mathbb{A}_f))^K$ respectively. Note that ϑ is not holomorphic in τ. Then the quantity in (1.32) is given by

$$\Phi(z, h; F) = \int^{\bullet}_{SL_2(\mathbb{Z})\backslash\mathfrak{H}} \langle\!\langle\, f(\tau), \vartheta(\tau, z, h)\,\rangle\!\rangle\; v^{-2}\,du\,dv \tag{1.39}$$

for a suitable choice of measure on $G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}$.

Let M be a $\mathbb{Z}$–lattice in V, on which the quadratic form $Q(x) = \frac{1}{2}(x, x)$ takes integral values, and let $M^\sharp$ be the dual lattice. Let $S_M \subset S(V(\mathbb{A}_f))$ be the space of functions with support in $\hat{M}^\sharp := M^\sharp \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ and constant on cosets of $\hat{M} := M \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. We will use the characteristic functions of cosets as a basis for this finite dimensional space. The space S_M is stable under the action of K'. The restriction to S_M of the theta function $\vartheta(\tau, z, h)$, viewed as a linear functional, defines a (non-holomorphic) modular form of weight $\ell = \frac{n}{2} - 1$ valued in $(\omega^\vee, S_M^\vee)$, the dual of the representation (ω, S_M) of K'.

Suppose that F and, hence, f takes values in S_M and is meromorphic at the cusp in the following sense. Write

$$f(\tau) = \sum_{\varphi} f_\varphi(\tau)\cdot\varphi, \tag{1.40}$$

where φ runs over the coset basis for S_M, and let

$$f_\varphi(\tau) = \sum_{m\in\mathbb{Q}} c_\varphi(m)\, q^m \tag{1.41}$$

be the Fourier expansion of f_φ, where $q^m = e(m\tau)$. We will sometimes write $c_0(m)$ for the Fourier coefficients of f_{φ_0} where φ_0 is the characteristic function of $\hat{M}$; the constant term $c_0(0)$ will play a crucial role. The Fourier coefficients $c_\varphi(m)$ are nonzero only for $m \in \frac{1}{N}\mathbb{Z}$, for some integer N, and we require that only a finite number of $c_\varphi(m)$'s with $m < 0$ are nonzero. Then the pairing

$$(\!(\ f(\tau), \vartheta(\tau, z, h)\ )\!) = \sum_\varphi f_\varphi(\tau)\, \vartheta(\tau, z, h; \varphi) \tag{1.42}$$

defines an $SL_2(\mathbb{Z})$ invariant function on $\mathfrak{H}$. It can be very rapidly increasing on the standard fundamental domain for $\Gamma = SL_2(\mathbb{Z})$. The regularization used to define the integral (1.39) will be reviewed in detail below.

A basic result of Borcherds, [**2**], expressed in our present notation, is the following:

Theorem 1.2. (Theorem 13.3 of [**2**]) *Suppose that F (and hence f) takes values in S_M^K and that the Fourier coefficients $c_\varphi(m)$ for $m \le 0$ are integers. Then the regularized integral*

$$\Phi(z, h; F) = \int^\bullet_{\Gamma\backslash\mathfrak{H}} (\!(\ f(\tau), \vartheta(\tau, z)\ )\!)\, v^{-2}\, du\, dv$$

can be written in the form

$$\Phi(z, h; F) = -2\log|\Psi(z, h; F)|^2 - c_0(0)\big(\ 2\log|y| + \log(2\pi) + \Gamma'(1)\ \big)$$

for a meromorphic modular form $\Psi(F)$ on $D \times H(\mathbb{A}_f)$ of weight $k = \frac{1}{2}c_0(0)$.

More precisely, suppose that $c_0(0)$ is even, so that $k = \frac{1}{2}c_0(0) \in \mathbb{Z}$. Then, there is a unitary character ξ of $H(\mathbb{Q})$ such that, for all $\gamma \in H(\mathbb{Q})$,

$$\Psi(\gamma z, \gamma h; F) = \xi(\gamma)\, j(\gamma, z)^k\, \Psi(z, h; F). \tag{1.43}$$

Moreover, as a function of $h \in H(\mathbb{A}_f)$, $\Psi(F)$ is right K–invariant for any compact open subgroup $K \subset H(\mathbb{A}_f)$ for which the values of F lie in $S_M \subset S(V(\mathbb{A}_f))^K$, and hence, $\Psi(F)$ defines a meromorphic section of the bundle $\mathcal{L}^{\otimes k} \otimes \mathcal{V}_\xi$, where $\mathcal{V}_\xi$ is the flat bundle defined by ξ. Since our calculations only involve $\log||\Psi(F)||^2$, the character ξ, which, in fact, has finite order [**4**], will play no role in the present paper. If the coefficient $c_0(0)$ is odd, $\Psi(F)^2 = \Psi(2F)$ is an automorphic form of weight $2k$. Note that, in any case, it is the quantity $2\log|\Psi(z,h;F)|^2$ which occurs in $\Phi(z,h;F)$, so that the parity of $c_0(0)$ will not matter.

Borcherds also determines the divisor of $\Psi(F)$. To describe this in our setup, we first recall the definition of the special cycles in X_K, from [**26**]. For $x \in V(\mathbb{Q})$ with $Q(x) > 0$, let $V_x = x^\perp$, and

$$D_x = \{\ z \in D \mid x \perp z\ \}. \tag{1.44}$$

Let H_x be the stabilizer of x in H, and note that $H_x \simeq \mathrm{GSpin}(V_x)$. For $h \in H(\mathbb{A}_f)$, there is a natural map

$$\begin{aligned} H_x(\mathbb{Q})\backslash D_x \times H_x(\mathbb{A}_f)/(H(\mathbb{A}_f) \cap hKh^{-1}) &\longrightarrow H(\mathbb{Q})\backslash D \times H(\mathbb{A}_f)/K = X_K \\ (z, h_1) &\mapsto (z, h_1 h) \end{aligned} \tag{1.45}$$

which defines a divisor $Z(x,h,K)$ on X_K. This divisor is rational over $\mathbb{Q}$. For a Schwartz function $\varphi \in S(V(\mathbb{A}_f))^K$, and a positive rational number $m \in \mathbb{Q}_{>0}$, we define a weighted linear combination $Z(m,\varphi,K)$ of these divisors as follows. Let

$$\Omega_m = \{x \in V \mid Q(x) = m\} \tag{1.46}$$

be the quadric determined by m, and fix $x_0 \in \Omega_m(\mathbb{Q})$, assuming that $\Omega_m(\mathbb{Q}) \neq \phi$. Then $\Omega_m(\mathbb{A}_f)$ is a closed subset of $V(\mathbb{A}_f)$, and we can write

$$\mathrm{supp}(\varphi) \cap \Omega_m(\mathbb{A}_f) = \coprod_r K \cdot \xi_r^{-1} x_0 \tag{1.47}$$

for some finite set of ξ_r's in $H(\mathbb{A}_f)$. Define

$$Z(m,\varphi,K) := \sum_r \varphi(\xi_r^{-1} x_0)\, Z(x_0, \xi_r, K). \tag{1.48}$$

If $\Omega_m(\mathbb{Q})$ is empty, then $Z(m,\varphi;K) = 0$. These cycles, which are defined for arbitrary codimension in [**26**], include the Heegner points, Hirzebruch–Zagier curves, and Humbert surfaces as particular cases. Various nice properties of the weighted cycles are described in [**26**]. For example, if $K' \subset K$ and if $\mathrm{pr} : X_{K'} \to X_K$ is the associated covering, then

$$\mathrm{pr}^* Z(m,\varphi,K) = Z(m,\varphi,K'), \tag{1.49}$$

so that the special cycles are defined on the full Shimura variety associated to (H,D), [**38**]. Because of this relation, we will frequently omit K and write simply $Z(m,\varphi)$ in place of $Z(m,\varphi,K)$. Also, if $h \in H(\mathbb{A}_f)$, then right multiplication by h^{-1} defines a natural morphism, rational over $\mathbb{Q}$,

$$r(h) : X_K \longrightarrow X_{hKh^{-1}}, \tag{1.50}$$

and

$$r(h)_* Z(m,\varphi,K) = Z(m,\omega(h)\varphi, hKh^{-1}), \tag{1.51}$$

where $\omega(h)\varphi(x) = \varphi(h^{-1}x)$. This relation describes the compatibility of the special cycles with the Hecke operators. Finally, by Proposition 5.4 of [**26**], we can give an explicit description of these cycles with respect to the decomposition (1.3) of the space X_K as a disjoint union of arithmetic quotients of D^+:

$$Z(m,\varphi;K) = \sum_j \sum_{\substack{x\in\Omega_m(\mathbb{Q})\\ \bmod \Gamma_j}} \varphi(h_j^{-1}x)\,\mathrm{pr}_j(D_x), \tag{1.52}$$

where $\mathrm{pr}_j : D^+ \to \Gamma_j\backslash D^+$ is the natural projection. Note that it follows from this formula that,

$$Z(m,\varphi;K) = Z(m,\varphi^\vee;K) \tag{1.53}$$

where $\varphi^\vee(x) = \varphi(-x)$.

Theorem 1.3. (Theorem 13.3 of [**2**]) *For f with Fourier expansion given by (1.40) and (1.41),*

$$\mathrm{div}(\Psi(F)^2) = \sum_\varphi \sum_{m>0} c_\varphi(-m)\, Z(m,\varphi,K).$$

Here φ runs over the coset basis for S_M.

Proof. The singular part of the function $\Phi(F)$ is computed in [**2**]. In our present notation, it is given as follows. Let $\beta_1(t) = -\mathrm{Ei}(-t)$ be the exponential integral, [**36**], and recall that $\beta_1(t) = -\log(t) + O(1)$ as $t \to 0$. The singular part of $\Phi(F)$ is given by

$$S(z,h;F) := \sum_{m>0}\sum_\varphi c_\varphi(-m) \sum_{\substack{x\in V(\mathbb{Q})\\ Q(x)=m}} \varphi(h^{-1}x)\,\beta_1(-2\pi R(x,z)). \tag{1.54}$$

By (1.16), this has the same singularity as the logarithm of the absolute value of the infinite product

$$\prod_\varphi \prod_{m>0} \prod_{\substack{x\in V(\mathbb{Q})\\ Q(x)=m}} (x,w(z))^{c_\varphi(-Q(x))\varphi(h^{-1}x)}. \tag{1.55}$$

Alternatively, in the neighborhood of any point $z_0 \in \mathbb{D}$, consider the divisor given by the finite product

$$\prod_\varphi \prod_{m>0} \prod_{\substack{x\in V(\mathbb{Q})\\ Q(x)=m\\ (x,w(z_0))=0}} (x,w(z))^{c_\varphi(-Q(x))\varphi(h^{-1}x)}. \tag{1.56}$$

Note that D_x is precisely the divisor defined by the equation $(x,w(z)) = 0$ on $\mathbb{D}$. Thus, since the singularities of $\Phi(F)$ on $\mathbb{D}$ are the same as those of $-2\log|\Psi(F)|^2$, we obtain the claimed result. $\square$

§2. Computation of a regularized integral.

Setting $k = \frac{1}{2}c_0(0)$, recall that the Petersson norm of the section defined by $\Psi(F)$ is

$$||\Psi(z,h;F)||^2 = |\Psi(z,h;F)|^2|y|^{2k}. \tag{2.1}$$

We view the function $||\Psi(z,h_j;F)||^2$ as a function on the component $\Gamma_j\backslash D^+$ of X_K and will write $||\Psi(z;F)||^2$ for the resulting function on the (possibly disconnected) complex manifold X_K.

The basic problem is to compute the following integral:

$$\begin{aligned}
\kappa(\Psi(F)) &:= -\frac{1}{\mathrm{vol}(X_K)}\int_{X_K} \log||\Psi(z;F)||^2\,d\mu(z)\\
&= -\frac{1}{\mathrm{vol}(X_K)}\int_{X_K} \log\left(\left|\Psi(z;F)\right|^2|y|^{2k}\right)d\mu(z)\\
&= \frac{1}{2}\,\frac{1}{\mathrm{vol}(X_K)}\int_{X_K} \Phi(z;F)\,d\mu(z) + k\left(\log(2\pi)+\Gamma'(1)\right) \qquad (2.2)\\
&= \frac{1}{2}\,\frac{1}{\mathrm{vol}(X_K)}\int_{X_K}\left(\int^{\bullet}_{\Gamma\backslash\mathfrak{H}} ⟪\, f(\tau),\vartheta(\tau,z)\,⟫\,v^{-2}\,du\,dv\right)d\mu(z) + k\,C_0\\
&= \frac{1}{2}\int^{\bullet}_{\Gamma\backslash\mathfrak{H}}\sum_{\varphi} f_{\varphi}(\tau)\,I(\tau;\varphi)\,v^{-2}\,du\,dv + k\,C_0,
\end{aligned}$$

where $C_0 = \log(2\pi)+\Gamma'(1)$, $\mathrm{vol}(X_K) = \mathrm{vol}(X_K, d\mu(z)\,dh)$ and

$$I(\tau;\varphi) = \frac{1}{\mathrm{vol}(X_K)}\int_{X_K}\vartheta(\tau,z;\varphi)\,d\mu(z). \tag{2.3}$$

In fact, the last interchange of order of integration (where one of the integrals regularized!) will be justified in the next section, *provided* the theta integral (2.3) converges. We will discuss this point in a moment. Here $d\mu(z)$ is a $H(\mathbb{R})$–invariant top degree form on D; the quantity $\kappa(\Psi(F))$ is independent of the normalization of this form.

We want to relate the integral $I(\tau;\varphi)$, over the complex manifold X_K, to the usual theta integral, over the adelic coset space $SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})$, appearing in the Siegel–Weil formula. This is done in detail in section 4, below, cf. Theorem 4.1. Note that there is an exact sequence

$$1 \longrightarrow Z \longrightarrow H \longrightarrow SO(V) \longrightarrow 1$$

where $H = \mathrm{GSpin}(V)$, as before. For simplicity, we assume that the compact open subgroup $K \subset H(\mathbb{A}_f)$ satisfies the condition:

$$Z_K := Z(\mathbb{A}_f)\cap K \simeq \hat{\mathbb{Z}}^{\times} \tag{2.4}$$

under the natural identification $Z(\mathbb{A}_f) \simeq \mathbb{A}_f^{\times}$. A slight variant of the proof of Proposition 4.17 yields:

Lemma 2.1. *Let φ_∞ be the Gaussian, as in (1.18) above. Then, for $\varphi \in S(V(\mathbb{A}_f))^K$,*

$$I(g';\varphi_\infty \otimes \varphi) := \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} \theta(g', h; \varphi_\infty \otimes \varphi)\, dh$$
$$= \frac{1}{\mathrm{vol}(X_K)} \int_{X_K} \theta(g', z; \varphi)\, d\mu(z).$$

Note that both sides are independent of the choice of $d\mu(z)$.

Corollary 2.2.

$$I(\tau;\varphi) = v^{-\ell/2}\, I(g'_\tau;\varphi_\infty \otimes \varphi).$$

Corollary 2.3. *Assume that F is valued in $S(V(\mathbb{A}_f))^K$. Then*

$$\kappa(\Psi(F)) = \frac{1}{2} \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\varphi} f_\varphi(\tau)\, I(\tau;\varphi)\, v^{-2}\, du\, dv + k\, C_0$$
$$= \frac{1}{2} \int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \sum_{\varphi} f_\varphi(\tau)\, v^{-\ell/2}\, I(g'_\tau;\varphi_\infty \otimes \varphi)\, v^{-2}\, du\, dv + k\, C_0,$$

with $C_0 = \log(2\pi) + \Gamma'(1)$.

Remark 2.4. Exceptional cases: By Weil's criterion, [**51**], p.75, Proposition 8, the theta integral $I(g'_\tau;\varphi_\infty \otimes \varphi)$ is absolutely convergent whenever $n - r > 0$, where $r = 0$, 1, or 2 is the Witt index of $V(\mathbb{Q})$, i.e., the dimension of a maximal isotropic subspace of $V(\mathbb{Q})$. Note that $r = 0$ is only possible when $n \le 2$. The only *exceptional cases* will thus be $n = 1$ with V isotropic ($r = 1$) and $n = 2$ with V split ($r = 2$). We will exclude these cases for now – although they can be handled by the regularization process used in [**31**].

We consider the regularized integral in the expression for $\kappa(\Psi(F))$ in Corollary 2.3. Note that $\kappa(\Psi(F))$ is independent of the choice of the lattice M and of K.

Recall that, for a $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ invariant function ϕ on $\mathfrak{H}$, the regularized integral

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \phi(\tau)\, d\mu(\tau), \tag{2.5}$$

used by Borcherds, is defined by taking the constant term in the Laurent expansion at $\sigma = 0$ of the function defined, for $\mathrm{Re}(\sigma)$ sufficiently large, by

$$\lim_{T\to\infty} \int_{\mathcal{F}_T} \phi(\tau)\, v^{-\sigma-2}\, du\, dv. \tag{2.6}$$

Here $\mathcal{F}$ is the standard fundamental domain for the action of Γ on $\mathfrak{H}$, and $\mathcal{F}_T$ is the intersection of this with the region $\mathrm{Im}(\tau) \le T$. This procedure can be applied provided that (i) the limit as T goes to infinity exists in a halfplane $\mathrm{Re}(\sigma) > \sigma_0$, and (ii) the resulting holomorphic function of σ has a meromorphic analytic continuation to a neighborhood of the point $\sigma = 0$. In short,

$$\int_{\Gamma\backslash\mathfrak{H}}^{\bullet} \phi(\tau)\ d\mu(\tau) := \mathop{\mathrm{CT}}_{\sigma=0}\left\{ \lim_{T\to\infty} \int_{\mathcal{F}_T} \phi(\tau)\, v^{-\sigma}\, d\mu(\tau)\right\}, \tag{2.7}$$

where $\mathrm{CT}_{\sigma=0}$ denotes the constant term of the Laurent expansion at the point $\sigma = 0$.

The following result will be proved in the next section.

