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DERIVATIVES OF EISENSTEIN SERIES AND GENERATING FUNCTIONS FOR ARITHMETIC CYCLES

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The classical formula of Siegel and Weil identifies the values of Siegel–Eisenstein series at certain critical points as integrals of theta functions. When the critical point is the center of symmetry for the functional equation, the Fourier coefficients of the values of the 'even' Siegel–Eisenstein series thus contain arithmetic information about the representations of quadratic forms. It is natural to ask for an arithmetic interpretation of the *derivative* of the 'odd' series at their center of symmetry.

I would like to report on my work on a family of identities relating the Fourier expansions of the derivatives of certain Siegel–Eisenstein series at their center of symmetry, on one side, and generating functions for the degrees of 0–cycles on moduli schemes for abelian varieties, on the other. On the one hand, such identities can be viewed as generalizations of the Siegel–Weil formula to the case of the derivative. On the other hand, the identities imply that the generating functions in question, which are given as power series in q with coefficients arising from arithmetical algebraic geometry, are in fact the q-expansions of modular forms. This work grows out of results obtained in collaboration with Steve Rallis [18], [19], [20] and with John Millson [15], [16], [17]. More recent progress has been made in collaboration with Michael Rapoport [21], [22], [23] and Tonghai Yang [24], [25]. At present, the identities have been fully established only in certain special cases as explained below. Nonetheless, these examples, together with partial results in higher dimensions, suggest the outline of a more extensive theory.

An additional origin of the investigation described here was the study of the triple product L-function at the center of the critical strip, in collaboration with Michael Harris [9] and with Benedict Gross [7]. In particular, a Siegel–Eisenstein series is a key ingredient in the Rankin-Selberg integral representation of this L–function. Thus the occurrence of arithmetic geometric quantities in the central derivatives of the Eisenstein series should reflect their appearance in the central derivative of the L–function, and hence should provide a relation to the Gross–Zagier formula [8].

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Section 1 contains two examples, one recalling the work of Hirzebruch and Zagier on the modular generating functions for curves on a Hilbert–Blumenthal surface and the second illustrating a generating function in the simplest arithmetic case, involving the derivative of a classical Eisenstein series of weight 1. In section 2, the incoherent Siegel–Eisenstein series, which should be related to arithmetic generating series, are defined in general. Section 3 reviews the results of [15],[16],[17] on generating functions for cycles on locally symmetric spaces. These results suggest what one should hope for in the arithmetic case. In section 4, the generating function for 0–cycles on an arithmetic surface attached to a Shimura curve is defined, and Conjecture 4.7 relates it to the central derivative of an incoherent Eisenstein series of weight $\frac{3}{2}$ and genus 2. The comparison of the nonsingular Fourier coefficients of the two objects is discussed in sections 5 and 6. Section 7 contains a brief survey of results in higher dimensional cases as well as a second look at the simplest example of section 1. Some speculations about further developments are made in section 8.

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0.1. **Notation**

$\mathbb{Q}$, $\mathbb{A}$, $\mathbb{A}_f$, $\mathbb{A}^\times$ denote the rational numbers, the adèles, the finite adèles, and the idèles of $\mathbb{Q}$ respectively.
$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$, and, for any $\mathbb{Z}$–module M, $\hat{M} = M \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.
$\mathbb{F}_q$ denotes the finite field with q elements, and $\bar{\mathbb{F}}_q$ denotes an algebraic closure of it.
$(\ ,\ )_p$ (resp. $(\ ,\ )_{\mathbb{A}}$) denotes the quadratic Hilbert symbol for $\mathbb{Q}_p$ (resp. $\mathbb{A}$).
ψ is a fixed nontrivial character of $\mathbb{A}/\mathbb{Q}$.
$\mathfrak{H}_g = \{\tau = u + iv \in \mathrm{Sym}_g(\mathbb{C}) \mid v > 0\}$ is the Siegel space of genus g;
$\mathfrak{H} = \mathfrak{H}_1$ is the upper half–plane.
$e(x) = e^{2\pi i x}$
$\mathrm{Sym}_n(R) = \{x \in M_n(R) \mid {}^t x = x\} =$ the space of $n \times n$ symmetrix matrices.

