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Derivatives of Eisenstein Series and Arithmetic Geometry*

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Abstract

We describe connections between the Fourier coefficients of derivatives of Eisenstein series and invariants from the arithmetic geometry of the Shimura varieties M associated to rational quadratic forms (V, Q) of signature $(n, 2)$. In the case $n = 1$, we define generating series $\hat{\phi}_1(\tau)$ for 1-cycles (resp. $\hat{\phi}_2(\tau)$ for 0-cycles) on the arithmetic surface $\mathcal{M}$ associated to a Shimura curve over $\mathbb{Q}$. These series are related to the second term in the Laurent expansion of an Eisenstein series of weight $\frac{3}{2}$ and genus 1 (resp. genus 2) at the Siegel–Weil point, and these relations can be seen as examples of an 'arithmetic' Siegel–Weil formula. Some partial results and conjectures for higher dimensional cases are also discussed.

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1. Introduction

In this report, we will survey results about generating functions for arithmetic cycles on Shimura varieties defined by rational quadratic forms of signature $(n, 2)$. For small values of n, these Shimura varieties are of PEL type, i.e., can be identified with moduli spaces for abelian varieties equipped with polarization, endomorphisms, and level structure. By analogy with CM or Heegner points on modular curves, cycles are defined by imposing additional endomorphisms. Relations between the heights or arithmetic degrees of such cycles and the Fourier coefficients of *derivatives* of Siegel Eisenstein series are proved in [10] and in subsequent joint work with Rapoport, [14], [15], [16], and with Rapoport and Yang [17], [18]. These relations may be viewed as an arithmetic version of the classical Siegel-Weil formula, which identifies the Fourier coefficients of *values* of Siegel Eisenstein

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series with representation numbers of quadratic forms. The most complete example is that of anisotropic ternary quadratic forms ($n = 1$), so that the cycles are curves and 0-cycles on the arithmetic surfaces associated to Shimura curves. Other surveys of the material discussed here can be found in [11] and [12].

2. Shimura curves

Let B be an indefinite quaternion algebra over $\mathbb{Q}$, and let $D(B)$ be the product of the primes p for which $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra. The rational vector space

$$V = \{ \ x \in B \mid \mathrm{tr}(x) = 0 \ \}$$

with quadratic form given by $Q(x) = -x^2 = \nu(x)$, where $\mathrm{tr}(x)$ (resp. $\nu(x)$) is the reduced trace (resp. norm) of x, has signature $(1, 2)$. The action of $B^\times$ on V by conjugation gives an isomorphism $G = \mathrm{GSpin}(V) \simeq B^\times$. Let

$$D = \{ \ w \in V(\mathbb{C}) \mid (w, w) = 0, \ (w, \bar{w}) < 0 \ \}/\mathbb{C}^\times \simeq \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$$

be the associated symmetric space. Let O_B be a maximal order in B and let $\Gamma = O_B^\times$ be its unit group. The quotient $M(\mathbb{C}) = \Gamma\backslash D$ is the set of complex points of the Shimura curve M (resp. modular curve, if $D(B) = 1$) determined by B. This space should be viewed as an orbifold $[\Gamma\backslash D]$. For a more careful discussion of this and of the stack aspect, which we handle loosely here, see [18]. The curve M has a canonical model over $\mathbb{Q}$. From now on, we assume that $D(B) > 1$, so that M is projective. Drinfeld's model $\mathcal{M}$ for M over $\mathrm{Spec}\,(\mathbb{Z})$ is obtained as the moduli stack for abelian schemes (A, ι) with an action $\iota : O_B \hookrightarrow \mathrm{End}(A)$ satisfying the 'special' condition, [3]. It is proper of relative dimension 1 over $\mathrm{Spec}\,(\mathbb{Z})$, with semi-stable reduction at all primes and is smooth at all primes p at which B splits, i.e., for $p \nmid D(B)$. We view $\mathcal{M}$ as an arithmetic surface in the sense of Arakelov theory and consider its arithmetic Chow groups with real coefficients $\widehat{CH}^r(\mathcal{M}) = \widehat{CH}^r_{\mathbb{R}}(\mathcal{M})$, as defined in [2]. Recall that these groups are generated by pairs $(\mathcal{Z}, g)$, where $\mathcal{Z}$ is an $\mathbb{R}$-linear combination of divisors on $\mathcal{M}$ and g is a Green function for $\mathcal{Z}$, with relations given by $\mathbb{R}$-linear combinations of elements $\widehat{\mathrm{div}}\ (f) = (\mathrm{div}(f), -\log|f|^2)$ where $f \in \mathbb{Q}(\mathcal{M})^\times$ is a nonzero rational function on $\mathcal{M}$. These real vector spaces come equipped with a geometric degree map $\deg_{\mathbb{Q}} : \widehat{CH}^1(\mathcal{M}) \to CH^1(\mathcal{M}_{\mathbb{Q}}) \overset{\deg}{\to} \mathbb{R}$, where $\mathcal{M}_{\mathbb{Q}}$ is the generic fiber of $\mathcal{M}$, an arithmetic degree map $\widehat{\deg}\ : \widehat{CH}^2(\mathcal{M}) \to \mathbb{R}$, and the Gillet-Soulé height pairing, [2],