Proposition 2.5.

$$\mathop{\mathrm{CT}}_{\sigma=0}\left\{ \lim_{T\to\infty} \int_{\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, I(\tau;\varphi)\, v^{-\sigma-2}\, du\, dv\right\}$$

$$= \lim_{T\to\infty} \left[\ \int_{\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, I(\tau;\varphi)\, v^{-2}\, du\, dv - c_0(0)\, \log(T)\ \right].$$

Thus we need to evaluate the basic integral

$$\int_{\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, I(\tau;\varphi)\, v^{-2}\, du\, dv \tag{2.8}$$

where $f_\varphi(\tau)$ is holomorphic on $\mathcal{F}$ and where σ has been set equal to zero.

Following the suggestion of section 9 of [**2**], we would *like* to define an automorphic function $J(\tau;\varphi)$ on $\mathfrak{H}$ for which

$$\frac{\partial}{\partial\bar\tau}\{J(\tau;\varphi)\} = I(\tau;\varphi)\, v^{-2}. \tag{2.9}$$

Then, by a simple Stokes' Theorem argument, we would have

$$\begin{aligned}
&\int_{\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, I(\tau;\varphi)\, v^{-2}\, du\wedge dv \\
&= \frac{1}{2i}\int_{\mathcal{F}_T} d\Big(\sum_{\varphi} f_\varphi(\tau)\, J(\tau;\varphi)\, d\tau\Big) \\
&= \frac{1}{2i}\int_{\partial\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, J(\tau;\varphi)\, d\tau \\
&= \frac{1}{2i}\int_{1/2+iT}^{-1/2+iT} \sum_{\varphi} f_\varphi(\tau)\, J(\tau;\varphi)\, du \\
&= -\frac{1}{2i}\ \text{constant term of}\ \Big(\sum_{\varphi} f_\varphi(\tau)\, J(\tau;\varphi)\Big)\Big|_{v=T}.
\end{aligned} \tag{2.10}$$

In the next to last step, we have used the invariance of $\sum_\varphi f_\varphi(\tau)\, J(\tau;\varphi)\, d\tau$ under $\tau \mapsto \tau+1$ and under $\tau \mapsto -1/\tau$.

To obtain a relation like (2.9), we apply the Maass operators and the Siegel–Weil formula. Let

$$X_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}). \tag{2.11}$$

Recall that, if $\phi : G'_\mathbb{R} \to \mathbb{C}$ is a smooth function with $\phi(g'k') = \chi_\ell(k')\phi(g')$, i.e., of weight ℓ, and if $f(\tau) = v^{-\frac{\ell}{2}}\,\phi(g'_\tau)$ is the corresponding function on $\mathfrak{H}$, then $X_\pm\phi$ has weight $\ell \pm 2$, and the corresponding function on $\mathfrak{H}$ is

$$v^{-\frac{1}{2}(\ell\pm 2)} X_\pm \phi(g'_\tau) = \begin{cases} \big(2i\frac{\partial f}{\partial\tau} + \frac{\ell}{v} f\big)(\tau) & \text{for } +, \\ -2iv^2\ \frac{\partial f}{\partial\bar\tau}(\tau) & \text{for } -. \end{cases} \tag{2.12}$$

We now take advantage of the Siegel–Weil formula; the facts we need are reviewed in the first part of section 4. For $\varphi \in S(V(\mathbb{A}_f))$, let $E(g', s, \Phi^r_\infty \otimes \lambda(\varphi))$ be the Eisenstein series of weight r on $G'_\mathbb{A}$ associated to φ. If $\varphi_\infty \in S(V(\mathbb{R}))$ is the Gaussian, then

$$\lambda(\varphi_\infty) = \Phi^\ell_\infty(s_0), \tag{2.13}$$

where $\ell = \frac{n}{2} - 1$, as above. By the Siegel–Weil formula, Theorem 4.1, we have the following.

Proposition 2.6. *Exclude the exceptional cases of Remark 2.4 above, so that the theta integral is absolutely convergent. Then*

$$I(\tau;\varphi) = v^{-\ell/2}\, I(g'_\tau;\varphi_\infty\otimes\varphi) = v^{-\ell/2}\, E(g'_\tau, s_0; \Phi^\ell_\infty \otimes \lambda(\varphi)),$$

where $s_0 = \frac{n}{2} = \ell + 1$.

On the other hand, an easy computation in the induced representation $I_\mathbb{R}(s,\chi)$ of $G'_\mathbb{R}$ shows:

Lemma 2.7. *Let $\Phi^r(s) \in I_\mathbb{R}(s,\chi)$ be the normalized eigenvector of weight r for the action of $K'_\mathbb{R}$. Then*

$$X_\pm \Phi^r(s) = \frac{1}{2}(s+1\pm r)\,\Phi^{r\pm 2}(s).$$

□

Therefore, we have the basic relation:

$$X_- E(g', s; \Phi^{\ell+2}\otimes\lambda(\varphi)) = \frac{1}{2}(s-\ell-1)\, E(g', s; \Phi^\ell \otimes \lambda(\varphi)). \tag{2.14}$$

Pushing this down to $\mathfrak{H}$ we obtain:

$$-2iv^2\,\frac{\partial}{\partial\bar\tau}\Big\{v^{-\frac{1}{2}(\ell+2)}\,E(g'_\tau,s;\Phi^{\ell+2}\otimes\lambda(\varphi))\Big\}=\frac{1}{2}(s-s_0)\,v^{-\frac{1}{2}\ell}E(g'_\tau,s;\Phi^{\ell}\otimes\lambda(\varphi)). \tag{2.15}$$

For convenience, we now write

$$E(\tau,s;\varphi,\ell)=v^{-\ell/2}\,E(g'_\tau,s_0;\Phi^{\ell}_\infty\otimes\lambda(\varphi)), \tag{2.16}$$

so that (2.15) becomes

$$-2iv^2\,\frac{\partial}{\partial\bar\tau}\Big\{E(\tau,s;\varphi,\ell+2)\Big\}=\frac{1}{2}(s-s_0)\,E(\tau,s;\varphi,\ell). \tag{2.17}$$

Of course,the vanishing of the right hand side of (2.17) at $s=s_0=\frac{n}{2}$ just shows the holomorphy of the special value

$$E(\tau,s_0;\varphi,\ell+2)=\varphi(0)+\mathrm{vol}(X)^{-1}\sum_{m>0}\deg(Z(m,\varphi))\cdot q^m, \tag{2.18}$$

cf. Theorem 4.23. Here we have written $\deg(Z(m,\varphi))$ in place of

$$\deg_{\mathcal{L}^\vee}(Z(m,\varphi;K))$$

and $\mathrm{vol}(X)$ in place of $\mathrm{vol}(X_K,\Omega^n)$ to lighten the notation.

Remark 2.8. The vanishing of the right side of (2.17) depends on the fact that $E(\tau,s;\varphi,\ell)$ has no pole at $s=s_0=\frac{n}{2}$. In the exceptional cases, $n=1$, $r=1$ and $n=2$, $r=2$ a pole can occur, and its residue accounts for a non–holomorphic component occuring in (2.14), cf. [**15**].

We write:

$$E(\tau,s;\varphi,\ell)\,v^{-2}=\frac{-4i}{s-s_0}\,\frac{\partial}{\partial\bar\tau}\Big\{E(\tau,s;\varphi,\ell+2)\Big\}. \tag{2.19}$$

Now, to evaluate (2.8), we use the Siegel–Weil formula (Proposition 2.6) and write

$$\int_{\mathcal{F}_T}\sum_{\varphi}f_\varphi(\tau)\,I(\tau;\varphi)\,v^{-2}\,du\,dv=\int_{\mathcal{F}_T}\sum_{\varphi}f_\varphi(\tau)\,E(\tau,s;\varphi,\ell)\,v^{-2}\,du\wedge dv\,\Big|_{s=s_0}.$$

Then, for general s, we can use the relation (2.19) and the Stoke's Theorem argument (2.10) to obtain the following basic identity.

$$
\begin{aligned}
I(s,T) &:= \int_{\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, E(\tau,s;\varphi,\ell)\, v^{-2}\, du\wedge dv \\
&= \frac{1}{2i}\int_{\mathcal{F}_T} d\Big(\sum_{\varphi} f_\varphi(\tau)\,\frac{-4i}{s-s_0} E(\tau,s;\varphi,\ell+2)\,d\tau\Big) \\
&= \frac{-2}{s-s_0}\int_{\partial\mathcal{F}_T} \sum_{\varphi} f_\varphi(\tau)\, E(\tau,s;\varphi,\ell+2)\,d\tau \qquad (2.20)\\
&= \frac{-2}{s-s_0}\int_{1/2+iT}^{-1/2+iT} \sum_{\varphi} f_\varphi(\tau)\, E(\tau,s;\varphi,\ell+2)\,du \\
&= \frac{2}{s-s_0}\cdot \text{constant term of } \Big(\sum_{\varphi} f_\varphi(\tau)\, E(\tau,s;\varphi,\ell+2)\Big)\Big|_{v=T}.
\end{aligned}
$$

By Corollary 2.3, and Proposition 2.5,

$$
\begin{aligned}
\kappa(\Psi(F)) &= \frac{1}{2}\lim_{T\to\infty}\Big[\int_{\mathcal{F}_T}\sum_{\varphi} f_\varphi(\tau)\, I(\tau;\varphi)\, v^{-2}\,du\,dv - c_0(0)\,\log(T)\Big] + k\,C_0 \\
&= \frac{1}{2}\lim_{T\to\infty}\Big[\, I(s_0,T) - c_0(0)\,\log(T)\,\Big] + k\,C_0,
\end{aligned}
$$

It will be convenient to introduce the following additional notation. Write

$$E(\tau,s;\varphi,\ell+2) = \sum_m A_\varphi(s,m,v)\, q^m, \tag{2.22}$$

where the Fourier coefficients have Laurent expansions

$$A_\varphi(s,m,v) = a_\varphi(m) + b_\varphi(m,v)(s-s_0) + O((s-s_0)^2). \tag{2.23}$$

where the $a_\varphi(m)$'s are given by (2.16). With this notation,

$$
\begin{aligned}
I(s,T) &= \frac{2}{s-s_0}\cdot\text{constant term of }\Big(\sum_{\varphi} f_\varphi(\tau)\, E(\tau,s;\varphi,\ell+2)\Big)\Big|_{v=T} \\
&= \frac{2}{s-s_0}\sum_{\varphi}\sum_{m} c_\varphi(-m)\, A_\varphi(s,m,T).
\end{aligned} \tag{2.24}
$$

We consider the individual terms. For $m=0$, we have

$$\frac{2}{s-s_0}\sum_{\varphi} c_\varphi(0)\,\big(\,\varphi(0) + b_\varphi(0,T)(s-s_0)\,\big) + O(s-s_0). \tag{2.25}$$

so that the contribution of such terms to the constant coefficient in the Laurent expansion at $s = s_0$ is

$$2\sum_{\varphi} c_\varphi(0)\, b_\varphi(0,T). \tag{2.26}$$

We will return to the polar part occuring in (2.25) in a moment. Similarly, from the $m < 0$ terms, we have the contribution

$$2\sum_{\varphi}\sum_{m<0} c_\varphi(-m)\, b_\varphi(m,T). \tag{2.27}$$

Finally, for the finite sum of terms with $m > 0$, we have, initially:

$$\begin{aligned}&\frac{1}{(s-s_0)}\frac{2}{\mathrm{vol}(X)}\sum_{\varphi}\sum_{m>0} c_\varphi(-m)\ \deg(Z(m,\varphi))\\ &\qquad + 2\sum_{\varphi}\sum_{m>0} c_\varphi(-m)\, b_\varphi(m,T) + O(s-s_0).\end{aligned} \tag{2.28}$$

Since our whole integral $I(s,T)$ does not have a pole at $s = s_0$, the polar part here must cancel the one which occurred earlier, i.e., we must have

$$2\sum_{\varphi} c_\varphi(0)\,\varphi(0) + \frac{2}{\mathrm{vol}(X)}\sum_{\varphi}\sum_{m>0} c_\varphi(-m)\ \deg(Z(m,\varphi)) = 0. \tag{2.29}$$

Since

$$\mathrm{div}(\Psi(F)^2) = \sum_{m>0} c_\varphi(-m)\, Z(m,\varphi), \tag{2.30}$$

this amounts to

$$\deg(\mathrm{div}(\Psi(F)^2) = \sum_{m>0} c_\varphi(-m)\ \deg(Z(m,\varphi)) = -\mathrm{vol}(X)\, c_0(0). \tag{2.31}$$

Recall that we are using the coset basis for S_M, so that $\varphi_0(0) = 1$ and $\varphi(0) = 0$ for $\varphi \neq \varphi_0$. Also note that, since Ω is the negative of a Kähler form, $\mathrm{vol}(X)$ and $\deg(Z(m,\varphi))$ will have opposite signs (for a coset function φ), cf. (4.49).

Example 2.9. Suppose that $n = 1$ and $r = 0$, i.e., V is anisotropic over $\mathbb{Q}$ of dimension 3 and X_K is a (disjoint union of) projective curves. Suppose that the image of K in $SO(V)(\mathbb{A}_f)$ is neat, so that all of the Γ_j's act without fixed points on $D^+ \simeq \mathfrak{H}$. Then, since $\Omega = -\frac{1}{2\pi}\, y^{-2}\, dx \wedge dy$, $\mathrm{vol}(X) = 2-2g$, where g is the genus of X_K, and hence we have

$$\deg(\mathrm{div}(\Psi(F)^2) = 2(g-1)\, c_0(0), \tag{2.32}$$

as expected. Here one must keep in mind the fact that $\Psi(F)^2$ has 'classical weight' $2\, c_0(0)$.

Collecting the contributions of (2.26), (2.27), and (2.28), we obtain

Proposition 2.10.

$$I(s_0,T) = \int_{\mathcal{F}_T} \sum_{\varphi} f_{\varphi}(\tau)\, I(\tau,\varphi)\, v^{-2}\, du\, dv$$

$$= 2\sum_{\varphi}\sum_{m} c_{\varphi}(-m)\, b_{\varphi}(m,T).$$

The following result will be proved in the next section.

Proposition 2.11. *(i) For $m<0$, $b_{\varphi}(m,T)$ decays exponentially as $T\to\infty$.*
(ii)

$$\lim_{T\to\infty}\left(2\sum_{\varphi}\sum_{m<0} c_{\varphi}(-m)\, b_{\varphi}(m,T) \right) = 0.$$

(iii) For $m=0$,

$$\lim_{T\to\infty}\left(b_0(0,T) - \frac{1}{2}\log(T)\right) = 0,$$

and, for $\varphi \neq \varphi_0$,

$$\lim_{T\to\infty} b_{\varphi}(0,T) = 0.$$

Thus, we obtain an explicit expression for the quantity $\kappa(\Psi(F))$. The following result summarizes the relations between the geometry of the Borcherds form $\Psi(F)$ and the family of Eisenstein series $E(\tau,s,;\varphi,\ell+2)$.

Main Theorem 2.12. *For $\varphi\in S(V(\mathbb{A}_f))$, let*

$$E(\tau,s;\varphi,\ell+2) = \sum_{m} A_{\varphi}(s,m,v)\, q^m,$$

with

$$A_{\varphi}(s,m,v) = a_{\varphi}(m) + b_{\varphi}(m,v)\,(s-s_0) + O((s-s_0)^2)$$

be the Laurent expansion at the point $s_0=\frac{n}{2}=\ell+1$ of the associated Eisenstein series of weight $\frac{n}{2}+1=\ell+2$. Let K be a compact open subgroup $K\subset H(\mathbb{A}_f)$ satisfying the condition (2.4), and let $X=X_K$. Exclude the cases $\dim V=3$, of Witt index 1 and $\dim V = 4$, of Witt index 2.
(i) Suppose that $\varphi\in S(V(\mathbb{A}_f))^K$. Then,

$$E(\tau,s_0;\varphi,\ell+2) = \varphi(0) + \mathrm{vol}(X)^{-1}\sum_{m>0}\deg_{\mathcal{L}^\vee}(Z(m,\varphi))\, q^m.$$

(ii) For any $\varphi \in S(V(\mathbb{A}_f))$, let

$$\kappa_\varphi(m) := \begin{cases} \lim\limits_{T\to\infty} b_\varphi(m,T) & \text{if } m>0, \text{ and} \\ \frac{1}{2} C_0\, \varphi(0) & \text{if } m=0, \end{cases}$$

where $C_0 = \log(2\pi) + \Gamma'(1)$. Suppose that $f : \mathfrak{H} \to S(V(\mathbb{A}_f))^K$ is a modular form of weight $1 - \frac{n}{2} = -\ell$ for $\mathrm{SL}_2(\mathbb{Z})$, with Fourier expansion

$$f(\tau) = \sum_\varphi \sum_m c_\varphi(m)\, q^m\, \varphi$$

where φ runs over the coset basis with respect to some lattice M and where $c_\varphi(m) \in \mathbb{Z}$ for $m \le 0$. Let $\Psi(f)$ be the associated Borcherds form of weight $c_0(0)/2$. Then

$$\mathrm{div}(\Psi(f)^2) = \sum_\varphi \sum_{m>0} c_\varphi(-m)\, Z(m,\varphi),$$

and

$$-\mathrm{vol}(X)\, c_0(0) = \sum_\varphi \sum_{m>0} c_\varphi(-m)\, \deg_{\mathcal{L}^\vee}(Z(m,\varphi)).$$

Moreover

$$\begin{aligned}\kappa(\Psi(f)) : &= -\frac{1}{\mathrm{vol}(X)} \int_{X_K} \log ||\Psi(z;f)||^2\, d\mu(z) \\ &= \sum_\varphi \sum_{m\ge 0} c_\varphi(-m)\, \kappa_\varphi(m).\end{aligned}$$

Here $\mathrm{vol}(X) = \mathrm{vol}(X_K, \Omega^n)$ and $\deg_{\mathcal{L}^\vee}(Z(m,\varphi)) = \int_{Z(m,\varphi;K)} \Omega^{n-1}$ are computed with respect to the invariant $(1,1)$–form $\Omega = dd^c \log(\rho)$, where $\rho = \rho(z) = -\frac{1}{2}(w(z), w(\bar{z}))$, cf. Proposition 4.10.