$$\mathrm{diag}(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{Sym}_n(R).$$

1. TWO EXAMPLES

To fix ideas, it may be useful to consider two examples in classical language. The first of these illustrates the construction of generating functions for curves on a complex surface. More precisely, it gives a compact quotient version of the Hirzebruch–Zagier generating function for curves on a Hilbert–Blumenthal surface and a similar generating function for 0–cycles on such a surface. The second example illustrates the arithmetic case where cycles on moduli spaces for abelian varieties are defined by imposing extra endomorphisms. The example involves CM elliptic curves and the generating function is identified as the central derivative of an Eisenstein series of weight 1.

1.1. The case of a complex surface.

The results of this section are special cases of joint work with John Millson [15], [16], [17]. Let V, $(\ ,\ )$ be a 4–dimensional rational vector space with a symmetric bilinear form of signature $(2, 2)$. Let $Q(x) = \frac{1}{2}(x, x)$ be the associated quadratic form. Fix a lattice L in V on which the form is $\mathbb{Z}$–valued and let

$$\Gamma \subset \{\gamma \in SO(V)(\mathbb{Q})^+ \mid \gamma L = L\}$$

be a subgroup of finite index, where $SO(V)(\mathbb{R})^+$ is the identity component of the Lie group $SO(V)(\mathbb{R})$ and $SO(V)(\mathbb{Q})^+ = SO(V)(\mathbb{Q}) \cap SO(V)(\mathbb{R})^+$. The space D of negative 2–planes in $V(\mathbb{R})$ is isomorphic to the product $\mathfrak{H} \times \mathfrak{H}$ of two copies of the upper half–plane $\mathfrak{H}$, and the quotient $S = \Gamma\backslash D$ is (the complex points of) a quasi–projective variety. Now assume that the space V is anisotropic so that S is projective. This assumption eliminates complications coming from the compactification of the cusps, which are a significant issue in the Hilbert–Blumenthal case considered by Hirzebruch and Zagier.

For a vector $x \in V(\mathbb{Q})$ with $Q(x) > 0$, let

$$D_x = \{z \in D \mid z \perp x\}$$

be the set of negative 2–planes orthogonal to x, so that $D_x \subset D$ is isomorphic to an embedded upper half–plane $\mathfrak{H} \subset \mathfrak{H} \times \mathfrak{H}$. The image $Z(x, \Gamma)$ of D_x in the quotient S is a compact curve, the image of the quotient $\Gamma_x\backslash D_x$, where Γ_x is the stabilizer of x. Note that the curve $Z(x, \Gamma)$ depends only on the Γ orbit of x. Associated to a positive integer

t, there is a finite sum of such curves

$$Z(t,L) = \sum_{\substack{x\in L,\ Q(x)=t\\ \bmod \Gamma}} Z(x,\Gamma), \tag{1.1}$$

parametrized by the Γ–orbits in the set of lattice vectors of length t. This is the analogue of the Hirzebruch–Zagier curve T_N on a Hilbert–Blumenthal surface [11]. Let $[Z(t,L)] \in H^2(S,\mathbb{Q})$ be the cohomology class of $Z(t,L)$, and let $[Z(0,L)] \in H^2(S,\mathbb{Q})$ be the cohomology class of the invariant form*

$$\omega = -\frac{1}{4\pi}\big(\operatorname{Im}(z_1)^{-2}\,dz_1\wedge d\bar z_1 + \operatorname{Im}(z_2)^{-2}\,dz_2\wedge d\bar z_2\big) \tag{1.2}$$

on $\mathfrak{H}\times\mathfrak{H}$. As in [11], one can form a generating function

$$\phi_1(\tau,L) = [Z(0,L)] + \sum_{t\in\mathbb{Z}_{>0}} [Z(t,L)]\,q^t, \tag{1.3}$$

where $q = e(\tau)$, for an auxillary variable $\tau = u+iv \in \mathfrak{H}$. The analogue of the result of Hirzebruch–Zagier [11] is a special case of [15], [16], [17], see Theorem 3.1 below.