$$\langle \ , \ \rangle : \widehat{CH}^1(\mathcal{M}) \times \widehat{CH}^1(\mathcal{M}) \longrightarrow \mathbb{R}.$$

Let $\mathcal{A}$ be the universal abelian scheme over $\mathcal{M}$. Then the Hodge line bundle $\omega = \epsilon^*(\Omega^2_{\mathcal{A}/\mathcal{M}})$ determined by $\mathcal{A}$ has a natural metric, normalized as in [18], section 3, and defines an element $\hat{\omega} \in \widehat{\mathrm{Pic}}(\mathcal{M})$, the group of metrized line bundles on

$\mathcal{M}$. We also write $\hat{\omega}$ for the image of this class in $\widehat{CH}^1(\mathcal{M})$ under the natural map, which sends a metrized line bundle $\hat{\mathcal{L}} = (\mathcal{L}, || \, ||) \in \widehat{\text{Pic}}(\mathcal{M})$ to the class of $(\text{div}(s), -\log ||s||^2)$, for any nonzero section s of $\mathcal{L}$.

Arithmetic cycles in $\mathcal{M}$ are defined by imposing additional endomorphisms of the following type.

Definition 1. ([10]) *The space of special endomorphisms $V(A, \iota)$ of an abelian scheme (A, ι), as above, is*

$$V(A, \iota) = \{ \ x \in \text{End}(A) \mid x \circ \iota(b) = \iota(b) \circ x, \ \forall \, b \in O_B, \ \text{and} \ \text{tr}(x) = 0 \ \},$$

with $\mathbb{Z}$-valued quadratic form given by $-x^2 = Q(x) \, \text{id}_A$.

2.1. Divisors

To obtain divisors on $\mathcal{M}$, we impose a single special endomorphism. For a positive integer t, let $\mathcal{Z}(t)$ be the divisor on $\mathcal{M}$ determined by the moduli stack of triples (A, ι, x) where (A, ι) is as before and where $x \in V(A, \iota)$ is a special endomorphism with $Q(x) = t$. Note that, for example, the complex points $\mathcal{Z}(t)(\mathbb{C})$ of $\mathcal{Z}(t)$ correspond to abelian surfaces (A, ι) over $\mathbb{C}$ with an 'extra' action of the order $\mathbb{Z}[\sqrt{-t}]$ in the imaginary quadratic field $\mathbb{Q}(\sqrt{-t})$, i.e., to CM points on the Shimura curve $M(\mathbb{C})$. On the other hand, the cycles $\mathcal{Z}(t)$ can have vertical components in the fibers of bad reduction $\mathcal{M}_p$ for $p \mid D(B)$. More precisely, in joint work with M. Rapoport we show:

Proposition 1. ([15]) *For $p \mid D(B)$, $\mathcal{Z}(t)$ contains components of the fiber of bad reduction $\mathcal{M}_p$ if and only if $\text{ord}_p(t) \geq 2$ and no prime $\ell \mid D(B)$, $\ell \neq p$, splits in $\boldsymbol{k}_t := \mathbb{Q}(\sqrt{-t})$.*

The precise structure of the vertical part of $\mathcal{Z}(t)$ is determined in [15] using the Drinfeld-Cherednik p-adic uniformization of $\mathcal{M}_p$. For example, for $p \mid D(B)$, the multiplicities of the vertical components in the fiber $\mathcal{M}_p$ of the cycle $\mathcal{Z}(p^{2r}t)$ grow with r, while the horizontal part of this cycle remains unchanged.