Remark 2.13: The quantity $\kappa(\Psi(f))$ is completely determined by the collection of integers $\{\, c_\varphi(-m)\,\}$ for $m \ge 0$. The universal quantities $\kappa_\varphi(m)$ are independent of $\Psi(f)$. They can be computed explicitly, cf. section 5 for an example and [**33**] for a more systematic discussion.

§3. Convergence estimates.

In this section we prove the crucial fact that the integration over X_K can be interchanged with the Borcherds' regularization.

Theorem 3.1. *Suppose that the integral of the theta function converges, i.e., suppose that V is not a ternary isotropic space of signature $(1,2)$ or a quaternary space of signature $(2,2)$ and $\mathbb{Q}$–rank 2, the exceptional cases of Remark 2.4 above. Then*

$$\int_{X_K} \int^\bullet_{\Gamma\backslash\mathfrak{H}} ((\ F(\tau), \vartheta(\tau,z)\ ))\ d\mu(\tau)\, d\mu(z), = \int^\bullet_{\Gamma\backslash\mathfrak{H}} ((\ F(\tau), \int_{X_K} \vartheta(\tau,z)\, d\mu(z)\ ))\, d\mu(\tau),$$

where $\int^{\bullet}$ *denotes the regularized integral.*

Writing $\mathcal{F}_T = \mathcal{F}_1 \cup \mathcal{B}_T$, where $\mathcal{B}_T = \mathcal{F}_T - \mathcal{F}_1$, we consider the first expression:

$$\begin{aligned}
&\int_{X_K} \int^{\bullet}_{\Gamma\backslash\mathfrak{H}} ((\, F(\tau), \vartheta(\tau,z)\,))\, d\mu(\tau)\, d\mu(z) \\
&= \int_{X_K} \operatorname*{CT}_{\sigma=0} \left\{ \lim_{T\to\infty} \int_{\mathcal{F}_T} ((\, F(\tau), \vartheta(\tau,z)\,))\, v^{-\sigma}\, d\mu(\tau) \right\} d\mu(z) \\
&= \int_{X_K} \operatorname*{CT}_{\sigma=0} \left\{ \lim_{T\to\infty} \int_{\mathcal{B}_T} ((\, F(\tau), \vartheta(\tau,z)\,))\, v^{-\sigma}\, d\mu(\tau) \right\} d\mu(z) \\
&\qquad + \int_{X_K} \int_{\mathcal{F}_1} ((\, F(\tau), \vartheta(\tau,z)\,))\, d\mu(\tau)\, d\mu(z) \\
&= \int_{X_K} \operatorname*{CT}_{\sigma=0} \left\{ \lim_{T\to\infty} \int_1^T C(v,z)\, v^{-\sigma-1}\, dv \right\} d\mu(z) \\
&\qquad + \int_{\mathcal{F}_1} \int_{X_K} ((\, F(\tau), \vartheta(\tau,z)\,))\, d\mu(z)\, d\mu(\tau),
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
C(v,z) = C(v,z,h) &:= v^{-1} \int_{-\frac{1}{2}}^{\frac{1}{2}} ((\, F(\tau), \vartheta(\tau,z,h)\,))\, du \\
&= \sum_{\varphi} \sum_{m\in\mathbb{Q}} c_\varphi(-m) \sum_{\substack{x \\ Q(x)=m}} \varphi(h^{-1}x)\, e^{-2\pi v R(x,z)}
\end{aligned} \tag{3.2}$$

is the constant term of $v^{-1}((\, F(\tau), \vartheta(\tau,z)\,))$. Here, in the term arising from integration over $\mathcal{F}_1$, we have used the integrability of $\vartheta(\tau,z)$ over X_K. It now suffices to show that the term

$$A := \int_{X_K} \operatorname*{CT}_{\sigma=0} \left\{ \lim_{T\to\infty} \int_1^T C(v,z)\, v^{-\sigma-1}\, dv \right\} d\mu(z) \tag{3.3}$$

in the last expression can be rewritten as

$$B := \operatorname*{CT}_{\sigma=0} \left\{ \lim_{T\to\infty} \int_{X_K} \int_1^T C(v,z)\, v^{-\sigma-1}\, dv\, d\mu(z) \right\}. \tag{3.4}$$

To see this, observe that the integral in B is then equal to

$$\begin{aligned}
&\int_{X_K} \int_1^T C(v,z)\, v^{-\sigma-1}\, dv\, d\mu(z) \\
&= \int_{X_K} \int_{\mathcal{B}_T} ((\, F(\tau), \vartheta(\tau,z)\,))\, v^{-\sigma}\, d\mu(\tau)\, d\mu(z) \\
&= \int_{\mathcal{B}_T} \int_{X_K} ((\, F(\tau), \vartheta(\tau,z)\,))\, d\mu(z)\, v^{-\sigma}\, d\mu(\tau),
\end{aligned} \tag{3.5}$$

again using the integrability of $\vartheta(\tau,z)$. Substituting the resulting expression for B in place of A in the last expression of (3.1), we obtain

$$
\begin{aligned}
&\operatorname*{CT}_{\sigma=0}\left\{\lim_{T\to\infty}\int_{\mathcal{B}_T}\int_{X_K}((\ F(\tau),\vartheta(\tau,z)\ ))\,d\mu(z)\,v^{-\sigma}\,d\mu(\tau)\right\}\\
&\qquad\qquad+\int_{\mathcal{F}_1}\int_{X_K}((\ F(\tau),\vartheta(\tau,z)\ ))\,d\mu(z)\,d\mu(\tau)\\
&=\operatorname*{CT}_{\sigma=0}\left\{\lim_{T\to\infty}\int_{\mathcal{F}_T}\int_{X_K}((\ F(\tau),\vartheta(\tau,z)\ ))\,d\mu(z)\,v^{-\sigma}\,d\mu(\tau)\right\}\\
&=\int_{\Gamma\backslash\mathfrak{H}}^{\bullet}\int_{X_K}((\ F(\tau),\vartheta(\tau,z)\ ))\,d\mu(z)
\end{aligned}
\tag{3.6}
$$

as required.

To show the equality of A and B, we break the function $C(v,z)$ into pieces.

$$
\begin{aligned}
C_+(v,z) &:= \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\sum_{\substack{x\\ Q(x)=m}}\varphi(x)\,e^{-2\pi vR(x,z)}\\
C_0(v,z) &:= \sum_{\varphi} c_\varphi(0)\sum_{\substack{x\\ Q(x)=0,x\neq 0}}\varphi(x)\,e^{-2\pi vR(x,z)}\\
C_{00}(v,z) &:= \sum_{\varphi} c_\varphi(0)\varphi(0) = c_0(0) \qquad \text{(for the coset basis)}\\
C_-(v,z) &:= \sum_{\varphi}\sum_{m<0} c_\varphi(-m)\sum_{\substack{x\\ Q(x)=m}}\varphi(x)\,e^{-2\pi vR(x,z)}.
\end{aligned}
\tag{3.7}
$$

We will write A_+, A_0, A_{00}, and A_- (resp. B_+, etc.) for the corresponding contributions to A (resp. B).

For the C_{00} term, we have

$$\int_1^T v^{-\sigma-1}\,dv = \frac{1}{\sigma}(1-T^{-\sigma}), \tag{3.8}$$

so that

$$\operatorname*{CT}_{\sigma=0}\left\{\lim_{T\to\infty}\int_1^T C_{00}(v,z)\,dv\right\} = 0. \tag{3.9}$$

This gives $A_{00} = B_{00} = 0$.

Next consider the quatities A_+ and B_+ arising from $C_+(v,z)$. Note that the sum on $m>0$ in $C_+(v,z)$ is finite, since there are only finitely many nonvanishing negative Fourier coefficients

$c_\varphi(-m)$. For a given coset representative $h = h_j$, we write $\Gamma = \Gamma_j = H(\mathbb{Q}) \cap hKh^{-1}$, so that $\Gamma\backslash D^+$ is the associated component of X_K. For a fixed $m > 0$ and φ and on the chosen component of X_K, the sum in $C_+(v, z)$ involves

$$\{x \in V(\mathbb{Q}) \mid Q(x) = m,\ \varphi(h^{-1}x) \neq 0\}. \tag{3.11}$$

This set consists of a finite number of Γ orbits. The contribution to A of a single such orbit is

$$c_\varphi(-m)\,\varphi(h^{-1}x) \int_{\Gamma\backslash D^+} \mathop{\mathrm{CT}}_{\sigma=0} \left\{ \lim_{T\to\infty} \int_1^T \sum_{\gamma\in\Gamma_x\backslash\Gamma} e^{-2\pi v R(x,\gamma z)}\, v^{-\sigma-1}\, dv \right\} d\mu(z). \tag{3.12}$$

To prove the finiteness of this expression, it will suffice to prove the finiteness of

$$\int_{\Gamma\backslash D^+} \lim_{T\to\infty} \int_1^T \sum_{\gamma\in\Gamma_x\backslash\Gamma} e^{-2\pi v R(x,\gamma z)}\, v^{-\sigma-1}\, dv\ d\mu(z), \tag{3.13}$$

for $\sigma = \sigma_0$ for some real $\sigma_0 < 0$. Indeed, such finiteness implies that (3.13) defines a holomorphic function of σ in the half plane $\mathrm{Re}(\sigma) > \sigma_0$. If z lies in the set

$$D - \bigcup_{\gamma\in\Gamma_x\backslash\Gamma} \gamma^{-1} D_x, \tag{3.14}$$

then none of the $R(x, \gamma z)$'s vanish and the limit on T inside the integral is finite. Note that the excluded set of z's has measure zero. The following result will be proved at the end of this section.

Proposition 3.2. *Let*

$$\beta_{\sigma+1}(t) = \int_1^\infty e^{-tv}\, v^{-\sigma-1}\, dv.$$

Then, if $Q(x) > 0$, the integral

$$\begin{aligned} &\int_{\Gamma\backslash D^+} \lim_{T\to\infty} \int_1^T \sum_{\gamma\in\Gamma_x\backslash\Gamma} e^{-2\pi v R(x,\gamma z)}\, v^{-\sigma-1}\, dv\ d\mu(z) \\ &= \int_{\Gamma\backslash D^+} \sum_{\gamma\in\Gamma_x\backslash\Gamma} \beta_{\sigma+1}(2\pi R(x,\gamma z))\, d\mu(z) \\ &= \int_{\Gamma_x\backslash D^+} \beta_{\sigma+1}(2\pi R(x,z))\, d\mu(z) \end{aligned}$$

is holomorphic in the halfplane $\mathrm{Re}(\sigma) > -1$.

Recall that $\beta_1(t) = O(-\log(t))$ as $t \to 0$ and $\beta_1(t) = O(e^{-t})$ as $t \to \infty$. Thus, when $\sigma = 0$, the integrand $\beta_1(2\pi R(x,z))$ has a logarithmic singularity on the 'waist' $\Gamma_x\backslash D_x^+$ of the tube $\Gamma_x\backslash D^+$. Also note that this 'waist' can be noncompact.

Corollary 3.3. $A_+ = B_+$.

Next we consider the terms A_0 and B_0 associated to the nonzero null vectors. Again for a given h and φ, the associated terms in $C_0(v,z)$ will be

$$c_\varphi(0) \sum_{\substack{x\neq 0 \\ Q(x)=0}} \varphi(h^{-1}x)\, e^{-2\pi v R(x,z)}. \tag{3.15}$$

There are a finite number of Γ orbits in the space of null lines in $V(\mathbb{Q})$. For a given null line $\ell \subset V$, we have the contribution to A_0:

$$c_\varphi(0) \int_{\Gamma\backslash D^+} \underset{\sigma=0}{\mathrm{CT}} \left\{ \lim_{T\to\infty} \int_1^T \sum_{\gamma\in\Gamma_\ell\backslash\Gamma} \sum_{x\in\ell(\mathbb{Q}),\ x\neq 0} \varphi(h^{-1}x)\, e^{-2\pi v R(x,\gamma z)}\, v^{-\sigma-1}\, dv \right\} d\mu(z). \tag{3.16}$$

Again, the following result, to be proved below, will suffice.

Proposition 3.4. *Suppose that $n > 1$. Then the integral*

$$\int_{\Gamma_\ell\backslash D^+} \sum_{x\in\ell(\mathbb{Q}),\ x\neq 0} \varphi(h^{-1}x)\, \beta_{\sigma+1}(2\pi v R(x,z))\ d\mu(z)$$

is holomorphic in the halfplane $\mathrm{Re}(\sigma) > -\frac{n}{2}$.

Corollary 3.5. $A_0 = B_0$.

Finally, we turn to the terms where $m < 0$. Note that the sum on m in $C_-(v,z)$ is now infinite so that we will need information about the growth of the Fourier coefficients $c_\varphi(-m)$. In fact, these can grow very fast!

As before, we fix φ and h, and, taking the limit with respect to T, we consider

$$\int_{\Gamma\backslash D^+} \sum_{m<0} c_\varphi(-m) \sum_{\substack{x \\ Q(x)=m}} \int_1^\infty \varphi(h^{-1}x)\, e^{-2\pi v R(x,z)}\, v^{-\sigma-1}\, dv\, d\mu(z). \tag{3.17}$$

Here we can push the integral over $\Gamma\backslash D^+$ inside the sum on m, and again use the fact that, for each m, there are only a finite number of Γ orbits in the set

$$\{x \in V(\mathbb{Q}) \mid Q(x) = m,\ \varphi(h^{-1}x) \neq 0\ \}. \tag{3.18}$$

Thus, it will suffice to show:

Proposition 3.6. *The sum*

$$\sum_{m<0} c_\varphi(-m) \sum_{\substack{x \\ Q(x)=m \\ \bmod \Gamma}} \varphi(h^{-1}x) \int_{\Gamma_x\backslash D^+} \int_1^\infty e^{-2\pi v R(x,z)}\, v^{-\sigma-1}\, dv\, d\mu(z)$$

defines an entire function of σ.

Corollary 3.7. $A_- = B_-$.

Proof of Proposition 3.2.

To show the finiteness of the integral

$$\int_{\Gamma_x\backslash D^+} \beta_{\sigma+1}(2\pi R(x,z))\, d\mu(z) \tag{3.19}$$

in the case $Q(x) > 0$, we introduce coordinates. We choose a basis for $V(\mathbb{R})$ so that the inner product has matrix $I_{n,2}$ and so that $x = 2\alpha v_1$ is a nonzero multiple of the first basis vector. Then $SO(V)(\mathbb{R})^+ \simeq SO^+(n,2) = G$ and the subgroup stabilizing x is isomorphic to $SO^+(n-1,2) = G_x$. Let $z_0 \in D^+$ be the oriented negative 2–plane spanned by v_{n+1} and v_{n+2} and let $K = SO(n) \times SO(2)$ be its stabilizer in $SO^+(n,2)$. The plane spanned by v_1 and v_{n+1}, the first negative basis vector, has signature $(1,1)$. The identity component of the special orthogonal group of this plane is a 1-parameter subgroup

$$A = \{a_t \mid t \in \mathbb{R}\}, \tag{3.20}$$

where $a_t v_1 = \cosh(t)v_1 + \sinh(t)v_{n+1}$. Let A_+ be the subset of a_t's with $t \geq 0$. Then, from the general theory of semisimple symmetric spaces – a convenient reference is [**13**] – one has a double coset decomposition

$$G = G_x A_+ K \tag{3.21}$$

and the integral formula

$$\int_G \phi(g)\, dg = \int_{G_x} \int_{A_+} \int_K \phi(g_x a_t k)\, |\sinh(t)| \cosh(t)^{n-1}\, dg_x\, dt\, dk. \tag{3.22}$$

For $z = g_x a_t \cdot z_0 \in D^+$, we have

$$R(x,z) = 2m \sinh^2(t), \tag{3.23}$$

since $Q(x) = 2\alpha^2 = m$. Then, our integral becomes (up to a positive constant depending on normalization of invariant measures)

$$\begin{aligned} &\int_{\Gamma_x\backslash D^+} \beta_{\sigma+1}(2\pi R(x,z))\, d\mu(z) \\ &= C \operatorname{vol}(\Gamma_x\backslash G_x) \operatorname{vol}(K) \int_0^\infty \beta_{\sigma+1}(4\pi m\, \sinh^2(t))\, \sinh(t) \cosh(t)^{n-1}\, dt. \end{aligned} \tag{3.24}$$

Lemma 3.8. *(i) The function*

$$\beta_{\sigma+1}(t) = \int_1^\infty e^{-tu}\, u^{-\sigma-1}\, du$$

is $O(e^{-t})$ as $t \to \infty$.
(ii) If $\sigma < 0$, then $\beta_{\sigma+1}(t) = O(t^{\sigma})$ as $t \to 0$.
(iii) If $\sigma = 0$, then

$$\beta_1(t) = -\mathrm{Ei}(-t) = -\log(t) + \gamma + \int_0^t \frac{e^u - 1}{u}\, du$$

is the exponential integral and this function has a logarithmic singularity, $-\log(t)$, as $t \to 0$.
(iv) If $\sigma > 0$, then $\beta_{\sigma+1}(t) = O(1)$ as $t \to 0$. $\square$

The integral (3.24) is finite for $\sigma > -1$, since, near the lower endpoint it looks like

$$\int_0 \sinh(t)^{2\sigma}\, \sinh(t)\cosh(t)^{n-1}\, dt = \int_0 u^{2\sigma+1}\, (u^2+1)^{\frac{1}{2}(n-2)}\, du. \tag{3.25}$$

Note that the signature of $x^{\perp}$ is $(n-1,2)$, so that the volume $\mathrm{vol}(\Gamma_x\backslash G_x)$ is also always finite. This proves Proposition . $\square$ $\square$

Proof of Proposition 3.4.