Theorem 1.1. — *The generating function $\phi_1(\tau,L)$ is an elliptic modular form of weight 2, valued in $H^2(S,\mathbb{C})$.*

Taking the intersection product with the cohomology class of a arbitrary curve C on S one obtains:

Corollary 1.2. — *For a curve C on S, the generating function for intersection numbers*

$$\phi_1(\tau,L)\cdot[C] = \operatorname{vol}(C) + \sum_{t>0}[Z(t,L)]\cdot[C]\,q^t$$

is a modular form of weight 2. Here $\operatorname{vol}(C) = \int_C \omega$.

One can also define a generating functions for 0–cycles on S as follows. For a pair of vectors $x = [x_1,x_2] \in V(\mathbb{Q})^2$ with matrix of inner products $Q(x) = \frac{1}{2}((x_i,x_j))_{i,j} \in \operatorname{Sym}_2(\mathbb{Q})$, there is an associated cycle $D_x = D_{x_1}\cap D_{x_2} \subset D$. If $Q(x)$ is nonsingular, then D_x is a point when $Q(x)$ is positive definite and is empty otherwise. If $Q(x)$ has rank 1, then $D_x = D_{x_1} = D_{x_2}$ is a curve when $Q(x)\geq 0$ (the components x_1 and x_2 are colinear since V is anisotropic) and is empty otherwise. Finally, if $Q(x) = 0$, then $x = 0$ and $D_x = D$. Let $Z(x,\Gamma)$ be the image of D_x in the quotient S; again, this depends only on

*This is twice the form used in [11], p.104, since the $Z(t,L)$'s of (1.1) are twice the corresponding cycles in [11].

the Γ–orbit of x. For $T \in \mathrm{Sym}_2(\mathbb{Z})$, let

$$Z(T,L) = \sum_{\substack{x \in L^2,\ Q(x)=T \\ \bmod \Gamma}} Z(x,\Gamma). \tag{1.4}$$

If $T > 0$ is positive definite, $Z(T,L)$ is either empty or a finite sum of points on S. If $T \geq 0$, has rank 1, then $Z(T,L)$ is either empty or a finite sum of curves on S, and if $T = 0$, then $Z(0,L) = S$. Let $[Z(T,L)] \in H^{2r(T)}(S,\mathbb{Q})$ be the cohomology class of $Z(T,L)$, where $r(T) = \mathrm{rank}(T)$. The generating function in this case is

$$\phi_2(\tau,L) = \sum_{T \in \mathrm{Sym}_2(Z)_{\geq 0}} [Z(T,L)] \cup [\omega]^{2-r(T)}\, q^T, \tag{1.5}$$

where $\tau \in \mathfrak{H}_2$, the Siegel space of genus 2, and $q^T = e(\mathrm{tr}(T\tau))$. Note that the terms for singular T's are obtained by shifting by suitable powers of $[\omega]$. The coefficients of this generating function lie in $H^4(S,\mathbb{C})$, and [15], [16], [17] yield the following result.

Theorem 1.3. — *The generating function $\phi_2(\tau,L)$ is a Siegel modular form of weight 2 and genus 2 valued in $H^4(S,\mathbb{C})$.*

Applying the degree map $H^4(S,\mathbb{C}) \to \mathbb{C}$, one obtains a scalar valued Siegel modular form.

Corollary 1.4. — *The generating function*

$$\deg(\phi_2(\tau,L)) = \mathrm{vol}(S) + \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Z})_{\geq 0} \\ r(T)=1}} \mathrm{vol}(Z(T,L))\, q^T + \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}} \deg(Z(T,L))\, q^T$$

is a Siegel modular form of weight 2 and genus 2.

In particular, the positive definite Fourier coefficients of $\deg(\phi_2(\tau,L))$ are the degrees of the 0–cycles $Z(T,L)$ on the surface S. The volumes of curves on S are taken with respect to the restriction of the invariant (1,1)–form ω of (1.2) and