To obtain classes in $\widehat{CH}^1(\mathcal{M})$, we construct Green functions by the procedure introduced in [10]. Let $L = O_B \cap V$. For $t \in \mathbb{Z}_{>0}$ and $v \in \mathbb{R}_{>0}$, define a function $\Xi(t, v)$ on $M(\mathbb{C})$ by

$$\Xi(t, v)(z) = \sum_{x \in L(t)} \beta_1(2\pi v R(x, z)),$$

where $L(t) = \{x \in L \mid Q(x) = t\}$, and, for $z \in D$ with preimage $w \in V(\mathbb{C})$, $R(x, z) = |(x, w)|^2 \, |(w, \bar{w})|^{-1}$. Here

$$\beta_1(r) = \int_1^\infty e^{-ru} \, u^{-1} \, du = -\text{Ei}(-r)$$

is the exponential integral. Recall that this function has a log singularity as r goes to zero and decays exponentially as r goes to infinity. In fact, as shown in [10], section 11, for any $x \in V(\mathbb{R})$ with $Q(x) \neq 0$, the function

$$\xi(x,z) := \beta_1(2\pi R(x,z))$$

can be viewed as a Green function on D for the divisor $D_x := \{z \in D \mid (x,z) = 0\}$. A simple calculation, [10], shows that, for $t > 0$, $\Xi(t,v)$ is a Green function of logarithmic type for the cycle $\mathcal{Z}(t)$, while, for $t < 0$, $\Xi(t,v)$ is a smooth function on $M(\mathbb{C})$.

Definition 2. *(i) For $t \in \mathbb{Z}$ and $v > 0$, the class $\widehat{\mathcal{Z}}(t,v) \in \widehat{CH}^1(\mathcal{M})$ is defined by:*

$$\widehat{\mathcal{Z}}(t,v) = \begin{cases} (\mathcal{Z}(t), \Xi(t,v)) & \text{if } t > 0, \\ -\hat{\omega} + (0, \mathbf{c} - \log(v)) & \text{if } t = 0, \\ (0, \Xi(t,v)) & \text{if } t < 0. \end{cases}$$

Here $\hat{\omega}$ is the metrized Hodge line bundle, as above, and the real constant $\mathbf{c}$ is given by

$$\frac{1}{2}\deg_{\mathbb{Q}}(\hat{\omega})\cdot\mathbf{c} = \langle\hat{\omega},\hat{\omega}\rangle - \zeta_{D(B)}(-1)\left[2\,\frac{\zeta'(-1)}{\zeta(-1)} + 1 - \log(4\pi) - \gamma - \sum_{p|D(B)} \frac{p\log(p)}{p-1}\right],$$

where $\zeta_{D(B)}(s) = \zeta(s)\prod_{p|D(B)}(1-p^{-s})$ and γ is Euler's constant.
(ii) For $\tau = u + iv \in \mathfrak{H}$ and $q = e(\tau) = e^{2\pi i\tau}$, the 'arithmetic theta function' $\hat{\phi}_1(\tau)$ is given by the generating series

$$\hat{\phi}_1(\tau) := \sum_{t\in\mathbb{Z}} \widehat{\mathcal{Z}}(t,v)\, q^t.$$

It is conjectured in [18] that the constant $\mathbf{c}$ occurring in the definition of $\widehat{\mathcal{Z}}(0,v)$ is, in fact, zero. It may be possible to use recent work of Bruinier and Kühn, [4], on the heights of curves on Hilbert modular surfaces to show that that $\langle\hat{\omega},\hat{\omega}\rangle$ has the predicted value and hence verify this conjecture.

Some justification for the terminology 'arithmetic theta function' is given by the following result, which is closely related to earlier work of Zagier, [25], and recent results of Borcherds, [1], cf. also [20].

Theorem 1. *The arithmetic theta function $\hat{\phi}_1(\tau)$ is a (nonholomorphic) modular form of weight $\frac{3}{2}$, valued in $\widehat{CH}^1(\mathcal{M})$, for a subgroup $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$.*

The proof of Theorem 1 depends on Borcherd's result [1] and on the modularity of various complex valued q-expansions obtained by taking height pairings of $\hat{\phi}_1(\tau)$

with other classes in $\widehat{CH}^1(\mathcal{M})$. We now describe some of these in terms of values and derivatives of a certain Eisenstein series, [18], of weight $\frac{3}{2}$

$$\mathcal{E}_1(\tau, s, D(B)) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (c\tau + d)^{-\frac{3}{2}} |c\tau + d|^{-(s-\frac{1}{2})} \, v^{\frac{1}{2}(s-\frac{1}{2})} \, \Phi_1(s, \gamma, D(B)),$$

associated to B and the lattice L, and normalized so that it is invariant under $s \mapsto -s$. The main result of joint work with M. Rapoport and T. Yang is the following:

Theorem 2. ([18]) (i)

$$\mathcal{E}_1(\tau, \frac{1}{2}; D(B)) = \deg(\hat{\phi}_1(\tau)) = \sum_t \deg_{\mathbb{Q}}(\widehat{\mathcal{Z}}(t, v)) \, q^t.$$

(ii)

$$\mathcal{E}_1'(\tau, \frac{1}{2}; D(B)) = \langle \, \hat{\phi}_1(\tau), \hat{\omega} \, \rangle = \sum_t \langle \, \widehat{\mathcal{Z}}(t, v), \hat{\omega} \, \rangle \, q^t.$$

Note that this result expresses the Fourier coefficients of the first two terms in the Laurent expansion at the point $s = \frac{1}{2}$ of the Eisenstein series $\mathcal{E}_1(\tau, s; D(B))$ in terms of the geometry and the arithmetic geometry of cycles on $\mathcal{M}$.