Finally, we consider the integral

$$\int_{\Gamma_\ell\backslash D^+} \sum_{x\in\ell(\mathbb{Q}),\ x\neq 0} \varphi(h^{-1}x)\,\beta_{\sigma+1}(2\pi v R(x,z))\ d\mu(z) \tag{3.26}$$

In this case, we choose a basis for V such that the matrix for the inner product is

$$\begin{pmatrix} & & 1 \\ & I_{n-1,1} & \\ 1 & & \end{pmatrix} \tag{3.27}$$

and such that ℓ is spanned by the first basis vector. Moreover, we assume that

$$\{\ x \in \ell(\mathbb{Q}) \mid \varphi(h^{-1}x) \neq 0\ \} \subset 2\mathbb{Z}v_1. \tag{3.28}$$

The parabolic subgroup P_ℓ stabilizing the line ℓ then has Levi decomposition $P_1 = U_1MA$ with $A \simeq GL(\ell)$, $M \simeq SO^+(n-1,1)$, and unipotent radical U_1. We take z_0 to be the oriented negative 2–plane spanned by $\frac{1}{2}(v_1 - v_{n+2})$ and v_{n+1} and let K be its stabilizer. Then

$$G = SO^+(V)(\mathbb{R}) = UMAK \tag{3.29}$$

and we have the integral formula

$$\int_G \phi(g)\,dg = \int_U \int_M \int_A \int_K \phi(uma_r k)\, r^{-n-1}\, du\, dm\, dr\, dk, \tag{3.30}$$

where $a_r v_1 = r v_1$. For $z = uma_r \cdot z_0$, and $x = 2\alpha v_1$,

$$R(x,z) = R(a_r^{-1}x, z_0) = 2\alpha^2 r^{-2}, \tag{3.31}$$

since

$$a_r^{-1}x = 2\alpha r^{-1} v_1 = 2\alpha r^{-1}\left(\frac{1}{2}(v_1 + v_{n+2}) + \frac{1}{2}(v_1 - v_{n+2})\right) \tag{3.32}$$

has $\alpha\, r^{-1}\,(v_1 - v_{n+2})$ as its z_0 component. The integral (3.26) is then majorized by a constant times

$$\begin{aligned} &\int_0^\infty \sum_{\alpha\in\mathbb{Z},\ \alpha\neq 0} \beta_{\sigma+1}(4\pi\alpha^2\, r^{-2})\, r^{-n-1}\, dr \\ &= 2(4\pi)^{-n}\,\zeta(n) \int_0^\infty \beta_{\sigma+1}(r^2)\, r^{n-1}\, dr. \end{aligned} \tag{3.33}$$

The integral here is finite provided $2\sigma + n \geq 0$, so we obtain the required convergence provided $n \geq 2$ and $\mathrm{Re}(\sigma) > -\frac{n}{2}$, i.e., in all isotropic cases except $n = 1$ (which was an exceptional case). □

Proof of Proposition 3.6. For x with $Q(x) = m < 0$, we write $x = \mathrm{pr}_z(x) + x'$ so that

$$R(x,z) = (x', x') - 2m \geq 2|m|. \tag{3.34}$$

We let $R'(x,z) = (x', x')$, and note that $R'(x,z) = 0$ if and only if $x \in z$. Then we have the easy estimate:

$$\begin{aligned} \int_1^\infty e^{-2\pi v R(x,z)}\, v^{-\sigma-1}\, dv &\leq e^{-2\pi R'(x,z)} \int_1^\infty e^{-4\pi|m|v}\, v^{-\sigma-1}\, dv \\ &\leq e^{-2\pi R'(x,z)} \int_1^\infty e^{-\epsilon v}\, e^{(\epsilon-4\pi|m|)v}\, v^{-\sigma-1}\, dv \\ &\leq e^{-2\pi R'(x,z)}\, e^{\epsilon-4\pi|m|} \int_1^\infty e^{-\epsilon v}\, v^{-\sigma-1}\, dv \\ &\leq C(\epsilon,\sigma)\, e^{-2\pi R'(x,z)}\, e^{-4\pi|m|}, \end{aligned} \tag{3.35}$$

for any ϵ with $0 < \epsilon < 4\pi|m|$, where the constant $C(\epsilon,\sigma)$ is uniform in any σ–halfplane and independent of m. Note that there is a positive lower bound for the quantity $|m|$ where $m < 0$ has $c_\varphi(-m) \neq 0$. This leads to the expression

$$\sum_{m<0} |c_\varphi(-m)|\, e^{-4\pi|m|} \sum_{\substack{x \\ Q(x)=m \\ \mathrm{mod}\ \Gamma}} \varphi(h^{-1}x) \int_{\Gamma_x\backslash D^+} e^{-2\pi R'(x,z)}\, d\mu(z). \tag{3.36}$$

Recall that the modular form f_φ with Fourier coefficients $c_\varphi(-m)$ has weight $1-\frac{n}{2}$, with some real multiplier for a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, and is holomorphic in the upper halfplane with possible poles at the cusps. Then it is known that

$$c_\varphi(-m) = O\Big(\ |m|^{-\frac{n+1}{4}}\, e^{C\sqrt{|m|}}\ \Big), \tag{3.38}$$

i.e., these coefficients grow at most subexponentially. The (explicit) constant C depends only on the order of the pole of f_φ and on the multiplier. If $n>2$, so that f_φ has negative weight, this fact follows from the classical work of Rademacher [**41**], Rademacher–Zuckermann [**42**], Zuckermann [**53**], and Petersson[**40**], cf. also Hejhal [**22**]. The cases $n=1$ and 2 are covered by Hejhal [**22**] and Niebur [**39**].

Finally, it remains to estimate the quantity

$$\sum_{\substack{x\\ Q(x)=m\\ \mathrm{mod}\ \Gamma}} \varphi(h^{-1}x)\int_{\Gamma_x\backslash D^+} e^{-2\pi R'(x,z)}\,d\mu(z). \tag{3.39}$$

To estimate the integral here, we choose basis for V so that the inner product has matrix $-I_{2,n}$ and such that $x=2\alpha v_1$. Let z_0 be the span of v_1 and v_2, and let $A=\{a_t\}$ be the 1–parameter subgroup which is the identity component of the special orthogonal group of the plane spanned by v_1 and v_3. In this case $a_tv_1 = \cosh(t)v_1+\sinh(t)v_3$. Again we have the decomposition (3.29) and an integral formula analogous to (3.30), but with the cosh and sinh switched in the modulus factor. For $z=g_xa_t\cdot z_0$, we also have

$$R(x,z) = 2|m|\cosh^2(t). \tag{3.40}$$

and

$$R'(x,z) = 2|m|\cosh^2(t) - 2|m| = 2|m|\sinh^2(t). \tag{3.41}$$

Then we have

$$\begin{aligned}
&\int_{\Gamma_x\backslash D^+} e^{-2\pi R'(x,z)}\,d\mu(z)\\
&= C'\,\mathrm{vol}(\Gamma_x\backslash G_x)\,\mathrm{vol}(K)\int_0^\infty e^{-4\pi|m|\ \sinh^2(t)}\,\sinh(t)^{n-1}\cosh(t)\,dt\\
&= C'\,\mathrm{vol}(\Gamma_x\backslash G_x)\,\mathrm{vol}(K)\,\frac{1}{2}\,(4\pi|m|)^{-\frac{n}{2}}\,\Gamma(\frac{n}{2}).
\end{aligned} \tag{3.42}$$

Using this in (3.39), we are left with

$$\sum_{m<0}|m|^{-\frac{3n+1}{4}}\,e^{C\sqrt{|m|}-4\pi|m|}\sum_{\substack{x\\ Q(x)=m\\ \mathrm{mod}\ \Gamma}}\varphi(h^{-1}x)\,\mathrm{vol}(\Gamma_x\backslash G_x). \tag{3.43}$$

Here m runs over the negative elements of $N^{-1}\mathbb{Z}$ for a suitable N depending on φ and h. The resulting expression is finite since, [**46**],

$$\sum_{\substack{x \\ Q(x)=m \\ \bmod \Gamma}} \varphi(h^{-1}x)\,\mathrm{vol}(\Gamma_x\backslash G_x) = O(|m|^{\frac{n}{2}+\epsilon}). \tag{3.44}$$

□

This completes the proof of Theorem 3.1. □ □ □

There are several more things which need to be proved.

Proof of Proposition 2.5. By (2.3), the left hand side of the identity

$$\underset{\sigma=0}{\mathrm{CT}}\left\{\lim_{T\to\infty}\int_{\mathcal{F}_T}\sum_{\varphi} f_\varphi(\tau)\,I(\tau,\varphi^C)\,v^{-\sigma-2}\,du\,dv\right\}$$

$$= \lim_{T\to\infty}\left[\int_{\mathcal{F}_T}\sum_{\varphi} f_\varphi(\tau)\,I(\tau,\varphi^C)\,v^{-2}\,du\,dv - c_0(0)\,\log(T)\right].$$

to be proved can be written as

$$\begin{aligned}
&\mathrm{vol}(X)^{-1}\,\underset{\sigma=0}{\mathrm{CT}}\left\{\lim_{T\to\infty}\int_{X_K}\int_{\mathcal{F}_T}((\,f(\tau),\vartheta(\tau,z)\,))\,v^{-\sigma-2}\,du\,dv\,d\mu(z)\right\}\\
&= \mathrm{vol}(X)^{-1}\int_{X_K}\int_{\mathcal{F}_1}((\,f(\tau),\vartheta(\tau,z)\,))\,v^{-2}\,du\,dv\,d\mu(z)\\
&\qquad + \mathrm{vol}(X)^{-1}\,\underset{\sigma=0}{\mathrm{CT}}\left\{\lim_{T\to\infty}\int_{X_K}\int_1^T C(v,z)\,v^{-\sigma-1}\,dv\,d\mu(z)\right\}
\end{aligned}$$

The analysis made in the proof of Theorem 3.1 above shows that the integral

$$\int_{X_K}\int_1^\infty\Big[C(v,z) - C_{00}(v,z)\Big]\,v^{-\sigma-2}\,dv\,d\mu(z)$$

defines a holomorphic function of σ in the half plane $\mathrm{Re}(\sigma) > -1$. Note that in the case $n = 1$ there are no C_0 terms, since V is then assumed to be anisotropic. The remaining term is

$$\mathrm{vol}(X)^{-1}\int_{X_K}\int_1^T C_{00}(v,z)\,v^{-\sigma-1}\,dv\,d\mu(z) = c_0(0)\,\frac{1}{\sigma}\big(1-T^{-\sigma}\big) = c_0(0)\log(T) + O(\sigma).$$

This term makes no contribution when we take the limit as T goes to infinity followed by the constant term at $\sigma = 0$. Thus, once the term $c_0(0)\,\log(T)$ has been removed, we can pass to the limit on T with $\sigma = 0$, and this proves Proposition 2.5. □

Proof of Proposition 2.11. In the Fourier expansion (2.21) for $E(\tau, s; \varphi, \ell+2)$ for a factorizable function $\varphi = \otimes_p \varphi_p \in S(V(\mathbb{A}_f))$, the mth coefficient, for $m \neq 0$, has a product formula

$$E_m(\tau, s; \varphi, \ell+2) = A_\varphi(s, m, v)\, q^m = W_{m,\infty}(\tau, s; \ell+2) \cdot \prod_p W_{m,p}(s, \varphi_p).$$

The following facts are well known, cf. [**33**] for more details. For $s = s_0 = \ell + 1 = \frac{n}{2}$,

$$W_{m,\infty}(\tau, \frac{n}{2}; \frac{n}{2}+1) = \frac{(-2i)^{\frac{n}{2}+1}}{\Gamma(\frac{n}{2}+1)}\, m^{\frac{n}{2}}\, q^m$$

if $m > 0$,

$$W_{m,\infty}(\tau, \frac{n}{2}; \frac{n}{2}+1) = 0$$

if $m < 0$, and

$$W'_{m,\infty}(\tau, \frac{n}{2}; \frac{n}{2}+1) = \pi(-i)^{-\frac{n}{2}-1}\, 2^{-\frac{n}{2}}\, q^m\, v^{-\frac{n}{2}} \int_1^\infty e^{-4\pi|m|vr}\, r^{-\frac{n}{2}-1}\, dr.$$

On the other hand, for any $m \neq 0$, the product over the finite primes is

$$C(m) := \Big(\prod_p W_{m,p}(s, \varphi_p)\Big)_{s=s_0} = O(1).$$

Therefore, for $m < 0$, we have

$$b_\varphi(m, v) = \pi(-i)^{-\frac{n}{2}-1}\, 2^{-\frac{n}{2}}\, C(m)\, v^{-\frac{n}{2}} \int_1^\infty e^{-4\pi|m|vr}\, r^{-\frac{n}{2}-1}\, dr,$$

where $C(m) = O(1)$. Thus

$$|b_\varphi(m, v)| = O\big(v^{-\frac{n}{2}-1}\, |m|^{-1}\, e^{-4\pi|m|v}\big).$$

Using (3.38), this proves part (i) and (ii) of Proposition 2.11.

Finally, the constant term has the form

$$E_0(\tau, s; \varphi, \ell+2) = v^{\frac{1}{2}(s+1-\ell)}\, \varphi(0) + W_{0,\infty}(\tau, s; \ell+2) \prod_p W_{0,p}(s, \varphi_p),$$

where

$$W_{0,\infty}(\tau, s; \ell+2) = 2\pi\, (-i)^{\frac{n}{2}+1}\, v^{-\frac{1}{2}(s+\frac{n}{2})}\, 2^{-s} \frac{\Gamma(s)\frac{1}{2}(s-\frac{n}{2})}{\Gamma(\frac{1}{2}(s+\frac{n}{2}+2))\Gamma(\frac{1}{2}(s-\frac{n}{2}+2))}.$$

Then, the derivative at $s = s_0 = \frac{n}{2}$ is

$$E'_0(\tau, \frac{n}{2}; \varphi, \frac{n}{2}+1) = \frac{1}{2}\, \log(v)\, \varphi(0) + \pi\, (-i)^{\frac{n}{2}+1}\, v^{-\frac{n}{2}}\, 2^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+1)}\, C(0).$$

This yields (iii) of Proposition 2.11. $\square$

§4. Formulas for degrees.

In this section, we explain how the Siegel–Weil formula can be applied to yield formulas for the degrees of certain divisors on the quasiprojective varieties attached to orthogonal groups of signature $(n, 2)$ over $\mathbb{Q}$. More precisely, these degrees occur as the Fourier coefficients of certain (special values of) Eisenstein series. The basic idea will be to apply the Siegel–Weil formula for two different quadratic spaces to describe a special value of the same Eisenstein series! Comparison of the Fourier coefficients of the two theta integrals and the Eisenstein series yields nontrivial identities, several of which occur in the classical literature, [**52**], [**11**], [**47**].

The Siegel-Weil formula.

For convenience of the reader, we briefly review the Siegel–Weil formula for the dual pair $(SL_2, O(V))$ needed in this section and in section 2.

Let V be a nondegenerate quadratic space over $\mathbb{Q}$, and let $G = SL_2$. As before, let $G'_{\mathbb{A}}$ be the metaplectic cover of $G(\mathbb{A}) = SL_2(\mathbb{A})$. We identify $G'_{\mathbb{A}} = SL_2(\mathbb{A}) \times \{\pm 1\}$, where multiplication on the right is given by $[g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2)]$, for the cocycle as in [**49**], [**16**]. In particular, we have subgroups

$$N_{\mathbb{A}} = \{n = [n(b), 1] \mid b \in \mathbb{A}\}, \qquad n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \tag{4.1}$$

and

$$M_{\mathbb{A}} = \{\underline{m}(a) = [m(a), \epsilon] \mid a \in \mathbb{A}^\times, \epsilon = \pm 1\}, \qquad m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}. \tag{4.2}$$

An idele character χ of $\mathbb{Q}^\times \backslash \mathbb{A}^\times$ determines a character χ^ψ of $M_{\mathbb{A}}$ by

$$\chi^\psi([m(a), \epsilon]) = \epsilon\, \chi(a)\, \gamma(a, \psi)^{-1} \tag{4.3}$$

where $\gamma(\cdot, \psi)$ is the global Weil index.

The group $G'_{\mathbb{A}}$ acts on the Schwartz space $S(V(\mathbb{A}))$ via the Weil representation $\omega = \omega_\psi$ determined by our fixed additive character ψ of $\mathbb{A}/\mathbb{Q}$, and this action commutes with the linear action of $O(V)(\mathbb{A})$. For $g' \in G'_{\mathbb{A}}$, $h \in O(V)(\mathbb{A})$, and $\varphi \in S(V(\mathbb{A}))$, the theta series

$$\theta(g', h; \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g')\varphi(h^{-1}x), \tag{4.4}$$

defines a smooth function on $G'_{\mathbb{A}} \times O(V)(\mathbb{A})$, left invariant under $G'_{\mathbb{Q}} \times O(V)(\mathbb{Q})$, and slowly increasing on the quotient $\big(G'_{\mathbb{Q}} \times O(V)(\mathbb{Q})\big)\backslash\big(G'_{\mathbb{A}} \times O(V)(\mathbb{A})\big)$.

By Weil's criterion [**27**] in the present case, the theta integral

$$I(g';\varphi) = \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} \theta(g',h;\varphi)\,dh, \tag{4.5}$$

where $\mathrm{vol}(O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A}),dh) = 1$, is absolutely convergent whenever either V is anisotropic or $\dim(V) - r > 2$, where r is the Witt index of V. The resulting automorphic form $I(\varphi)$ on $G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}$ is identified, by the Siegel–Weil formula, with a special value of an Eisenstein series, defined as follows.