$$\mathrm{vol}(S) = \int_S \omega^2.$$

Theorems 1.1 and 1.3 are proved by constructing a theta function $\theta_1(\tau,L)$ for $\tau \in \mathfrak{H}$, resp. $\theta_2(\tau,L)$ for $\tau \in \mathfrak{H}_2$, valued in *closed* $(1,1)$–forms, resp. closed $(2,2)$–forms on S. The generating function is the cohomology class of this theta function, i.e., $\phi_i(\tau,L) = [\theta_i(\tau,L)]$ for $i = 1$, 2, and hence is modular. In addition, the generating function of Corollary 1.2 resp. Corollary 1.4 is obtained as the integral of $\theta_1(\tau,L)$ over the curve C, resp. $\theta_2(\tau,L)$ over S. For suitable Γ's, this last integral over S is a constant multiple of the group theoretic integral of the theta function which occurs in the Siegel–Weil formula, and hence coincides with a special value of a Siegel–Eisenstein series of genus 2 at the point $s = \frac{1}{2}$, [13] and Proposition 3.2 below.

Corollary 1.5. — *There is a nonzero constant c such that*

$$\deg(\phi_2(\tau, L)) = c \cdot E(\tau, \frac{1}{2}, L)$$

for a suitable Siegel–Eisenstein series $E(\tau, s, L)$ of genus 2 and weight 2.

In the case in which S is a product of modular curves, such a geometric interpretation of the Fourier coefficients of a Siegel–Eisenstein series was observed by Gross and Keating [6].

1.2. **Interlude.**

More general results of this type, [15], [16], [17], and [13], concerning generating functions for cycles of codimension n on Shimura varieties X defined by rational quadratic forms of signature $(m-2, 2)$ are discussed in section 3 below. Note that the complex dimension of X is $m-2$. The main aim of this report is to explain the first steps in establishing a similar theory in the arithmetic case. Roughly speaking, this means the following. First, one wants to consider cycles of codimension n on integral models $\mathfrak{X}$ of the Shimura varieties X and their classes in the arithmetic Chow groups $\widehat{CH}^r(\mathfrak{X})$, [4]. For $2 \le m \le 4$, integral models can be obtained as moduli spaces of suitable abelian varieties, and cycles can be defined as the loci where the abelian varieties in question are equipped with additional endomorphisms of a certain type. The theta functions valued in the deRham complex are not available in this context, and so it is not clear at present how to define generating functions for cycles of arbitrary codimension. If one considers 0–cycles, however, one may apply the arithmetic degree map $\widehat{\deg} : \widehat{CH}^{m-1}(\mathfrak{X}) \to \mathbb{R}$. One may then look for an analogue of Corollary 1.5 and Proposition 3.2, where the Siegel–Eisenstein series will now have genus $n = m-1$ and the critical point will be $s_0 = \frac{m}{2} - \frac{n+1}{2} = 0$, i.e., the central point on the real axis for the functional equation of the Eisenstein series. Moreover, it turns out that the 'correct' Eisenstein series of genus n and weight $\frac{n+1}{2}$ will have a zero at this point, so that one should look at its first derivative $E'(\tau, 0, L)$. The case of the arithmetic surfaces attached to Shimura curves is discussed at length below in sections 4–6, and the analogue of Corollary 1.5 is given in Conjecture 4.7. The simplest example, however, occurs for $m = 2$. This case involves only classical objects, e.g., elliptic curves with complex multiplication and Eisenstein series of weight 1 for SL_2.

1.3. **Derivatives of Eisenstein series of weight 1.**

This section describes the simplest case in which the derivative at $s = 0$ of an Eisenstein series can be identified with a generating function for the arithmetic degrees of 0–cycles on a moduli scheme. These results are joint work with Michael Rapoport and Tonghai Yang [24]. Fix a prime $d \equiv 3 \mod 4$ with $d > 3$, and let $\boldsymbol{k} = \mathbb{Q}(\sqrt{-d})$ be the corresponding imaginary quadratic field with ring of integers $\mathcal{O}_{\boldsymbol{k}}$ and associated Dirichlet character χ_d. Let

$$\Lambda(s,\chi_d) = \pi^{-\frac{s+1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)\,L(s,\chi_d)$$

where $L(s,\chi_d)$ is the Dirichlet L–series of χ_d. For a nonzero integer $n\in\mathbb{Z}$, let $\rho(n)$ be the number of ideals in $\mathcal{O}_{\boldsymbol{k}}$ of norm n. For example, note that, for a prime p, $\rho(p)=0$ if p is inert in $\boldsymbol{k}$, $\rho(p)=2$ is p is split in $\boldsymbol{k}$, and $\rho(d)=1$.