Next consider the image of

$$\hat{\phi}_1(\tau) - \mathcal{E}_1(\tau, \frac{1}{2}; D(B)) \cdot \deg(\hat{\omega})^{-1} \cdot \hat{\omega}$$

in $CH^1(\mathcal{M}_{\mathbb{Q}})$, the usual Chow group of the generic fiber. By (i) of Theorem 2, it lies in the Mordell-Weil space $CH^1(\mathcal{M}_{\mathbb{Q}})^0 \otimes \mathbb{C} \simeq \mathrm{Jac}(M)(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{C}$. In fact, it is essentially the generating function defined by Borcherds, [1], for the Shimura curve M, and hence is a holomorphic modular of weight $\frac{3}{2}$. For the case of modular curves, such a modular generating function, whose coefficients are Heegner points, was introduced by Zagier, [25]. By the Hodge index theorem for $\widehat{CH}^1(\mathcal{M})$, [2], the proof of Theorem 1 is completed by showing that the pairing of $\hat{\phi}_1(\tau)$ with each class of the form $(Y_p, 0)$, for Y_p a component of the fiber $\mathcal{M}_p$, $p \mid D(B)$ and each class of the form $(0, \phi)$, where $\phi \in C^\infty(M(\mathbb{C}))$, is modular.

2.2. 0-cycles

We next consider a generating function for 0-cycles on $\mathcal{M}$. Recall that the arithmetic Chow group $\widehat{CH}^2(\mathcal{M})$, with real coefficients, is generated by pairs $(\mathcal{Z}, g)$, where $\mathcal{Z}$ is a real linear combination of 0-cycles on $\mathcal{M}$ and g is a real smooth $(1,1)$-form on $\mathcal{M}(\mathbb{C})$. In fact, the arithmetic degree map, as defined in [2],

$$\widehat{\deg} \, : \widehat{CH}^2(\mathcal{M}) \to \mathbb{R}, \qquad \widehat{\deg} \, ((\mathcal{Z}, g)) = \sum_i n_i \log |k(P_i)| + \frac{1}{2} \int_{\mathcal{M}(\mathbb{C})} g,$$

where $\mathcal{Z} = \sum_i n_i\, P_i$ for closed points P_i of $\mathcal{M}$ with residue field $k(P_i)$, is an isomorphism.

Let $\tau = u + iv \in \mathfrak{H}_2$, the Siegel space of genus 2, and for $T \in \mathrm{Sym}_2(\mathbb{Z})$, let $q^T = e^{2\pi i \mathrm{tr}(T\tau)}$. To define the generating series

$$\hat{\phi}_2(\tau) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})} \widehat{\mathcal{Z}}(T, v)\, q^T,$$

we want to define classes $\widehat{\mathcal{Z}}(T, v) \in \widehat{CH}^2(\mathcal{M})$ for each $T \in \mathrm{Sym}_2(\mathbb{Z})$ and $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$.

We begin by considering cycles on $\mathcal{M}$ which are defined by imposing pairs of endomorphisms. For $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ a positive definite integral symmetric matrix, let $\mathcal{Z}(T)$ be the moduli stack over $\mathcal{M}$ consisting of triples $(A, \iota, \mathbf{x})$ where (A, ι) is as before, and $\mathbf{x} = [x_1, x_2] \in V(A, \iota)^2$ is a pair of special endomorphisms with matrix of inner products $Q(\mathbf{x}) = \frac{1}{2}((x_i, x_j)) = T$. We call T the fundamental matrix of the triple $(A, \iota, \mathbf{x})$. The following result of joint work with M. Rapoport describes the cases in which $\mathcal{Z}(T)$ is, in fact, a 0-cycle on $\mathcal{M}$.