Let $\chi = \chi_V$ be the quadratic character of $\mathbb{A}^\times/\mathbb{Q}^\times$ defined by

$$\chi(x) = (x, (-1)^{m(m-1)/2}\det(V)), \tag{4.6}$$

where $m = \dim(V)$ and $\det(V) \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$ is the determinant of the matrix for the quadratic form Q on V. For $s \in \mathbb{C}$, let $I(s,\chi)$ be the principal series representation of $G'_{\mathbb{A}}$ consisting of smooth functions $\Phi(s)$ on $G'_{\mathbb{A}}$ such that

$$\Phi(n\,\underline{m}(a)g',s) = \begin{cases} \chi^{\psi}(\underline{m}(a))\,|a|^{s+1}\,\Phi(g',s) & \text{if } n \text{ is odd,} \\ \\ \chi(m(a))\,|a|^{s+1}\,\Phi(g',s) & \text{if } n \text{ is even.} \end{cases} \tag{4.7}$$

There is then a $G'_{\mathbb{A}}$ intertwining map

$$\lambda = \lambda_V : S(V(\mathbb{A})) \longrightarrow I(s_0,\chi_V), \qquad \lambda(\varphi)(g') = \omega(g')\varphi(0), \tag{4.8}$$

where $s_0 = \frac{m}{2} - 1$. As in section 1, let $K'_\infty K'$ be the full inverse image of $SO(2) \times \mathrm{SL}_2(\hat{\mathbb{Z}})$ in $G'_{\mathbb{A}}$. A section $\Phi(s) \in I(s,\chi)$ will be called standard if its restriction to $K'_\infty K'$ is independent of s. By the Iwasawa decomposition $G'_{\mathbb{A}} = N'_{\mathbb{A}}M'_{\mathbb{A}}K'_\infty K'$, the function $\lambda(\varphi) \in I(s_0,\chi)$ has a unique extension to a standard section $\Phi(s) \in I(s,\chi)$, where $\Phi(s_0) = \lambda(\varphi)$. The Eisenstein series, defined by

$$E(g',s;\Phi) = E(g',s;\varphi) = \sum_{\gamma \in P'_{\mathbb{Q}}\backslash G'_{\mathbb{Q}}} \Phi(\gamma g',s) \tag{4.9}$$

for $\mathrm{Re}(s) > 1$ has a meromorphic analytic continuation to the whole s–plane.

Theorem 4.1. (Siegel–Weil formula) *(i) Assume that V is anisotropic or that* $\dim(V) - r > 2$*, where r is the Witt index of V, so that the theta integral (4.5) is absolutely convergent. Then $E(g',s;\varphi)$ is holomorphic at the point $s = s_0 = \frac{m}{2} - 1$, where $m =$* $\dim(V)$*, and*

$$E(g',s_0;\varphi) = \kappa \cdot I(g',\ \varphi),$$

where $\kappa = 2$ when $m \leq 2$ and $\kappa = 1$ otherwise.
(ii) Suppose, in addition, that $m > 1$. Then

$$E(g', s_0; \varphi) = \kappa \cdot I(g'; \varphi) = \frac{\kappa}{2} \int_{SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})} \theta(g', h; \varphi)\, dh,$$

where dh is Tamagawa measure on $SO(V)(\mathbb{A})$.

When $m > 4$ part (i) is the classic result of Siegel and Weil, in Weil's formulation. The variants for $m \leq 4$ are also mostly classical, e.g., due to Hecke, Siegel, etc.. We do not attempt to give systematic references. Of course, the analogous result holds for any number field F.

The point of (ii) is that, we can almost always replace the integral over $O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})$ with the integral over $SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})$. The later is much more convenient, since $SO(V)$ is connected. In the range $m > 4$, this fact is again a very special case of the results of Weil, [**51**], pp.76–77, Théorèm 5.

Proof. We only prove part (ii). Let $I' : S(V(\mathbb{A})) \to \mathbb{C}$ be the linear functional given by

$$I'(\varphi) = \int_{SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})} \theta(h; \varphi)\, dh, \tag{4.10}$$

where $\theta(h; \varphi) = \theta(e, h; \varphi)$, so that I' defines an element of

$$\mathrm{Hom}_{SO(V)(\mathbb{A})}\big(\, S(V(\mathbb{A})), \mathbb{C}\,\big), \tag{4.11}$$

where $SO(V)(\mathbb{A})$ acts trivially on $\mathbb{C}$. The group

$$C(\mathbb{A}_f) = O(V)(\mathbb{A})/SO(V)(\mathbb{A}) \simeq \mu_2(\mathbb{A}) \tag{4.12}$$

acts on this space of such functionals. In fact one has:

Proposition 4.2. *If* $\dim(V) > 1$, *then the action of $C(\mathbb{A}_f)$ on the space of $SO(V)(\mathbb{A})$–invariant linear functionals on $S(V(\mathbb{A}))$ (4.11) is trivial.*

Proof. For any prime $p \leq \infty$, consider the analogous local space

$$Hom_{SO(V_p)}(S(V_p), \mathbb{C}), \tag{4.13}$$

with its action of C_p. If the sign character ϵ_p of $C_p = O(V_p)/SO(V_p)$ occurs, then the sign representation sgn_p of $O(V_p)$ occurs in the local theta correspondence for the dual pair

$(\widetilde{\mathrm{SL}}_2(\mathbb{Q}_p), O(V_p))$. But it is known, [**43**], that the sign representation does not occur for such a dual pair if $\dim(V) = m > 1$. Thus C_p acts trivially on (4.13), and a standard argument then shows that $C(\mathbb{A})$ acts trivially on (4.11), as claimed. $\square$

On the other hand, it is clear that

$$
\begin{aligned}
I(g', \varphi) &= \int_{C(\mathbb{Q})\backslash C(\mathbb{A})} I'\big(\omega(g')\omega(h)\varphi\big)\, dc \\
&= \frac{1}{2} \int_{C(\mathbb{A})} I'\big(\omega(g')\omega(h)\varphi\big)\, dc \\
&= \frac{1}{2}\, I'\big(\omega(g')\varphi\big),
\end{aligned} \tag{4.14}
$$

where $h \in O(V)(\mathbb{A})$ projects to $c \in C(\mathbb{A})$ and where $\mathrm{vol}(C(\mathbb{A}), dc) = 1$. The factor $1/2$ occurs as the volume of $C(\mathbb{Q})\backslash C(\mathbb{A})$. $\square$ $\square$

A matching principle.

For a nondegenerate quadratic space V over $\mathbb{Q}$ of dimension m, let

$$\Pi(V) = \ \mathrm{image}(\lambda_V) \subset I(s_0, \chi) \tag{4.15}$$

be the resulting $G'_{\mathbb{A}}$-submodule of the principal series, where $\chi = \chi_V$ and $s_0 = \frac{m}{2} - 1$. There are analogous local maps

$$\lambda_p : S(V_p) \longrightarrow I_p(s_0, \chi_p), \tag{4.16}$$

with images

$$\Pi_p(V_p) = \ \mathrm{image}(\lambda_p) \subset I_p(s_0, \chi_p), \tag{4.17}$$

the local components of $\Pi(V)$ for the corresponding local induced representations. Note that $\Pi(V)$ and the $\Pi_p(V_p)$'s are not always irreducible. The key idea is that the Eisenstein series (4.9) associated to $\varphi = \otimes_p \varphi_p \in S(V(\mathbb{A}))$ depends only on the the collection $\{\lambda_p(\varphi_p)\}$ of local components.

Definition 4.3: Let V_p and V'_p be quadratic spaces over $\mathbb{Q}_p$ of dimension m and fixed character $\chi_{V_p} = \chi_{V'_p} = \chi_p$. Functions $\varphi_p \in S(V_p)$ and $\varphi'_p \in S(V'_p)$ are said to **match** if

$$\lambda_p(\varphi_p) = \lambda'_p(\varphi'_p).$$

Remark 4.4: This matching is analogous to that which occurs in the trace formula and relative trace formula, and our identity of theta integrals can be viewed as an analogue of a comparison of trace formulas.

Examples 4.5: If $m > 4$, or if $m = 4$ and $\chi_p \neq 1$, then the nonarchimedean local principal series $I_p(s_0, \chi_p)$ are irreducible and hence, for any pair V_p and V'_p, every $\varphi_p \in S(V_p)$ has a matching $\varphi'_p \in S(V'_p)$.

If $m = 4$, and $\chi_p = 1$, then $s_0 = 1$ and $I_p(s_0, \chi_p)$ has the special representation as irreducible submodule and the trivial representation as quotient. The split 4 dimensional quadratic space V_p has $\Pi_p(V_p) = I_p(s_0, \chi_p)$, while the anisotropic space V'_p given by the reduced norm on the division quaternion algebra over $\mathbb{Q}_p$ has $\Pi_p(V'_p)$ the irreducible special. Therefore the space of φ_p's in $S(V_p)$ which have matching φ'_p's has codimension 1.

If $m = 3$, then $s_0 = \frac{1}{2}$ and $I_p(s_0, \chi_p)$ always has length 2 with a special representation of G'_p as the irreducible subrepresentation and an irreducible Weil representation – playing the role of the trivial representation for the metaplectic group G'_p – as irreducible quotient, [**44**]. The ternary quadratic space V_p of trace 0 elements in $M_2(\mathbb{Q}_p)$ with a scalar multiple (determined by χ_p) of the determinant form has $\Pi_p(V_p) = I_p(s_0, \chi_p)$, and the analogous space V'_p of trace 0 elements in the division quaternion algebra over $\mathbb{Q}_p$ has $\Pi_p(V'_p) \subset I_p(s_0, \chi_p)$ the unique irreducible subrepresentation. Now the subspace of φ_p's in $S(V_p)$ which have matching φ'_p's in $S(V'_p)$ has infinite codimension.

Remark 4.6: If $m = 2$ and $\chi_p \neq 1$, then the spaces $\Pi_p(V_p)$ and $\Pi_p(V'_p)$ are irreducible and distinct, while, if $m = 1$, there is a unique space with a given χ_p, so the matching phenomenon of interest here will not occur globally.

Note that the cases $m = 3$ and 4 are precisely those for which s_0 is in or at the edge of the critical strip $|\mathrm{Re}(s)| \leq 1$.

Over $\mathbb{R}$, the situation is the following. For $r \in \frac{1}{2}\mathbb{Z}$, satisfing a suitable parity condition, let $\Phi^r(s)$ be the (unique) function in $I_\infty(s, \chi_\infty)$ such that

$$\Phi^r(k', s) = \chi_r(k'), \tag{4.18}$$

for the character χ_r of K'_∞. The space of K'_∞–finite vectors in $I_\infty(s, \chi_\infty)$ is then spanned by the $\Phi^r(s)$'s for $r \in r_0 + 2\mathbb{Z}$.

Lemma 4.7. *Suppose that V_∞ and V'_∞ are quadratic spaces over $\mathbb{R}$ of dimension m and with the same quadratic character, i.e., with signatures (p, q) and (p', q') with $q \equiv q'$* mod (2).

Suppose that $\varphi_\infty \in S(V_\infty)$ and $\varphi'_\infty \in S(V'_\infty)$ are eigenfunctions for K'_∞ with eigencharacter χ_r and with $\varphi_\infty(0) = \varphi'_\infty(0)$. Then φ_∞ and φ'_∞ match and $\lambda_\infty(\varphi_\infty) = \lambda'_\infty(\varphi'_\infty) = \Phi^r(s_0)$.

Matching Principle 4.8. *Suppose that V and V' are quadratic spaces over $\mathbb{Q}$ of the same dimension and with the same quadratic character $\chi_V = \chi_{V'} = \chi$. Suppose that $\varphi \in S(V(\mathbb{A}))$ and $\varphi' \in S(V'(\mathbb{A}))$ match, i.e., $\lambda(\varphi) = \lambda'(\varphi') = \Phi(s_0)$. Assume that the convergence condition of the Siegel–Weil formula is fulfilled by the spaces V and V'. Then*

$$I(g', \varphi) = E(g', s_0, \Phi) = I(g', \varphi').$$

Remark 4.9. The definition of matching and the resulting equality of theta integrals can be extended to dual pairs $(Sp(r), O(V))$, $(Sp(r), O(V'))$ for any $r \geq 1$ over a number field, dual pairs for unitary groups etc., etc.

Of course, the matching principle is a trivial observation, but, while the Eisenstein series is built from purely local date, the theta integrals involved depend on global arithmetic. In particular, their equality can yield some highly nontrivial identities. We now describe one of these.

A geometric example.

Let V be a quadratic space over $\mathbb{Q}$ of signature $(n, 2)$, and let V' be a quadratic space over $\mathbb{Q}$ with $\chi_{V'} = \chi_V = \chi$ but with signature $(n+2, 0)$. Suppose that $\varphi \in S(V(\mathbb{A}_f))$ and $\varphi' \in S(V'(\mathbb{A}_f))$ are matching functions. By the discussion above, when $n > 2$, any φ has such a matching φ'. We next construct matching functions over $\mathbb{R}$.

As explained in section 1 above, the Gaussian for $V(\mathbb{R})$ is the function $\varphi_\infty \in S(V(\mathbb{R}))\otimes A^{(0,0)}(D)$ given by

$$\varphi_\infty(x, z) = e^{-\pi(x,x)_z} = e^{-2\pi R(x,z)}\, e^{-2\pi Q(x)}. \tag{4.19}$$

It has weight $\ell = \frac{n}{2} - 1$ and $\varphi_\infty(0, z) = 1$, so that

$$\lambda_\infty(\varphi_\infty) = \Phi^\ell(s_0). \tag{4.20}$$

Let $V'(\mathbb{R})$ be a quadratic space of signature $(n+2, 0)$. The Gaussian $\varphi'_\infty \in S(V'(\mathbb{R}))$ is given by

$$\varphi'_\infty(x) = e^{-2\pi Q'(x)}. \tag{4.21}$$

It has weight $\frac{n+2}{2} = \ell + 2$ and $\varphi'_\infty(0) = 1$, so that

$$\lambda'_\infty(\varphi'_\infty) = \Phi^{\ell+2}(s_0). \tag{4.22}$$

In particular, the Gaussians of $V(\mathbb{R})$ and $V'(\mathbb{R})$ *do not match*, and we will need to find another function for $V(\mathbb{R})$.

One of the main results of [**28**] was the construction of a Schwartz *form* for V,

$$\varphi_{KM} \in S(V(\mathbb{R})) \otimes A^{(1,1)}(D), \tag{4.23}$$

where $A^{(1,1)}(D)$ is the space of smooth $(1,1)$–forms on D, with the following properties:

(i) For all $h \in O(V(\mathbb{R}))$,

$$h^*\varphi_{KM}(h^{-1}x) = \varphi_{KM}(x), \tag{4.24}$$

where h^* indicated the action of h on the space $A^{(1,1)}(D)$ by pullback.

(ii) φ_{KM} has weight $\ell + 2$ for K'_∞, i.e.,

$$\omega(k')\varphi_{KM} = \chi_{\ell+2}(k')\,\varphi_{KM}, \tag{4.25}$$

for the Weil representation action of K' on $S(V(\mathbb{R}))$.

(iii) φ_{KM} is closed:

$$d\varphi_{KM} = 0 \tag{4.26}$$

for exterior differentiation d on D.

Note that it follows from properties (i) and (iii) above that $\varphi_{KM}(x) \in A^{(1,1)}(D)$ is a closed $O(V(\mathbb{R}))_x$ invariant form. For example,

$$\Omega := \varphi_{KM}(0) \tag{4.27}$$

is an $O(V(\mathbb{R}))$ invariant $(1,1)$–form on D, which we will identify in a moment.

In the present situation, φ_{KM} is obtained as follows. Recall from Lemma 3.8 that for $t \in \mathbb{R}_{>0}$, the exponential integral $\beta_1(t)$ has a logarithmic singularity, $-\log(t)$, as $t \to 0$ and decays like e^{-t} as $t \to \infty$. For $x \in V(\mathbb{R})$, $x \neq 0$, and $z \in D$, let

$$\xi(x,z) = \beta_1(2\pi R(x,z))\, e^{-2\pi Q(x)}. \tag{4.28}$$

This function is smooth away from the incidence locus

$$\{[x,z] \in V(\mathbb{R}) \times D \mid \mathrm{pr}_z(x) = 0\}. \tag{4.29}$$

For example, if $x \in V(\mathbb{R})$ is fixed, then $\xi(x)$ is a smooth function on $D - D_x$, where

$$D_x = \{\ z \in D \mid z \perp x\}, \tag{4.30}$$

as in (1.44). Moreover, $\xi(x,z)$ decays exponentially as z goes to infinity away from D_x. Note that D_x is nonempty if and only if $Q(x) > 0$. The crucial fact then is that, for $x \neq 0$,

$$\varphi_{KM}(x) = dd^c\xi(x), \tag{4.31}$$

where, $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. In fact, as in [**27**], we have the stronger assertion:

Proposition 4.10. *As currents on D,*

$$dd^c\xi(x) + e^{-2\pi Q(x)}\,\delta_{D_x} = [\varphi_{KM}(x)].$$

Proof. Omitted $\square$

We can recover the explicit formula for Ω from this result.

Proposition 4.11. *On the tube domain $\mathbb{D}$, let $\rho = \rho(z) = -\frac{1}{2}(w(z), w(\bar{z}))$, be the norm of the section $z \mapsto w(z)$ of $\mathcal{L}_D$, as in (1.10). Then*

$$\begin{aligned}\Omega &= dd^c \log(\rho)\\ &= -\frac{1}{2\pi i}\Big[\; -(y,y)^{-2}(y,dz)\wedge(y,d\bar{z}) + (y,y)^{-1}\frac{1}{2}(dz,d\bar{z})\;\Big].\end{aligned}$$

Proof. We compute

$$\begin{aligned}dd^c\xi(x) &= -\frac{1}{2\pi i}\partial\bar{\partial}\Big\{\beta_1(2\pi R)\Big\}\cdot e^{-2\pi Q(x)}\\ &= \frac{1}{2\pi i}\partial\Big\{e^{-2\pi R}\bar{\partial}\log(R)\Big\}\cdot e^{-2\pi Q(x)}\\ &= \frac{1}{2\pi i}\Big[-2\pi\partial R\wedge\bar{\partial}\log(R) + \partial\bar{\partial}\log(R)\Big]\cdot e^{-2\pi R-2\pi Q(x)}\\ &= \varphi_{KM}(x)\end{aligned} \tag{4.32}$$

For a moment, we write $\alpha = (x, w(z))$ and $\rho = |y|^2 = -(y,y)$, as in (1.10), so that, by (1.16), $R = \rho^{-1}|\alpha|^2$. Then

$$\partial\bar{\partial}\log(R) = -\partial\bar{\partial}\log(\rho), \tag{4.33}$$

and

$$\partial R\wedge\bar{\partial}\log(R) = \rho^{-1}d\alpha\wedge d\bar{\alpha} - \rho^{-2}\,\bar{\alpha}\,d\alpha\wedge\bar{\partial}\rho - \rho^{-2}\,\alpha\,\partial\rho\wedge d\bar{\alpha} + \rho^{-3}|\alpha|^2\partial\rho\wedge\bar{\partial}\rho. \tag{4.34}$$

Notice that this last expression defines a smooth form on D.