There are two normalized Eisenstein series of weight 1 for $\Gamma=\mathrm{SL}_2(\mathbb{Z})$ attached to $\boldsymbol{k}$. For $\tau=u+iv\in\mathfrak{H}$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, let

$$E^*_{\pm}(\tau,s) = v^{\frac{s}{2}}\,d^{\frac{s+1}{2}}\,\Lambda(s+1,\chi_d)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}(c\tau+d)^{-1}|c\tau+d|^{-s}\,\Phi_{\pm}(\gamma),$$

where, for $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma$,

$$\Phi_{\pm}(\gamma)=\begin{cases}\chi_d(a) & \text{if } c\equiv 0 \mod (d)\\ \pm i\,d^{-\frac12}\,\chi_d(c) & \text{if } c \text{ is prime to } d.\end{cases}$$

The entire analytic continuation of these series in s satisfy the functional equations

$$E^*_{\pm}(\tau,-s)=\pm E^*_{\pm}(\tau,s).$$

A general construction of series of this sort is described in section 2 below. A case of the Siegel–Weil formula due to Hecke describes the value at $s=0$ of the even series:

$$E^*_{+}(\tau,0)=2h_{\boldsymbol{k}}+4\sum_{t=1}^{\infty}\rho(t)\,q^t = 2\sum_{\mathfrak{a}}\vartheta(\tau,\mathfrak{a}),$$

where $h_{\boldsymbol{k}}$ is the class number of $\boldsymbol{k}$, the ideal $\mathfrak{a}$ runs over representatives of the ideal classes of $\boldsymbol{k}$, and $\theta(\tau,\mathfrak{a})$ is the binary theta series attached to $\mathfrak{a}$. For the odd series, $E^*_{-}(\tau,0)=0$, and the function of interest is the (negative of the) leading term

$$\phi(\tau,d) = -\frac{\partial}{\partial s}\{E^*_{-}(\tau,s)\}\big|_{s=0}.$$

THEOREM 1.6. — *The modular form $\phi(\tau,d)$ of weight 1 has Fourier expansion*

$$\phi(\tau,d)=a_0(v)+\sum_{t<0}a_t(v)\,q^t+\sum_{t=1}^{\infty}a_t\,q^t,$$

where, for $t>0$,

$$a_t = 2\log(d)\,(\mathrm{ord}_d(t)+1)\,\rho(t)+2\sum_{p\neq d}\log(p)\,(\mathrm{ord}_p(t)+1)\,\rho(t/p),$$

where the sum runs over primes p inert in $\boldsymbol{k}$,

$$a_0 = -h_{\boldsymbol{k}}\left(\log(d)+\log(v)+2\frac{\Lambda'(1,\chi_d)}{\Lambda(1,\chi_d)}\right),$$

and, for $t < 0$,

$$a_t(v) = -2\mathrm{Ei}(-4\pi|t|v)\,\rho(-t).$$

Here

$$-\mathrm{Ei}(-x) = \int_1^\infty u^{-1}\,e^{-ux}\,du$$

is the exponential integral.

The idea now is to give an interpretation of these coefficients as the degrees in the sense of arithmetic geometry of certain 0–cycles on the the coarse moduli scheme $\mathcal{M}$ for elliptic curves (E, ι) with complex multiplication $\iota : \mathcal{O}_{\boldsymbol{k}} \hookrightarrow \mathrm{End}(E)$ by $\mathcal{O}_{\boldsymbol{k}}$. This scheme over $\mathcal{O}_{\boldsymbol{k}}$ can be identified with $\mathrm{Spec}(\mathcal{O}_H)$, where H is the Hilbert class field of $\boldsymbol{k}$. For such a curve (E, ι), the space of special endomorphisms is the $\mathbb{Z}$-module

(1.6) $$V(E, \iota) = \{x \in \mathrm{End}(E) \mid x\,\iota(a) = \iota(\bar{a})\,x \text{ for all } a \in \mathcal{O}_{\boldsymbol{k}}\ \}.$$