Proposition 2. ([15]) *Suppose that $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$. (i) The cycle $\mathcal{Z}(T)$ is either empty or is supported in the set of supersingular points in a fiber $\mathcal{M}_p$ for a unique prime p determined by T. In particular, $\mathcal{Z}(T)_{\mathbb{Q}} = \emptyset$. The prime p is determined by the condition that T is represented by the ternary quadratic space $V^{(p)} = \{\, x \in B^{(p)} \mid \mathrm{tr}(x) = 0 \,\}$, with $Q^{(p)}(x) = -x^2$, where $B^{(p)}$ is the definite quaternion algebra over $\mathbb{Q}$ with $B^{(p)}_\ell \simeq B_\ell$ for all primes $\ell \neq p$. If there is no such prime, then $\mathcal{Z}(T)$ is empty.*
(ii) (T regular) Let p be as in (i). Then, if $p \nmid D(B)$ or if $p \mid D(B)$ but $p^2 \nmid T$, then $\mathcal{Z}(T)$ is a 0-cycle in $\mathcal{M}_p$.
(iii) (T irregular) Let p be as in (i). If $p \mid D(B)$ and $p^2 \mid T$, then $\mathcal{Z}(T)$ is a union, with multiplicities, of components of $\mathcal{M}_p$, cf. [15], *176.*

For $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ regular, as in (ii) of Proposition 2, we let

$$\widehat{\mathcal{Z}}(T, v) := \widehat{\mathcal{Z}}(T) = (\mathcal{Z}(T), 0) \in \widehat{CH}^2(\mathcal{M}).$$

For $T = \begin{pmatrix} t_1 & m \\ m & t_2 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ irregular, we use the results of [15], section 8 (where the quadratic form is taken with the opposite sign). We must therefore assume that $p \neq 2$, although the results of the appendix to section 11 of [18] suggest that it should be possible to eliminate this restriction. In this case, the vertical cycle $\mathcal{Z}(T)$ in the fiber $\mathcal{M}_p$ is the union of those connected components of the intersection $\mathcal{Z}(t_1) \times_{\mathcal{M}} \mathcal{Z}(t_2)$ where the 'fundamental matrix', [15], is equal to T. Here $\mathcal{Z}(t_1)$ and $\mathcal{Z}(t_2)$ are the codimension 1 cycles defined earlier. Note that, by Proposition 1, they can share some vertical components. We base change to $\mathbb{Z}_p$ and set

$$\widehat{\mathcal{Z}}(T, v) := \chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2)}) \cdot \log(p) \quad \in \mathbb{R} \simeq \widehat{CH}^2(\mathcal{M}),$$

where χ is the Euler-Poincaré characteristic of the derived tensor product of the structure sheaves $\mathcal{O}_{\mathcal{Z}(t_1)}$ and $\mathcal{O}_{\mathcal{Z}(t_2)}$, cf. [15], section 4. Note that the same definition could have been used in the regular case.

Next we consider nonsingular $T \in \mathrm{Sym}_2(\mathbb{Z})$ of signature $(1,1)$ or $(0,2)$. In this case, $\mathcal{Z}(T)$ is empty, since the quadratic form on $V(A,\iota)$ is positive definite, and our 'cycle' should be viewed as 'vertical at infinity'. For a pair of vectors $\mathbf{x} = [x_1, x_2] \in V(\mathbb{Q})^2$ with nonsingular matrix of inner products $Q(\mathbf{x}) = \frac{1}{2}\big((x_i, x_j)\big)$, the quantity

$$\Lambda(\mathbf{x}) := \int_D \xi(x_1) * \xi(x_2),$$

where $\xi(x_1) * \xi(x_2)$ is the $*$-product of the Green functions $\xi(x_1)$ and $\xi(x_2)$, [6], is well defined and depends only on $Q(\mathbf{x})$. In addition, $\Lambda(\mathbf{x})$ has the following remarkable invariance property.

Theorem 3. ([10, Theorem 11.6]) *For $k \in O(2)$, $\Lambda(\mathbf{x} \cdot k) = \Lambda(x)$.*

For $T \in \mathrm{Sym}_2(\mathbb{Z})$ of signature $(1,1)$ or $(0,2)$ and for $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$, choose $a \in GL_2(\mathbb{R})$ such that $v = a\,{}^t a$, and define

$$\widehat{\mathcal{Z}}(T,v) := \sum_{\mathbf{x} \in L^2,\ Q(\mathbf{x})=T,\ \bmod \Gamma} \Lambda(\mathbf{x}\, a) \quad \in \mathbb{R} \simeq \widehat{CH}^2(\mathcal{M}).$$

Here $L = O_B \cap V$ and $\Gamma = O_B^\times$, as before. Note that the invariance property of Theorem 3 is required to make the right side independent of the choice of a.

We omit the definition of the terms for singular T's, cf. [11].