Setting $x = 0$, we obtain:

$$\Omega = \varphi_{KM}(0) = dd^c\log(\rho). \tag{4.35}$$

But now, writing

$$\rho = -(y,y) = \frac{1}{4}(z-\bar{z}, z-\bar{z}), \tag{4.36}$$

we have

$$\begin{aligned}\Omega &= -\frac{1}{2\pi i}\partial\bar{\partial}\log(\rho)\\ &= -\frac{1}{2\pi i}\Big[\,-\rho^{-2}\,\partial\rho\wedge\bar{\partial}\rho + \rho^{-1}\partial\bar{\partial}\rho\,\Big]\\ &= -\frac{1}{2\pi i}\Big[\,-(y,y)^{-2}(y,dz)\wedge(y,d\bar{z}) + (y,y)^{-1}\frac{1}{2}(dz,d\bar{z})\,\Big]\end{aligned} \tag{4.37}$$

as claimed. $\square$

Corollary 4.12. *The form*

$$\Omega = \varphi_{KM}(0) = dd^c\log||s||^2$$

on X_K is the first Chern form for the holomorphic line bundle $\mathcal{L}^\vee$ dual to $\mathcal{L}$. In particular, $-\Omega$ is an invariant Kähler form on $\mathbb{D}$ and hence determines a Kähler form on X_K.

Examples 4.13: In the case $n=1$ we have $\mathbb{D}\simeq\mathbb{C}\setminus\mathbb{R} = \mathfrak{H}^+\cup\mathfrak{H}^-$ and $\Omega = -\frac{1}{2\pi}\,y^{-2}\,dx\wedge dy$. In the case $n=2$, we have $\mathbb{D}\simeq\mathfrak{H}\times\mathfrak{H}$ and

$$\Omega = -\frac{1}{4\pi}\bigg(y_1^{-2}\,dx_1\wedge dy_1 + y_2^{-2}\,dx_2\wedge dy_2\bigg), \tag{4.38}$$

(compare [**25**], p.104, [**48**], p.102.)

We now return to the theta integral and its geometric meaning. Write

$$\varphi_{KM}(x)\wedge\Omega^{n-1} = \tilde{\varphi}_{KM}(x)\,\Omega^n, \tag{4.39}$$

for a function $\tilde{\varphi}_{KM}\in S(V(\mathbb{R}))\otimes A^{(0,0)}(D)$. Note that, since Ω is $O(V(\mathbb{R}))$–invariant,

$$\tilde{\varphi}_{KM}(hx,hz) = \tilde{\varphi}_{KM}(x,z) \tag{4.40}$$

for all $h\in O(V(\mathbb{R}))$. Moreover, $\tilde{\varphi}_{KM}$ also has weight $\ell+2$ for the Weil representation action of K'_∞.

Lemma 4.14. *For all $z\in D$,*

$$\lambda(\varphi_{KM}(\cdot,z)) = \Phi^{\ell+2}(s_0)\,\Omega,$$

and

$$\lambda(\tilde{\varphi}_{KM}(\cdot,z)) = \Phi^{\ell+2}(s_0).$$

Corollary 4.15. *For all $z \in D$, the functions $\tilde{\varphi}_{KM}(\cdot, z) \in S(V(\mathbb{R}))$ and $\varphi'_0 \in S(V'(\mathbb{R}))$ match.*

We now return to the global situation, so that, for the matching functions $\varphi \in S(V(\mathbb{A}_f))$ and $\varphi' \in S(V'(\mathbb{A}_f))$ above,

$$\lambda(\tilde{\varphi}_{KM} \otimes \varphi) = \lambda'(\varphi'_0 \otimes \varphi) = \Phi(s_0), \tag{4.41}$$

for a standard section $\Phi(s) \in I(s, \chi)$. Hence, we have an equality of Eisenstein series:

$$E(g', s, \lambda(\tilde{\varphi}_{KM} \otimes \varphi)) = E(g', s, \lambda'(\varphi'_0 \otimes \varphi')) = E(g', s, \Phi). \tag{4.42}$$

Applying the Siegel–Weil formula, we have

Corollary 4.16.

$$I(g', \tilde{\varphi}_{KM} \otimes \varphi; V) = I(g', \varphi'_0 \otimes \varphi'; V') = E(g', s_0, \Phi).$$

Here, in forming the theta integral of V, we use the theta function

$$\theta(g', h; \tilde{\varphi}_{KM} \otimes \varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g')\tilde{\varphi}_{KM}(h_\infty^{-1}x, z_0)\, \varphi(h^{-1}x), \tag{4.43}$$

on $G'_{\mathbb{A}} \times O(V(\mathbb{A}))$, where $z_0 \in D$ is a fixed point. In particular, as a function on $O(V(\mathbb{A}))$ this function is right invariant under the stabilizer of z_0 in $O(V(\mathbb{R}))$.

Next we would like to explain the geometric content of the first of these expressions. The key point is to determine the relation between the integral of the theta function (4.5) over $O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})$ and the integral of the differential form $\theta(g', \varphi_{KM} \otimes \varphi) \wedge \Omega^{n-1}$ over X_K.

Proposition 4.17. *Assume that the compact open subgroup $K \subset H(\mathbb{A}_f)$ satisfies:*

$$Z_K := K \cap Z(\mathbb{A}) \simeq \hat{\mathbb{Z}}^\times, \tag{Z}$$

under the isomorphism $Z(\mathbb{A}) \simeq \mathbb{A}^\times$. Then

$$I(g', \tilde{\varphi}_{KM} \otimes \varphi; V) = (-1)^n \frac{1}{4} \operatorname{vol}(K) \int_{X_K} \theta(g', \varphi_{KM} \otimes \varphi) \wedge \Omega^{n-1}.$$

Proof. By (ii) of Theorem 4.1,

$$I(g'; \tilde{\varphi}_{KM} \otimes \varphi; V) = \frac{1}{2} \int_{SO(V)(\mathbb{Q})\backslash SO(V)(\mathbb{A})} \theta(g', h; \tilde{\varphi}_{KM} \otimes \varphi)\, dh \tag{4.44}$$

where dh is Tamagawa measure on $SO(V)(\mathbb{A})$. A factorization $dh = dh_\infty \times dh_f$ will be determined by the choice of dh'_∞ made below.

We fix the measure dz on Z_K with total mass 1, and we obtain a measure dk on K by requiring that dk/dz be the measure on the compact open subgroup $K/Z_K \subset SO(V)(\mathbb{A}_f)$ induced by dh_f. This provides a normalization of the Haar measure on $H(\mathbb{A}_f)$ and hence a measure dh' on $Z(\mathbb{R})\backslash H(\mathbb{A})$. Continuing the calculation above, and noting that $Z(\mathbb{A}) = Z(\mathbb{Q})Z(\mathbb{R})Z_K$, we have

$$
\begin{aligned}
I(g';\tilde{\varphi}_{KM}\otimes\varphi;V) &= \frac{1}{2}\int_{H(\mathbb{Q})Z(\mathbb{A})\backslash H(\mathbb{A})} \theta(g',h';\tilde{\varphi}_{KM}\otimes\varphi)\,dh \\
&= \frac{1}{2}\int_{H(\mathbb{Q})Z(\mathbb{R})\backslash H(\mathbb{A})} \theta(g',h;\tilde{\varphi}_{KM}\otimes\varphi)\,dh' \qquad (4.45)\\
&= \frac{1}{2}\sum_j \int_{H(\mathbb{Q})Z(\mathbb{R})\backslash H(\mathbb{Q})H(\mathbb{R})^+h_jKh_j^{-1}} \theta(g',hh_j;\tilde{\varphi}_{KM}\otimes\varphi)\,dh' \\
&= \frac{1}{4}\,\mathrm{vol}(K)\sum_j \int_{\Gamma_j Z(\mathbb{R})\backslash H(\mathbb{R})^+} \theta(g',h_\infty h_j;\tilde{\varphi}_{KM}\otimes\varphi)\,dh_\infty.
\end{aligned}
$$

Here we have used the fact that φ is K–invariant. The extra factor of $\frac{1}{2}$ in the last step arises from the fact that $Z(\mathbb{Q})\cap K \simeq \{\pm 1\}$. Finally, we normalize the measure dh_∞ on $Z(\mathbb{R})\backslash H(\mathbb{R}) = SO(V)(\mathbb{R})$ by requiring that for $\phi \in C_c(D)$,

$$
\int_{Z(\mathbb{R})\backslash H(\mathbb{R})} \phi(h_\infty z_0)\,dh_\infty = (-1)^n \int_D \phi\cdot\Omega^n, \qquad (4.46)
$$

where $z_0 \in D$ is the base point used in (4.43). Then, using (4.39), we have

$$
\begin{aligned}
I(g';\tilde{\varphi}_{KM}\otimes\varphi;V) &= (-1)^n\,\frac{1}{4}\,\mathrm{vol}(K)\sum_j \int_{\Gamma_j\backslash D^+} \theta(g',h_j;\tilde{\varphi}_{KM}\otimes\varphi)\,\Omega^n \\
&= (-1)^n\,\frac{1}{4}\,\mathrm{vol}(K)\sum_j \int_{\Gamma_j\backslash D^+} \theta(g',h_j;\varphi_{KM}\otimes\varphi)\wedge\Omega^{n-1} \qquad (4.47)\\
&= (-1)^n\,\frac{1}{4}\,\mathrm{vol}(K)\int_{X_K} \theta(g',\varphi_{KM}\otimes\varphi)\wedge\Omega^{n-1}.
\end{aligned}
$$

□

Remark 4.18. The same unfolding argument yields

$$
\begin{aligned}
1 &= \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} dh \\
&= (-1)^n \, \frac{1}{4} \operatorname{vol}(K) \sum_j \int_{\Gamma_j\backslash D^+} \Omega^n \qquad (4.48)\\
&= (-1)^n \, \frac{1}{4} \operatorname{vol}(K) \operatorname{vol}(X_K, \Omega^n),
\end{aligned}
$$

and thus the useful formula

$$
\operatorname{vol}(K) = (-1)^n \, \frac{4}{\operatorname{vol}(X_K, \Omega^n)}. \tag{4.49}
$$

The sign in (4.44) has been introduced to make $\operatorname{vol}(K)$ positive.

Viewed as a differential form on $D \times H(\mathbb{A}_f)/K$, the theta form is given by

$$
\theta(g', h; \varphi_{KM} \otimes \varphi) = \sum_{m\in\mathbb{Q}} \sum_{\substack{x\in V(\mathbb{Q})\\ Q(x)=m}} \omega(g')\varphi_{KM}(x)\, \varphi(h^{-1}x), \tag{4.50}
$$

on the set $D \times hK$. Here note that $\omega(g')\varphi_{KM}(x)$ is a $(1,1)$–form on D and that (4.48) is, in fact, the Fourier expansion of the theta form as a function on $G'_{\mathbb{A}}$. Let $\theta_m(g'; \varphi_{KM} \otimes \varphi)$ be the m–th Fourier coefficient, i.e., the partial sum over $x \in V(\mathbb{Q})$ with $Q(x) = m$, and note that, since this form is itself $H(\mathbb{Q})$–invariant, it defines a $(1,1)$–form on X_K.

Recall from section 1 that, for $m \in \mathbb{Q}_{>0}$ and for any $\varphi \in S(V(\mathbb{A}_f))^K$, there is a divisor $Z(m, \varphi) = Z(m, \varphi; K)$ on X_K, cf. (1.48) and (1.52). Also recall that the line bundle $\mathcal{L}_D^\vee$ on D descends to a line bundle $\mathcal{L}^\vee$ on X_K.

Definition 4.19. *The $\mathcal{L}^\vee$–degree of a cycle Z of codimension r in X_K is*

$$
\deg_{\mathcal{L}^\vee}(Z) := \int_Z \Omega^{n-r},
$$

where Ω is the first Chern form of $\mathcal{L}^\vee$, as in Proposition 4.11 and Corollary 4.12.

Note that if X_K were compact and smooth, this would be simply $c_1(\mathcal{L}^\vee)^{n-r}[Z]$, for the first Chern class $c_1(\mathcal{L}^\vee)$.

Also observe that, for Z an irreducible subvariety,

$$
(-1)^{n-r} \deg_{\mathcal{L}^\vee}(Z) > 0 \tag{4.51}
$$

since $-\Omega$ is a Kähler form on X_K.

The following result is a consequence of the Thom form property of φ_{KM} (Theorem 4.1 of [**29**], and Theorem 2.1 of [**30**]). As before, take $\tau = u + iv \in \mathfrak{H}$, and write $q^m = e(m\tau)$.

Theorem 4.20. *For $m > 0$, and for $g'_\tau \in G'_{\mathbb{R}}$,*

$$\int_{X_K} \theta_m(g'_\tau, \varphi_{KM} \otimes \varphi) \wedge \Omega^{n-1} = v^{(\ell+2)/2} \deg_{\mathcal{L}^\vee}(Z(m, \varphi; K)) \cdot q^m,$$

where $\ell = \frac{n}{2} - 1$.

Remark 4.21: A key point here is that the cycle $Z(m, \varphi)$ always has finite volume and the invariant form Ω^{n-1} is, in particular, bounded. Thus, Theorem 2.1 of [**30**] can be applied, even when X_K is non-compact. Alternatively, it is easy to obtain Theorem 4.20 by a direct calculation, using the integral formulas for the affine symmetric spaces, as used in the estimates in section 3 above.

We now turn to the theta integral for the space V'.

We fix a compact open subgroup $K' \subset O(V')(\mathbb{A}_f)$ such that $\varphi' \in S(V'(\mathbb{A}_f))^{K'}$, and write

$$O(V')(\mathbb{A}) = \coprod_j O(V')(\mathbb{Q})\, O(V')(\mathbb{R})\, h_j K'. \tag{4.52}$$

Note that, since V' is positive definite, the group

$$\Gamma_j = O(V')(\mathbb{Q}) \cap \Big(O(V')(\mathbb{R}) h_j K' h_j^{-1} \Big) \tag{4.53}$$

is finite; we set $e_j = |\Gamma_j|$. Again, we have a standard calculation, where we note that the Gaussian φ'_0 is invariant under $O(V')(\mathbb{R})$:

$$\begin{aligned} I(g', \varphi'_0 \otimes \varphi'; V') &= \int_{O(V')(\mathbb{Q})\backslash O(V')(\mathbb{A})} \theta(g', h; \varphi'_0 \otimes \varphi')\, dh \\ &= \sum_j \int_{\Gamma_j \backslash O(V')(\mathbb{R}) h_j K'} \theta(g', h; \varphi'_0 \otimes \varphi')\, dh \\ &= \operatorname{vol}(O(V')(\mathbb{R})K') \sum_j e_j^{-1} \theta(g', h_j; \varphi'_0 \otimes \varphi')\, dh. \end{aligned} \tag{4.54}$$

If we take $g' = g'_\tau$, then

$$\omega(g'_\tau)\varphi'_0(x) = v^{(\ell+2)/2}\, e(Q'(x)\tau). \tag{4.55}$$

Note that

$$\begin{aligned} 1 &= \operatorname{vol}(O(V')(\mathbb{Q})\backslash O(V')(\mathbb{A}), dh) \\ &= \operatorname{vol}(O(V')(\mathbb{R})K') \sum_j e_j^{-1}, \end{aligned} \tag{4.56}$$

we have

$$\mathrm{vol}(O(V')(\mathbb{R})K') = \Big(\sum_j e_j^{-1}\Big)^{-1} := \mu(K'), \tag{4.57}$$

the mass of the K'–genus. Thus we obtain the classical expression

$$I(g'_\tau, \varphi'_0 \otimes \varphi'; V') = v^{(\ell+2)/2}\,\mu(K')\,\sum_j e_j^{-1}\theta(g', h_j; \varphi'_0 \otimes \varphi'). \tag{4.58}$$

Proposition 4.22. *For $m \in \mathbb{Q}$, let $q^m = e(m\tau)$, and recall that $\ell = \frac{n}{2} - 1$. Then the Fourier expansion of $I(g'_\tau, \varphi'_0 \otimes \varphi'; V')$ is given by*

$$I(g'_\tau, \varphi'_0 \otimes \varphi'; V') = v^{(\ell+2)/2} \sum_{m \geq 0} r_{\varphi'}(m)\, q^m,$$

where

$$r_{\varphi'}(m) = \mu(K')\,\sum_j e_j^{-1}\Big(\sum_{\substack{x \in V'(\mathbb{Q})\\ Q'(x)=m}} \varphi'(h_j^{-1}x)\Big).$$

In particular, the constant term is

$$v^{(\ell+2)/2}\,\varphi'(0) = v^{(\ell+2)/2}\,\varphi(0),$$

via matching.