This space has a $\mathbb{Z}$-valued quadratic form Q defined by $x^2 = -Q(x)\cdot id_E$. For $t \in \mathbb{Z}$, let $Z(t)$ be the coarse moduli scheme whose points over an algebraically closed field correspond to triples (E, ι, x) where $x \in V(E, \iota)$ with $Q(x) = t$. The scheme $Z(t) \to \mathcal{M}$ is the locus of (E, ι)'s with an extra multiplication, anticommuting with the action of $\mathcal{O}_{\boldsymbol{k}}$. Such extra endomorphisms can only exist for a supersingular curve E in characteristic p for a prime p which is not split in $\boldsymbol{k}$. Then $Z(t) = \mathrm{Spec}(R(t))$ where $R(t)$ is an Artin algebra in which p is nilpotent. Let

$$\widehat{\deg}(Z(t)) = \log|R(t)|.$$

The second main result of [24] is a calculation of this degree; this calculation depends in an essential way on the results of Gross [5].

Theorem 1.7. — *For $t > 0$,*

$$\widehat{\deg}(Z(t)) = a_t,$$

and hence

$$\phi(\tau, d) = \sum_{t>0} \widehat{\deg}(Z(t))\,q^t + a_0(v) + \sum_{t<0} a_t(v)\,q^t.$$

A sort of geometric interpretation of the remaining terms will be discussed in section 7 below, cf. also [24].

2. CENTRAL DERIVATIVES OF SIEGEL–EISENSTEIN SERIES

A general construction of ‘even’ and ‘odd’ Siegel–Eisenstein series is best described in representation theoretic language, and is connected with the Siegel–Weil formula at the central critical point. These series, which will be called coherent and incoherent series respectively, for reasons explained below, will have integral or half–integral weight depending on the parity of the dimension of the relevant quadratic spaces. Hence it is necessary to work on the metaplectic group.

Let $G = \mathrm{Sp}_{2n}$ be the symplectic group of rank n over $\mathbb{Q}$ and let $P = MN$ be the maximal parabolic with Levi factor $M \simeq \mathrm{GL}_n$ and unipotent radical $N \simeq \mathrm{Sym}_n$. Let

$$G_{\mathbb{A}} = \begin{cases} \mathrm{M}p_{2n}(\mathbb{A}) & \text{if } n \text{ is even} \\ \mathrm{Sp}_{2n}(\mathbb{A}) & \text{if } n \text{ is odd,} \end{cases}$$

where $\mathrm{M}p_{2n}(\mathbb{A})$ is the twofold metaplectic cover of $\mathrm{Sp}_{2n}(\mathbb{A})$, and let $P_{\mathbb{A}}$ and $M_{\mathbb{A}}$ be the subgroups of $G_{\mathbb{A}}$ corresponding to P and M. Let $G_{\mathbb{Q}}$ be $\mathrm{Sp}_{2n}(\mathbb{Q})$ for n odd resp. the image of this group in $\mathrm{M}p_{2n}(\mathbb{A})$ under the canonical splitting, if n is even. For each place $p \le \infty$ of $\mathbb{Q}$, there are groups G_p, P_p and M_p, defined analogously.

A quadratic character χ of $\mathbb{A}^\times/\mathbb{Q}^\times$ determines a character $\chi = \chi^\psi$ of $M_{\mathbb{A}}$, trivial on $M_{\mathbb{Q}} = M_{\mathbb{A}} \cap G_{\mathbb{Q}}$, and for $s \in \mathbb{C}$, one has the degenerate principal series representation

$$I(s,\chi) = \mathrm{Ind}_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\chi|\ |^s)$$

of $G_{\mathbb{A}}$. (The character χ^ψ depends on the fixed choice of the nontrivial additive character ψ of $\mathbb{A}/\mathbb{Q}$ in the metaplectic case.)