By analogy with Theorem 1, we conjecture that, with this definition, the generating series $\hat{\phi}_2(\tau)$ is the q-expansion of a Siegel modular form of weight $\frac{3}{2}$ for a subgroup $\Gamma' \subset \mathrm{Sp}_2(\mathbb{Z})$. More precisely, there is a normalized Siegel Eisenstein series $\mathcal{E}_2(\tau, s; D(B))$ of weight $\frac{3}{2}$ attached to B, [10].

Conjecture 1.

$$\mathcal{E}_2'(\tau, 0; D(B)) \overset{?}{=} \hat{\phi}_2(\tau). \tag{$C1$}$$

This amounts to the family of identities

$$\mathcal{E}_{2,T}'(\tau, 0; D(B)) \overset{?}{=} \widehat{\mathcal{Z}}(T,v)\, q^T \tag{$C1_T$}$$

on Fourier coefficients, for all $T \in \mathrm{Sym}_2(\mathbb{Z})$. Here the isomorphism $\widehat{\deg}$ is being used.

Theorem 4. ([10], [15]) *The Fourier coefficient identity $(C1_T)$ holds in the following cases:*
(i) $T \in \mathrm{Sym}_2(\mathbb{Z})$ is not represented by V or by any of the spaces $V^{(p)}$ of Proposition 2.
(In this case both $\widehat{\mathcal{Z}}(T,v)$ and $\mathcal{E}_{2,T}'(\tau, 0; D(B))$ are zero.)

(ii) $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ *is regular and* $p \nmid 2D(B)$, [10].
(iii) $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$ *is irregular with* $p \neq 2$, *or regular with* $p \mid D(B)$ *and* $p \neq 2$, [15].
(iv) $T \in \mathrm{Sym}_2(\mathbb{Z})$ is nonsingular of signature $(1,1)$ or $(0,2)$, [10].

Theorem 4 is proved by a direct computation of both sides of $(C1_T)$. In case (ii), the computation of the Fourier coefficient $\mathcal{E}'_{2,T}(\tau,0;D(B))$ depends on the formula of Kitaoka, [8], for the local representation densities $\alpha_p(S,T)$ for the given T and a variable unimodular S. The computation of $\widehat{\mathcal{Z}}(T,v) = \widehat{\deg}\,((\mathcal{Z}(T),0))$ depends on a special case of a result of Gross and Keating, [7], about the deformations of a triple of isogenies between a pair of p-divisible formal groups of dimension 1 and height 2 over $\bar{\mathbb{F}}_p$. Their result is also valid for $p=2$, so it should be possible to extend (ii) to the case $p=2$ by extending the result of Kitaoka.

In case (iii), an explicit formula for the quantity $\chi(\mathcal{Z}(T), \mathcal{O}_{\mathcal{Z}(t_1)} \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(t_2)})$ is obtained in [15] using p-adic uniformization. The analogue of Kitaoka's result is a determination of $\alpha_p(S,T)$ for arbitrary S due to T. Yang, [22]. In both of these results, the case $p=2$ remains to be done.

Case (iv) is proved by directly relating the function Λ, defined via the $*$-product to the derivative at $s=0$ of the confluent hypergeometric function of a matrix argument defined by Shimura, [21]. The invariance property of Theorem 3 plays an essential role. The case of signature $(1,1)$ is done in [10]; the argument for signature $(0,2)$ is the same.

A more detailed sketch of the proofs can be found in [11].

As part of ongoing joint work with M. Rapoport and T. Yang, the verification of $(C1_T)$ for singular T of rank 1 is nearly complete.

3. Higher dimensional examples

So far, we have discussed the generating functions $\hat{\phi}_1(\tau) \in \widehat{CH}^1(\mathcal{M})$ and $\hat{\phi}_2(\tau) \in \widehat{CH}^2(\mathcal{M})$ attached to the arithmetic surface $\mathcal{M}$, and the connections of these series to derivatives of Eisenstein series. There should be analogous series defined as generating functions for arithmetic cycles for the Shimura varieties attached to rational quadratic spaces (V,Q) of signature $(n,2)$. At present there are several additional examples, all based on the accidental isomorphisms for small values of n, which allow us to identify the Shimura varieties in question with moduli spaces of abelian varieties with specified polarization and endomorphisms. Here we briefly sketch what one hopes to obtain and indicate what is known so far. The results here are joint work with M. Rapoport.