The matching identity now amounts to:

Theorem 4.23. *For $\varphi \in S(V(\mathbb{A}_f))$ and $\varphi' \in S(V'(\mathbb{A}_f))$ matching, and for the corresponding standard section $\Phi(s)$, with $\Phi(s_0) = \Phi_\infty^{\ell+2}(s_0) \otimes \lambda(\varphi)$,*

$$v^{-(\ell+2)/2}\,E(g'_\tau, s_0, \Phi) = \varphi(0) + \frac{1}{\mathrm{vol}(X_K, \Omega^n)} \sum_{m>0} \deg_{\mathcal{L}^\vee}(Z(m, \varphi; K)) \cdot q^m$$

$$= \varphi'(0) + \sum_{m>0} r_{\varphi'}(m)\, q^m.$$

Comparing coefficients, we have

Corollary 4.24. *(i) For $m > 0$,*

$$\deg_{\mathcal{L}^\vee}(Z(m,\varphi;K)) = \mathrm{vol}(X_K,\Omega^n)\, r_{\varphi'}(m).$$

(ii) If $m = 0$,

$$\int_{X_K} \theta_0(g'_\tau, \varphi_{KM}\otimes\varphi)\wedge\Omega^{n-1} = v^{(\ell+2)/2}\,\varphi(0)\,\mathrm{vol}(X_K,\Omega^n).$$

(iii) If $m < 0$,

$$\int_{X_K} \theta_m(g'_\tau, \varphi_{KM}\otimes\varphi) = 0.$$

Several special cases of these identities occur in the literature, cf. for example, [**48**], [**23**], [**24**]. Note that, whereas the Fourier coefficients of the theta integral for V involve degrees $\deg_{\mathcal{L}^\vee}(Z(m,\varphi))$, the Fourier coefficients of the theta integral for the positive definite space V' are weighted representation numbers and the Fourier coefficients of the special value of the Eisenstein series, at least for factorizable data, have a product formula, i.e., are 'multiplicative functions' in classical terminology.

§**5. Examples.**

In this section, we illustrate our results about integrals of Borcherds forms and about generating functions for degrees with some more or less explicit examples. The basic idea is the following. On the one hand, by using Hasse–Minkowski, one can construct even integral quadratic lattices M of signature $(n,2)$ with prescribed local behavior. Associated to such a lattice are global geometric objects, the quasi–projective variety $X_M = \Gamma_M\backslash D^+$, the divisors $Z(m,\varphi)$, $\varphi \in M^\sharp/M$, etc. On the other hand, associated to cosets $\varphi \in M^\sharp/M$ are the Eisenstein series $E(\tau,s;\varphi)$ of weight $\frac{n}{2}+1$. These series and their Fourier expansions depend directly on the local data defining M. The local and global objects are then related by the degree identity of Theorem 4.23,

$$\mathrm{vol}(X_M)\cdot E(\tau,\frac{n}{2};\varphi) = \mathrm{vol}(X_M)\cdot\varphi(0) + \sum_{m>0}\deg_{\mathcal{L}^\vee}(Z(m,\varphi))\cdot q^m,$$

giving the first term of the Laurent expansion at $s=\frac{n}{2}$, and by Theorem 2.12, expressing the (log–norm) integrals of all Borcherds forms $\Psi(F)^2$ for $\mathbb{C}[L^\sharp/L]$–valued F's of weight $1-\frac{n}{2}$ in terms of the $\kappa_\varphi(m)$'s arising from the second term of $E(\tau,s;\varphi)$'s at $s=\frac{n}{2}$.

A more systematic discussion of examples will be given in a sequel with Tonghai Yang, [**33**].

First, we recall an example due to Gritsenko and Nikulin [**19**]. Let

$$Q = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 2 & & \\ & & & & 1 \\ & & & 1 & \end{pmatrix}, \tag{5.1}$$

and let $M = \mathbb{Z}^5$ with quadratic form, of signature $(3,2)$, defined by

$$Q(x) = \frac{1}{2}\,{}^t x Q x. \tag{5.2}$$

The dual lattice of M is $M^\sharp = Q^{-1}M$ and $|M^\sharp/M| = 2$. Note that, if $x \in M^\sharp$ with $Q(x) = m$, then $m \in \frac{1}{4}\mathbb{Z}$ and $4m \equiv 0, 1 \mod (4)$, depending on the M coset $x + M$.

As explained in [**19**], pp186–188, and [**47**], there are compatible isomorphisms

$$\mathrm{GSpin}(M_{\mathbb{R}}) \simeq \mathrm{Sp}_4(\mathbb{R}), \qquad D^+ \simeq \mathfrak{H}_2, \tag{5.3}$$

such that

$$\Gamma = SO^+(M) \simeq \mathrm{Sp}_4(\mathbb{Z})/\{\pm 1_4\}. \tag{5.4}$$

Let

$$X = \Gamma\backslash D^+ \simeq \mathrm{Sp}_4(\mathbb{Z})\backslash\mathfrak{H}_2. \tag{5.5}$$

Recall that, in the tube domain model, our invariant form $\Omega = \varphi_{KM}(0)$ is given by

$$\Omega = -\frac{1}{4\pi i}\Big[\, -2(y,y)^{-2}(y,dz)\wedge(y,d\bar{z}) + (y,y)^{-1}\,(dz,d\bar{z})\,\Big]. \tag{5.6}$$

In the case $n = 3$, we write

$$z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathfrak{H}_2 \tag{5.7}$$

and take the inner product of a pair of 2×2 symmetric matrices to be $(a,b) = -\mathrm{tr}(ab^\iota)$, for ι the main involution on $M_2(\mathbb{Q})$. By an easy computation, noting that $(y,y) = -2\det(y)$, we find:

$$\Omega^3 = -\frac{3}{16\pi^3}\,\det(y)^{-3}\left(\frac{i}{2}\right)^3 dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\wedge dz_3\wedge d\bar{z}_3 \tag{5.8}$$

and so [**47**], p.331

$$\mathrm{vol}(X,\Omega^3) = \zeta(-1)\,\zeta(-3) = -\frac{1}{12}\,\zeta(-3) = -\frac{1}{1440}. \tag{5.9}$$

Let $V = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be the associated rational quadratic space, and let $\varphi_0,\ \varphi_1 \in S(V(\mathbb{A}_f))$ be the characteristic functions of the sets $\hat{M} = M \otimes \hat{\mathbb{Z}}$ and $y_1 + \hat{M}$ respectively, where y_1 is an element in $M^\sharp \setminus M$. As explained in [**19**] and [**47**], the divisors $Z(m, \varphi_\mu)$, for $\mu = 0,\, 1$, are then given by

$$Z(m, \varphi_\mu) = \begin{cases} \mathcal{G}_{4m} & \text{if } 4m \equiv \mu \mod (4), \\ \emptyset & \text{otherwise,} \end{cases} \tag{5.10}$$

where

$$\mathcal{G}_\Delta = \sum_{\substack{n \geq 1 \\ n^2 \mid \Delta}} \nu_{\Delta/n^2}\, \mathcal{H}_{\Delta/n^2}, \tag{5.11}$$

for $\mathcal{H}_\Delta$ the Humbert surface of discriminant Δ and with

$$\nu_\Delta = \begin{cases} \frac{1}{2} & \text{if } \Delta = 1 \text{ or } 4, \\ 1 & \text{otherwise.} \end{cases} \tag{5.12}$$

We can define a vector valued Eisenstein series

$$\mathbf{E}(\tau, s; M) = \begin{pmatrix} E(\tau, s; \varphi_0) \\ E(\tau, s; \varphi_1) \end{pmatrix} \tag{5.13}$$

of weight 5/2. The Fourier expansion of this series can be computed [**33**], and from this it is easy to derive the following information. Write

$$E(\tau, s; \varphi_\mu) = \sum_m A_\mu(s, m, v)\, q^m \tag{5.14}$$

as in (2.21), where the Fourier coefficients have Laurent expansions

$$A_\mu(s, m, v) = a_\mu(m) + b_\mu(m, v)(s - s_0) + O((s - s_0)^2), \tag{5.15}$$

as in (2.22).

Proposition 5.1. *The value of $\mathbf{E}(\tau, \frac{3}{2}; L_0)$ at the point $s_0 = \frac{3}{2}$ is given by the following expression.*

$$E(\tau, \frac{3}{2}; \varphi_0) = 1 + \zeta(-3)^{-1} \sum_{m=1}^{\infty} H(2, 4m)\, q^m$$

and

$$E(\tau, \frac{3}{2}; \varphi_1) = \qquad \zeta(-3)^{-1} \sum_{m-\frac{1}{4}=0}^{\infty} H(2, 4m)\, q^m$$

where $H(2,N)$ are as in Cohen [**11**].

In particular, for the value, observe that

$$E(4\tau, \frac{3}{2}; \varphi_0) + E(4\tau, \frac{3}{2}; \varphi_1) = \zeta(-3)^{-1}\mathcal{H}_2(\tau), \tag{5.16}$$

is Cohen's Eisenstein series of weight $\frac{5}{2}$. Also, for convenient reference, we recall some values from [**11**]:

(5.17)

N :	0	1	4	5	8	9	12	13	16	17	20	...
$-120\, H(2,N)$:	-1	10	70	48	120	250	240	240	550	480	528	...

Remark 5.2. Recall that the positive coefficients in Cohen's Eisenstein series $\mathcal{H}_r(\tau)$ of weight $r + \frac{1}{2}$ are given by

$$H(r, 4m) = L(1-r, \chi_d) \sum_{c|n} \mu(c)\, \chi_d(c)\, c^{r-1} \sigma_{2r-1}(n/c), \tag{5.18}$$

where $4m = (-1)^r n^2 d$ for a field discriminant $d \equiv 0,\ 1 \mod (4)$. The sum on c is a multiplicative function and it is easy to check that, in fact,

$$H(r, 4m) = L(1-r, \chi_d) \prod_p b_p(n, 1-r). \tag{5.19}$$

where $b_p(n,s)$ is given by

$$b_p(n, s) = \frac{1 - \chi_d(p)\, X + \chi_d(p)\, p^k X^{2k+1} - p^{k+1} X^{2k+2}}{1 - pX^2}, \tag{5.20}$$

with $X = p^{-s}$ and $k = \mathrm{ord}_p(n)$.

By Theorem 4.23, we have

$$E(\tau, \frac{3}{2}; \varphi_\mu) = \varphi_\mu(0) + \mathrm{vol}(X)^{-1} \sum_{m>0} \deg(Z(m, \varphi_\mu))\, q^m, \tag{5.21}$$

so we obtain, for $4m \equiv \mu \mod (4)$,

$$\deg(Z(m, \varphi_\mu)) = \deg(\mathcal{G}_{4m}) = -\frac{1}{12}\, H(2, 4m) \tag{5.22}$$

Thus, we recover the relation (1) of van der Geer, [**47**], p.346, as well as his Theorem 8.1 on the generating function for the volumes of the Humbert surfaces.

A nice example of a Borcherds form $\Psi(f)$ is discussed in [**19**].

Let $\phi_{12,1}(\tau, w)$, $\tau \in \mathfrak{H}_1$, $w \in \mathbb{C}$ be the holomorphic Jacobi form of weight 12 and index 1 of Eichler and Zagier [**12**], pp.38–39, so that

$$\phi_{12,1}(\tau, w) = \sum_{n,r} C_{12}(4n - r^2)\, q^n\, \zeta^r, \tag{5.23}$$

for $q = e(\tau)$ and $\zeta = e(w)$, where $c_{12}(n)$ is given by the table on p.141 of [**12**]:

(5.24)

n	0	3	4	7	8	11	12	15	16	19	20	...
$C_{12}(n)$	0	1	10	-88	-132	1275	736	-8040	-2880	24035	13080	...

(We write $C_{12}(n)$ in place of $c_{12}(n)$ to avoid confusion with the coefficients $c_\mu(m)$ which will occur in a moment.) Write

$$\phi_{12,1}(\tau, w) = \sum_{\mu = 0,1} h_\mu(\tau)\, \theta_{1,\mu}(\tau, w) \tag{5.25}$$

where

$$h_\mu(\tau) = \sum_{\substack{m \\ m \equiv -\mu \mod (4)}} C_{12}(m)\, q^{\frac{m}{4}} \tag{5.26}$$

has weight $\frac{23}{2}$ for $\Gamma_0(4)$ and $\theta_{1,\mu}(\tau, w)$ is the standard Jacobi theta series. Then, dividing by Δ to shift the weight, we have

$$\frac{\phi_{12,1}(\tau, w)}{\Delta(\tau)} = \sum_{\mu = 0,1} f_\mu(\tau)\, \theta_{1,\mu}(\tau, w) \tag{5.27}$$

where

$$f_\mu(\tau) = \sum_m c_\mu(m)\, q^m, \tag{5.28}$$

has weight $-\frac{1}{2}$ and

$$\begin{aligned} f_0(\tau) &= 10 + 108\, q + 808\, q^2 + ... \\ f_1(\tau) &= q^{-\frac{1}{4}} - 64\, q^{\frac{3}{4}} - 513\, q^{\frac{7}{4}} + ... \end{aligned} \tag{5.29}$$

Associated to the vector valued form (see [**2**], Example 2.3, p.500 and [**12**], Theorem 5.1, p.59)

$$\mathbf{f}_5(\tau) = \big(\ f_0(\tau), f_1(\tau)\ \big) = f_0(\tau)\, \varphi_0 + f_1(\tau)\, \varphi_1, \tag{5.30}$$

valued in $\mathbb{C}[M^\sharp/M]$, is a Borcherds form $\Psi(\mathbf{f}_5)$, identified explicitly by Gritsenko-Nikulin:

$$\Psi(\mathbf{f}_5) = 2^{-6}\,\Delta_5(z), \tag{5.31}$$

where $\Delta_5(z)$ is the Siegel cusp form of weight 5 (and character) for $\mathrm{Sp}_4(\mathbb{Z})$. Then $\Psi(\mathbf{f}_5)^2$ has weight 10 (and trivial character) and

$$\mathrm{div}(\Psi(\mathbf{f}_5)^2) = Z(\frac{1}{4}, \varphi_1). \tag{5.32}$$

Similarly, for any positive integer t, we can consider the form $j(\tau)^t \cdot f(\tau)$. For example, for $t = 1$, we get

$$\begin{aligned} j(\tau)\, f_0(\tau) &= 10\, q^{-1} + 7548 + O(q) \\ j(\tau)\, f_1(\tau) &= q^{-\frac{5}{4}} + 680\, q^{-\frac{1}{4}} + O(q^{\frac{1}{4}}) \end{aligned} \tag{5.33}$$

so that the associated $\Psi(\mathbf{f}_{3774})^2$ has weight 7548 and divisor

$$10\, Z(1, \varphi_0) + Z(\frac{5}{4}, \varphi_1) + 680\, Z(\frac{1}{4}, \varphi_1). \tag{5.34}$$

For $t = 2$, we get

$$\begin{aligned} j(\tau)^2\, f_0(\tau) &= 10\, q^{-2} + 14988\, q^{-1} + 9634552 + O(q) \\ j(\tau)^2\, f_1(\tau) &= q^{-\frac{9}{4}} + 1424\, q^{-\frac{5}{4}} + 851559\, q^{-\frac{1}{4}} + O(q^{\frac{1}{4}}) \end{aligned} \tag{5.35}$$

so that the associated $\Psi(\mathbf{f}_{4827376})^2$ has weight 9634552 and divisor

$$10\, Z(2, \varphi_0) + 14988\, Z(1, \varphi_0) + Z(\frac{9}{4}, \varphi_1) + 1424\; Z(\frac{5}{4}, \varphi_1) + 851559\, Z(\frac{1}{4}, \varphi_1). \tag{5.36}$$

It is amusing to check the weight/degree relation, (2.30),

$$\sum_{\mu} \sum_{m>0} c_\mu(-m)\, \frac{1}{12} H(2, 4m) = -\mathrm{vol}(X)\, c_0(0), \tag{5.37}$$

i.e.,

$$-\sum_{\mu} \sum_{m>0} c_\mu(-m)\, 120\, H(2, 4m) = c_0(0) \tag{5.38}$$

in these cases.

To compute the quantities $\kappa(\Psi(f))$ for these Borcherds forms, we need to determine the quantities $\kappa_\mu(m)$ derived from the second term in the Laurent expansion of $\mathbf{E}(\tau, s; M)$ at the point $s = \frac{3}{2}$.

Theorem 5.3. *(i) For $m > 0$, write $4m = n^2 d$ for d the discriminant of the real quadratic field $\mathbb{Q}(\sqrt{m})$, and let χ_d be the associated quadratic character*[5]*. Then, for $4m \equiv \mu \mod (4)$,*

$$b_\mu(m,v) = \zeta(-3)^{-1}\, H(2,4m)\left[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - \frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} - C\right.$$
$$\left. + \sum_{p|n}\left(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\right) + \frac{1}{2}\,J(\frac{3}{2}, 4\pi m v)\right].$$

where

$$2C = \log(4\pi) + \gamma,$$
$$J(\frac{3}{2}, t) = \int_0^\infty e^{-tr}\,\frac{(1+r)^{\frac{3}{2}} - 1}{r}\,dr,$$

and for $k = \operatorname{ord}_p(n)$,

$$-\frac{1}{\log(p)}\cdot\frac{b_p'(n,-1)}{b_p(n,-1)} = \frac{2p^3}{1-p^3} + \frac{-\chi_d(p)\,p + \chi_d(p)\,(2k+1)\,p^{3k+1} - (2k+2)\,p^{3k+3}}{1 - \chi_d(p)\,p + \chi_d(p)\,p^{3k+1} - p^{3k+3}}.$$

(ii) For $m < 0$,

$$b_\mu(m,v) = -\frac{\pi^2}{3}\,\frac{L(2,\chi_m)}{\zeta(4)}\,(\pi v)^{-\frac{3}{2}}\int_1^\infty e^{-4\pi|m|vr}\,r^{-\frac{3}{2}}\,dr.$$

(iii) For the constant term is given by

$$b_0(0,v) = \frac{1}{2}\,\log(v) - \frac{\pi}{6}\,\frac{\zeta(3)}{\zeta(4)}\,v^{-\frac{3}{2}}.$$

(iv) If $4m \not\equiv \mu \mod (4)$, then $b_\mu(m,v) = 0$.

For $m < 0$, the L–series $L(s,\chi_m)$ is a modified Dirichlet series analogous to that occurring in the definition of $H(r,4m)$. In any case, it is clear that, $\lim_{v\to\infty} b_\mu(m,v) = 0$ for $m < 0$. Similarly, for $m > 0$, $\lim_{v\to\infty} J(\frac{3}{2}, 4\pi m v) = 0$.

Corollary 5.4. *For $m > 0$ with $4m = n^2 d$ and with $4m \equiv \mu \mod (4)$,*

$$\kappa_\mu(m) = \zeta(-3)^{-1}\, H(2,4m)\left[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - \frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} - C\right.$$
$$\left. + \sum_{p|n}\left(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\right)\right].$$

[5]When $4m = n^2$, we take $\mathbb{Q}(\sqrt{m}) = \mathbb{Q}\oplus\mathbb{Q}$, $\chi_d = 1$ and $L(s,\chi_1) = \zeta(s)$

If $4m \not\equiv \mu \mod (4)$, *then* $\kappa_\mu(m) = 0$.

Now, in calculating $\kappa(\Psi(\mathbf{f}))$ via Theorem 2.12, we can use the degree relation:

$$\begin{aligned}
\kappa(\Psi(\mathbf{f})) &= \sum_{\mu}\sum_{m>0} c_\mu(-m)\,\kappa_\mu(m) + c_0(0)\,\frac{1}{2}\,C_0 \\
&= \sum_{\mu}\sum_{m>0} c_\mu(-m)\,120\,H(2,4m)\Big[-\frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} \\
&\qquad + \sum_{p|n}\Big(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\Big)\Big] \\
&\qquad - c_0(0)\left[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - C - \frac{1}{2}C_0\right].
\end{aligned} \tag{5.39}$$

In the first example above, where $m = \frac{1}{4}$, $d = 1$, $\chi_d = 1$ and $L(s,\chi_d) = \zeta(s)$, we obtain

$$\begin{aligned}
\kappa(\Psi(\mathbf{f}_5)) &= \zeta(-3)^{-1}\,H(2,1)\left[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} - C\right] + 10\cdot\frac{1}{2}\,C_0. \\
&= 10\left[-\frac{4}{3} - 2\,\frac{\zeta'(-3)}{\zeta(-3)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2}\,\log(2) + \log(\pi)\right].
\end{aligned} \tag{5.40}$$

Noting that $|y|^2 = 2\det(y)$ here, we have

$$||\Psi(\mathbf{f}_5)(z)||^2 = 2^{-12}\,|\Delta_5(z)|^2\,2^5\,\det(y)^5, \tag{5.41}$$

so that

(5.42)

$$-\mathrm{vol}(X)^{-1}\int_X \log\big(|\Delta_5(z)|^2\,\det(y)^5\big)\cdot\Omega^3 = 10\left[-\frac{4}{3}-2\,\frac{\zeta'(-3)}{\zeta(-3)}+\frac{\zeta'(-1)}{\zeta(-1)}+\frac{3}{2}\,\log(2)+\log(\pi)\right]-7\log(2).$$

In the second example, there are terms for $m = \frac{5}{4}$, 1 and $\frac{1}{4}$, and we obtain

$$\begin{aligned}
\kappa(\Psi(\mathbf{f}_{3774})) &= 700\left[\frac{\zeta'(-1)}{\zeta(-1)} + \frac{b_2'(2,-1)}{b_2(2,-1)} + \log(2)\right] \\
&\quad + 48\left[\frac{L'(-1,\chi_5)}{L(-1,\chi_5)} + \frac{1}{2}\,\log(5)\right] \\
&\quad + 6800\,\frac{\zeta'(-1)}{\zeta(-1)} \\
&\quad + 7548\left[-\frac{4}{3} - 2\,\frac{\zeta'(-3)}{\zeta(-3)} + \frac{3}{2}\,\log(2) + \log(\pi)\right].
\end{aligned} \tag{5.43}$$

where

$$\frac{b_2'(2,-1)}{b_2(2,-1)} = -\frac{9}{11}\,\log(2). \tag{5.44}$$

And so on.

In the next section, we explain why the values $\kappa_\mu(m)$ which occur here should be connected with the 'arithmetic volumes' of (suitable integral extensions of) the cycles $Z(m,\varphi_\mu)$.

§6. Speculations.

The integrals considered in this paper play a role in the arithmetic geometry of cycles on the $\mathrm{GSpin}(n,2)$ varieties discussed above. While these Shimura varieties have canonical models over $\mathbb{Q}$, for all n, we do not have a sufficient theory of the integral models to give a precise discussion of the integral extensions of the $Z(m,\varphi)$'s for general n. In addition, even for the archimedean theory, due to the non-compactness of X_K, one will need a suitable theory of line bundles with singular metrics, Green's currents with additional singularities, etc. Such problems are under consideration by Burgos, Kramer and Kühn [**10**]. For the case of arithmetic surfaces, i.e., $n=1$, see [**5**], [**35**]. Nonetheless, based on low dimensional calculations, it is possible to make some rough speculations, which provide a setting for the results of this paper.

A metrized line bundle $\hat\omega$ on a projective arithmetic variety $\mathfrak{X}$ over $\mathrm{Spec}(\mathbb{Z})$ defines a class $\hat\omega \in \widehat{\mathrm{Pic}}(\mathfrak{X}) \simeq \widehat{CH}^1(\mathfrak{X})$ and classes $\hat\omega^r \in \widehat{CH}^r(\mathfrak{X})$, the r-th arithmetic Chow group of $\mathfrak{X}$,with rational coefficients [**17**]. For a cycle $\mathfrak{Z}$ on $\mathfrak{X}$ of codimension r, there is a height $h_{\hat\omega}(\mathfrak{Z})$ with respect to $\hat\omega$, [**6**]. For example, for an integral horizontal $\mathfrak{Z}$ of codimension r, with normalization $j:\tilde{\mathfrak{Z}} \to \mathfrak{Z} \subset \bar{\mathfrak{X}}$, assumed to be itself regular over $\mathrm{Spec}(\mathbb{Z})$,

$$h_{\hat\omega}(\mathfrak{Z}) = \widehat{\deg}\, j^*(\hat\omega^{n+1-r}), \tag{6.1}$$

where $\widehat{\deg} : \widehat{CH}^{n+1-r}(\mathfrak{Z}) \to \mathbb{R}$ is the arithmetic degree map. Also, if $(\mathfrak{Z},g) \in \widehat{CH}^r(\mathfrak{X})$ is a codimension r cycle with Green's current g, then, for the height pairing $\langle\ ,\ \rangle$ between $\widehat{CH}^r(\mathfrak{X})$ and $\widehat{CH}^{n+1-r}(\mathfrak{X})$,

$$\langle (\mathfrak{Z},g), \hat\omega^{n+1-r}\rangle = h_{\hat\omega}(\mathfrak{Z}) + \frac{1}{2}\int_{\mathfrak{X}(\mathbb{C})} g\cdot c_1(\hat\omega)^{n+1-r}, \tag{6.2}$$

where $c_1(\hat\omega)$ is the first Chern form of $\hat\omega$ on $\mathfrak{X}(\mathbb{C})$.

For V of signature $(n,2)$, let $X = X_K$ be the canonical model over $\mathbb{Q}$ of the arithmetic quotient $\Gamma_K\backslash D^+$. Here we are assuming that K is large enough so that X is geometrically irreducible.

Suppose that we have a regular model $\mathfrak{X}$ of X over $\mathrm{Spec}(\mathbb{Z})$, with a regular compactification $\bar{\mathfrak{X}}$. Suppose that the metrized line bundle $\mathcal{L}^\vee$ dual to $\mathcal{L}$ (cf. (1.4) and (1.5)) on X is the restriction of a line bundle $\hat{\omega}$ on $\bar{\mathfrak{X}}$, where the metric on $\hat{\omega}$ is allowed to have singularities along $\bar{X}(\mathbb{C})\setminus X(\mathbb{C})$. Note that the first Chern form of $\hat{\omega}$ is the form Ω considered above. Suppose that one has a sufficiently extended theory of an arithmetic Chow ring (with rational coefficients) $\widehat{CH}^\bullet(\bar{\mathfrak{X}})$ so that the height construction can be applied. Thus, in particular, $\hat{\omega}$ defines a class in $\widehat{CH}^1(\bar{\mathfrak{X}})$ and powers $\hat{\omega}^r \in \widehat{CH}^r(\bar{\mathfrak{X}})$, etc.

Next, we consider a Borcherds form $\Psi = \Psi(f)^2$ of weight $c_0(0)$. Then Ψ is meromorphic function on $X(\mathbb{C}) \simeq \mathfrak{X}(\mathbb{C})$, whose divisor is rational over $\mathbb{Q}$. We suppose that, in fact, there is a (rational) section $\tilde{\Psi}$ of $(\omega^{-1})^{\otimes c_0(0)}$ whose restriction to $\mathfrak{X}(\mathbb{C}) \simeq X(\mathbb{C})$ is Ψ. It follows that $\widehat{\mathrm{div}}(\tilde{\Psi}) = -c_0(0)\,\hat{\omega} \in \widehat{CH}^1(\bar{\mathfrak{X}})$. Then, we would have

$$
\begin{aligned}
-c_0(0)\,\langle \hat{\omega}, \hat{\omega}^n\rangle &= \langle \widehat{\mathrm{div}}(\tilde{\Psi}), \hat{\omega}^n\rangle \\
&= h_{\hat{\omega}}(\mathrm{div}(\tilde{\Psi})) + \frac{1}{2}\int_{X(\mathbb{C})} \log ||\Psi||^{-2}\,\Omega^n \qquad (6.3) \\
&= h_{\hat{\omega}}(\mathrm{div}(\tilde{\Psi})) + \frac{1}{2}\mathrm{vol}(X)\,\kappa(\Psi).
\end{aligned}
$$

Recall that (Theorem 1.3), on $X(\mathbb{C})$,

$$
\mathrm{div}_X(\Psi) = \mathrm{div}_{\mathfrak{X}_\mathbb{Q}}(\Psi(f)^2) = \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\,Z(m,\varphi). \tag{6.4}
$$

Then, on the integral model, we would have

$$
\mathrm{div}_{\mathfrak{X}}(\tilde{\Psi}) = \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\,\mathfrak{Z}(m,\varphi) + \big(\text{vertical components}\big), \tag{6.5}
$$

where the $\mathfrak{Z}(m,\varphi)$'s have generic fibers $\mathfrak{Z}(m,\varphi)_\mathbb{Q} = Z(m,\varphi)$.

Using the expression in Theorem 2.12 for $\kappa(\Psi) = 2\kappa(\Psi(f))$, we obtain

(6.6)

$$
\begin{aligned}
-c_0(0)\,\langle \hat{\omega}, \hat{\omega}^n\rangle = \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\Big[h_{\hat{\omega}}(\mathfrak{Z}(m,\varphi)) + \mathrm{vol}(X)\,\kappa_\varphi(m)\Big]& \\
+ \mathrm{vol}(X)\,c_0(0)\,\kappa_0(0) + \text{contributions of vertical components}&.
\end{aligned}
$$

This (hypothetical) relation is suggestive. For example, if $c_0(0) = 0$ so that $\widehat{\mathrm{div}}(\tilde{\Psi}) = 0$, we obtain

(6.7)

$$
0 = \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\Big[h_{\hat{\omega}}(\mathfrak{Z}(m,\varphi)) + \mathrm{vol}(X)\,\kappa_\varphi(m)\Big] + \text{contributions of vertical components},
$$

which suggests a close relation between $\kappa_\varphi(m)$ and the height $h_{\hat{\omega}}(\mathfrak{Z}(m,\varphi))$.

In our example for $n=5$ from section 5, we can write

(6.8)

$$\kappa_\mu(m) = \mathrm{vol}(X)^{-1}\ \deg(Z(m,\varphi))\Big[-\frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} + \sum_{p|n}\Big(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\Big)\Big]$$
$$+\,\mathrm{vol}(X)^{-1}\ \deg(Z(m,\varphi))\Big[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - C\Big]$$

so that (6.6) can be written as

$$-c_0(0)\,\langle\hat{\omega},\hat{\omega}^3\rangle = \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\,\delta(m,\varphi_\mu)$$
$$+\,\mathrm{vol}(X)\,c_0(0)\Big[\kappa_0(0) - \frac{4}{3} - 2\,\frac{\zeta'(-3)}{\zeta(-3)} + C\Big] \tag{6.9}$$
$$+\text{ contributions of vertical components.}$$

where

(6.10)

$$\delta(m,\varphi_\mu) = h_{\hat{\omega}}(\mathfrak{Z}(m,\varphi_\mu)) + \deg(Z(m,\varphi))\Big[-\frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} + \sum_{p|n}\Big(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\Big)\Big].$$

Again, this suggests that

(6.11)

$$h_{\hat{\omega}}(\mathfrak{Z}(m,\varphi_\mu)) \equiv -\deg(Z(m,\varphi))\Big[-\frac{1}{2}\log(d) - \frac{L'(-1,\chi_d)}{L(-1,\chi_d)} + \sum_{p|n}\Big(\log|n|_p - \frac{b_p'(n,-1)}{b_p(n,-1)}\Big)\Big]$$

and

$$\langle\hat{\omega},\hat{\omega}^3\rangle \equiv \mathrm{vol}(X)\Big[\frac{4}{3} + 2\,\frac{\zeta'(-3)}{\zeta(-3)} - \frac{3}{2}\log(2) - \log(\pi)\Big], \tag{6.12}$$

where, in both relations, we have still to account for a possible linear combination of $\log(p)$'s coming from vertical components. In addition, it is possible to shift a term of the form

$$\mathrm{vol}(X)^{-1}\ \deg(Z(m,\varphi))\cdot A, \tag{6.13}$$

where A is a constant independent of μ and m, between the two terms in (6.8), so there is some further ambiguity. It seems reasonable to expect that A is a multiple of $\zeta'(-1)/\zeta(-1)$. This would be consistent with recent results of Bruinier and Kühn for certain Hilbert modular varieties, [**9**], Kühn's thesis [**34**], and conjectures of Maillot and Roessler, [**37**]. Recall that $\mathrm{vol}(X) = \zeta(-1)\,\zeta(-3)$.

Of course, this discussion is too vague with respect to integral models, compactifications, an extended theory of arithmetic Chow rings, and vertical contributions. Nonetheless, it explains the motivation for considering the quantities $\kappa(\Psi(f))$ and $\kappa_\varphi(m)$ and their possible applications.

Finally, it is worthwhile to compare the formula for $\Psi(f)$ of Theorem 2.12 with the following result of Rohrlich[6], [**45**]. Suppose that h is a meromorphic modular form of weight k for a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ having a cusp of width 1 at ∞ such that h is nonzero at every cusp of Γ and has constant term 1 at ∞. Let

$$\mathcal{E}(z,s) = \sum_{\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma(z))^s, \qquad \mathrm{Re}(s) > 1. \tag{6.14}$$

Write

$$\mathcal{E}(z,s) = \frac{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}{s-1} + g(z) + O(s-1). \tag{6.15}$$

Then, setting

$$r = \frac{2\pi}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}, \tag{6.16}$$

we have

$$g(z) = -\frac{1}{2\pi}\log(y^r|D(z)|e^{-r}) \tag{6.17}$$

for a holomorphic modular form D of weight $2r$ for Γ (with possible multiplier). Then

$$\begin{aligned} \kappa(h) :&= -\frac{1}{\mathrm{vol}(\Gamma\backslash\mathfrak{H})}\int_{\Gamma\backslash\mathfrak{H}} \log\left|h(z)y^{\frac{k}{2}}\right|^2 d\mu(z) \\ &= \sum_{z\in\Gamma\backslash\mathfrak{H}} 2\,\mathrm{ord}_z(h)\,\log(y^r|D(z)|). \end{aligned} \tag{6.18}$$

In the case of a Shimura curve case, i.e, for V anisotropic of signature $(1,2)$ and a Borcherds form $\Psi(f)$, we have

$$\mathrm{ord}_z(\Psi(f)^2) = \sum_{\varphi}\sum_{m>0} c_\varphi(-m)\,\mathrm{ord}_z\big(Z(m,\varphi)\big). \tag{6.19}$$

[6] I am indebted to Ulf Kuhn for calling this beautiful paper to my attention.

This suggests the following question. Is there a modular form D^B on X_K of weight $2r$ such that

(6.20)
$$\kappa_\varphi(m) = \sum_{z\in Z(m,\varphi)} \log\big(y^r|D^B(z)|e^{-\frac{1}{2}rC_0}\big),$$

for $r = \frac{2\pi}{\mathrm{vol}(\Gamma_M\backslash\mathfrak{H})}$?

One would then have

(6.21)
$$\begin{aligned}
&\sum_z 2\,\mathrm{ord}_z(\Psi(f))\,\log\big(y^r|D^B(z)|\big)\\
&= \sum_z \Big(\sum_\varphi\sum_{m>0} c_\varphi(-m)\,\mathrm{ord}_z\big(Z(m,\varphi)\big)\Big)\,\log\big(y^r|D^B(z)|e^{-\frac{1}{2}rC_0}\big) + rC_0\,\deg(\mathrm{div}(\Psi(f)))\\
&= \sum_\varphi\sum_{m>0} c_\varphi(-m)\Big(\sum_{z\in Z(m,\varphi)} \log\big(y^r|D^B(z)|e^{-\frac{1}{2}rC_0}\big)\Big) + \frac{1}{2}C_0c_0(0)\\
&= \sum_\varphi\sum_{m>0} c_\varphi(-m)\,\kappa_\varphi(m) + \frac{1}{2}C_0c_0(0)\\
&= \sum_\varphi\sum_{m\ge0} c_\varphi(-m)\,\kappa_\varphi(m)\\
&= \kappa(\Psi(f)),\\
&= -\frac{1}{\mathrm{vol}(X_K)}\int_{X_K} \log||\Psi(z;f)||^2\,d\mu(z)
\end{aligned}$$

just as in Rohrlich's case. The function D in Rohrlich arises in the Kronecker limit formula for the Fuchsian group Γ, [**18**]. The function D^B would be a kind of analogue in the compact quotient case, i.e., in the absence of the Eisenstein series!

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