Hilbert-Blumenthal varieties ($n=2$), [14]. When the rational quadratic space (V,Q) has signature $(2,2)$, the associated Shimura variety M is a quasi-projective surface with a canonical model over $\mathbb{Q}$. There is a model $\mathcal{M}$ of M over $\mathrm{Spec}\,(\mathbb{Z}[N^{-1}])$ defined as the moduli scheme for collections $(A,\lambda,\iota,\bar{\eta})$ where A is an abelian scheme of relative dimension 8 dimension with polarization λ, level structure $\bar{\eta}$, and an action of $O_C \otimes \mathcal{O}_{\boldsymbol{k}}$, where O_C is a maximal order in the Clifford algebra $C(V)$ of V and $\mathcal{O}_{\boldsymbol{k}}$ is the ring of integers in the quadratic field $\boldsymbol{k} = \mathbb{Q}(\sqrt{d})$ for $d = \mathrm{discr(V)}$, the discriminant field of V, [14]. Again, a space $V(A,\iota) = V(A,\lambda,\iota,\bar{\eta})$ of special endomorphisms is defined; it is a $\mathbb{Z}$-module of finite rank equipped with a positive definite quadratic form Q. For $T \in \mathrm{Sym}_r(\mathbb{Z})$, we let $\mathcal{Z}(T)$ be the locus of $(A,\lambda,\iota,\bar{\eta},\mathbf{x})$'s where $\mathbf{x} = [x_1,\dots,x_r]$, $x_i \in V(A,\iota)$ is a collection of r special endomorphisms with matrix of inner products $Q(\mathbf{x}) = \frac{1}{2}\big((x_i,x_j)\big) = T$.

One would *like* to define a family of generating functions according to the following conjectural chart. Again there is a metrized Hodge line bundle $\hat{\omega} \in \widehat{CH}^1(\mathcal{M})$.

$r=1,\quad \mathcal{Z}(t)_{\mathbb{Q}} = \mathrm{HZ-curve}, \hat{\phi}_1(\tau) = \hat{\omega} + ? + \sum_{t\neq 0} \widehat{\mathcal{Z}}(t,v)\, q^t,\quad \langle \hat{\phi}_1(\tau), \hat{\omega}^2\rangle \overset{?}{=} \mathcal{E}_1'(\tau,1).$

$r=2,\quad \mathcal{Z}(t)_{\mathbb{Q}} = 0\mathrm{-cycle},\quad \hat{\phi}_2(\tau) = \hat{\omega}^2 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\, q^T \langle \hat{\phi}_2(\tau), \hat{\omega}\rangle \overset{?}{=} \mathcal{E}_2'(\tau,\tfrac{1}{2}).$

$r=3,\quad \mathcal{Z}(T)_{\mathbb{Q}} = \emptyset,\quad \hat{\phi}_3(\tau) = \hat{\omega}^3 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\, q^T \widehat{\deg}\, \hat{\phi}_3(\tau) \overset{?}{=} \mathcal{E}_3'(\tau,0).$

Here, the generating function $\hat{\phi}_r(\tau)$ is valued in $\widehat{CH}^r(\mathcal{M})$, the rth arithmetic Chow group, $\mathcal{E}_r(\tau,s)$ is a certain normalized Siegel Eisenstein series of genus r, and the critical value of s in the identity in the last column is the Siegel-Weil point $s_0 = \frac{1}{2}(\dim(V) - r - 1)$. Of course, one would like the $\hat{\phi}_r(\tau)$'s to be Siegel modular forms of genus r and weight 2.

There are many technical problems which must be overcome to obtain such results. For example, one would like to work with a model over $\mathrm{Spec}\,(\mathbb{Z})$. If V is anisotropic, then M is projective, but if V is isotropic, e.g., for the classical Hilbert-Blumenthal surfaces where it has $\mathbb{Q}$-rank 1, then one must compactify. Since the metric on $\hat{\omega}$ is singular at the boundary a more general version of the Gillet-Soulé theory, currently being developed by Burgos, Kramer and Kühn, [5], [19], will be needed.

Nonetheless, the chart suggests many identities which can in fact be checked rigorously. For example, there are again rational quadratic spaces $V^{(p)}$ of dimension 4 and signature $(4,0)$ obtained by switching the Hasse invariant of V at p.

Theorem 5. [14], [11]. *(i) If $T \in \mathrm{Sym}_3(\mathbb{Z})_{>0}$ is not represented by any of the $V^{(p)}$'s, then $\mathcal{Z}(T) = \emptyset$ and $\mathcal{E}_{3,T}'(\tau,0) = 0$.*
(ii) If $T \in \mathrm{Sym}_3(\mathbb{Z})_{>0}$ is represented by $V^{(p)}$ where p is a prime of good reduction split in $\boldsymbol{k}$, then $\mathcal{Z}(T)$ is a 0-cycle in $\mathcal{M}_p$ and

$$\widehat{\deg}\,((\mathcal{Z}(T),0))\, q^T = \mathcal{E}_{3,T}'(\tau,0). \tag{$\star$}$$

(iii) If $T \in \mathrm{Sym}_3(\mathbb{Z})_{>0}$ is represented by $V^{(p)}$ and p is a prime of good reduction inert in $\boldsymbol{k}$, then $\mathcal{Z}(T)$ is a 0-cycle in $\mathcal{M}_p$ if and only if $p \nmid T$. If this is the case, then the Fourier coefficient identity ($\star$) again holds. If $p \mid T$, then $\mathcal{Z}(T)$ is a union of components of the supersingular locus of $\mathcal{M}_p$.

Finally, say if V is anisotropic, one can consider the image $\mathrm{cl}(\hat{\phi}_r(\tau)) \in H^{2r}(M,\mathbb{C})$ of $\hat{\phi}_r(\tau)$ in the usual (Betti) cohomology of $\mathcal{M}(\mathbb{C})$. Of course, $\mathrm{cl}(\hat{\phi}_3(\tau)) = 0$ for degree reasons. Joint work with J. Millson on generating functions for cohomology classes of special cycles yields:

Theorem 6. ([13], [9], [11]) *Suppose that* V *is anisotropic.* (i) $\mathrm{cl}(\hat{\phi}_r(\tau))$ *is a Siegel modular form of genus* r *and weight* 2 *valued in* $H^{2r}(M,\mathbb{C})$.
(ii) *For the cup product pairing,* $\big(\,\mathrm{cl}(\hat{\phi}_r(\tau)), \mathrm{cl}(\hat{\omega})\,\big) = \mathcal{E}_r(\tau, s_0)$, *where* $s_0 = \frac{1}{2}(3-r)$.

Part (ii) here generalizes (i) of Theorem 2 above, so that, again, the value at s_0 of the Eisenstein series $\mathcal{E}_r(\tau, s)$ involves the complex geometry, while, conjecturally, the second term involves the height pairing.

Siegel modular varieties ($n=3$), [16]. Here, an integral model $\mathcal{M}$ of the Shimura variety M attached to a rational quadratic space of signature $(3,2)$ can be obtained as a moduli space of polarized abelian varieties of dimension 16 with an action of a maximal order O_C in the Clifford algebra of V. We just give the relevant *conjectural* chart:

$$
\begin{array}{llll}
r=1, & \mathcal{Z}(t)_{\mathbb{Q}} = \begin{matrix}\text{Humbert}\\ \text{surface}\end{matrix}, & \hat{\phi}_1(\tau) = \hat{\omega} + ? + \sum_{t\neq 0} \widehat{\mathcal{Z}}(t,v)\, q^t, & \langle \hat{\phi}_1(\tau), \hat{\omega}^3\rangle \overset{?}{=} \mathcal{E}_1'(\tau, \tfrac{3}{2}). \\
r=2, & \mathcal{Z}(t)_{\mathbb{Q}} = \text{curve} & \hat{\phi}_2(\tau) = \hat{\omega}^2 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\, q^T & \langle \hat{\phi}_2(\tau), \hat{\omega}^2\rangle \overset{?}{=} \mathcal{E}_2'(\tau, 1). \\
r=3, & \mathcal{Z}(T)_{\mathbb{Q}} = \text{0-cycle}, & \hat{\phi}_3(\tau) = \hat{\omega}^3 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\, q^T & \langle \hat{\phi}_2(\tau), \hat{\omega}\rangle \overset{?}{=} \mathcal{E}_3'(\tau, \tfrac{1}{2}). \\
r=4, & \mathcal{Z}(T)_{\mathbb{Q}} = \emptyset, & \hat{\phi}_4(\tau) = \hat{\omega}^4 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\, q^T & \widehat{\deg}\, \hat{\phi}_4(\tau) \overset{?}{=} \mathcal{E}_4'(\tau, 0).
\end{array}
$$

Here the Eisenstein series and, conjecturally, the generating functions $\hat{\phi}_r(\tau)$ have weight $\frac{5}{2}$, and the values of the Eisenstein series should be related to the series $\mathrm{cl}(\hat{\phi}_r(\tau))$. In the case of a prime p of good reduction a model of M over $\mathrm{Spec}\,(\mathbb{Z}_p)$ is defined in [16], and cycles are defined by imposing special endomorphisms. For example, for $r=4$, the main results of [16] give a criterion for $\mathcal{Z}(T)$ to be a 0-cycle in a fiber $\mathcal{M}_p$ and show that, when this is the case, then $\widehat{\deg}((\mathcal{Z}(T), 0))\, q^T = \mathcal{E}_{4,T}'(\tau, 0)$. The calculation of the left hand side is again based on the result of Gross and Keating mentioned in the description of the proof of Theorem 4 above. This provides some evidence for the last of the derivative identities in the chart.

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