For $\Phi(s) \in I(s,\chi)$, the Siegel-Eisenstein series is defined for $\mathrm{Re}(s) > \frac{n+1}{2}$ by

$$E(g,s,\Phi) = \sum_{\gamma \in P_{\mathbb{Q}} \backslash G_{\mathbb{Q}}} \Phi(\gamma g, s).$$

From the standard theory of Eisenstein series one knows that this function has a meromorphic analytic continuation to the whole s–plane and satisfies a functional equation relating s and $-s$. In addition, it has no poles on the line $\mathrm{Re}(s) = 0$ (unitary axis). In particular, there is an intertwining map

$$E(0) : I(0,\chi) \longrightarrow \mathcal{A}(G), \qquad \Phi(0) \mapsto E(0,\Phi) \tag{2.1}$$

from the degenerate principal series at $s = 0$ to the space $\mathcal{A}(G)$ of automorphic forms on $G_{\mathbb{A}}$.

The image and kernel of this map can be described in terms of representations associated to quadratic forms as follows.

A rational vector space V of dimension m with a nondegenerate quadratic form Q determines a quadratic character χ_V of $\mathbb{A}^\times/\mathbb{Q}^\times$ by

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det(Q))_{\mathbb{A}},$$

where $(\ ,\ )_{\mathbb{A}}$ is the global quadratic Hilbert symbol. For such a space (V,Q), there is a Weil representation ω_V of $G_{\mathbb{A}}$ on the Schwartz space $S(V(\mathbb{A})^n)$, determined by ψ. This gives rise to a $G_{\mathbb{A}}$– intertwining map

$$\lambda_V : S(V(\mathbb{A})^n) \longrightarrow I(s_0,\chi_V), \qquad \lambda_V(\varphi)(g) = \big(\omega_V(g)\varphi\big)(0),$$

where $s_0 = \frac{m}{2} - \frac{n+1}{2}$. Specializing to the case $\chi_V = \chi$ and $m = n+1$, one obtains an irreducible constituent $\Pi(V) = \lambda_V\big(S(V(\mathbb{A})^n)\big)$ of $I(0,\chi)$.

Similarly, for each place $p \le \infty$, there is an analogous local construction which yields an irreducible constituent $\Pi_p(V_p)$ of the local induced representation $I_p(0,\chi_{V_p})$ of G_p associated to each quadratic space V_p over $\mathbb{Q}_p$ of dimension $n+1$ and character χ_{V_p}. Then, for a global space V, one has

$$\Pi(V) \simeq \otimes'_{p\le\infty}\Pi_p(V_p) \subset I(0,\chi) = \otimes'_{p\le\infty} I_p(0,\chi_p)$$

where $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and the primes indicate the restricted tensor products. For a finite prime p, there are precisely two possible quadratic spaces V_p^+ and V_p^- over $\mathbb{Q}_p$, for a fixed n and character χ_p. They are distinguished by their Hasse invariants $\epsilon_p(V_p^{\pm}) = \pm 1$, and, in fact, [19],

$$I_p(0,\chi_p) = \Pi_p(V_p^+) \oplus \Pi_p(V_p^-).$$

For $p = \infty$, the quadratic spaces of dimension $n+1$ and character χ_∞ are determined by their signature, and fall into two groups according to their Hasse invariant. The local induced representation $I_\infty(0,\chi_\infty)$ is the direct sum of the corresponding $\Pi_\infty(V_\infty)$'s, [18]. For example, for $n = 4$ the quadratic spaces over $\mathbb{R}$ of signatures $(5,0)$, and $(1,4)$ have Hasse invariant $+1$, while that of signature $(3,2)$ has Hasse invariant -1, and

$$I_\infty(0,\chi_\infty) = \Pi(5,0) \oplus \Pi(3,2) \oplus \Pi(1,4),$$

in the obvious notation. Here $\chi_\infty = 1$.

If a collection of local quadratic spaces $\mathcal{C} = \{V_p\}$ is the set of localizations of a global space V, then the product formula for the Hasse invariants asserts that

$$\epsilon(\mathcal{C}) := \prod_{p\le\infty} \epsilon_p(V_p) = 1.$$

Such a collection and the Eisenstein series associated to it will be called *coherent*. On the other hand, the collection $\mathcal{C}$ of local quadratic spaces obtained by choosing one prime p_0 (e.g., $p_0 = \infty$) and switching the space V_{p_0} to a space V'_{p_0} with the opposite Hasse invariant has

$$\epsilon(\mathcal{C}) := \epsilon_{p_0}(V_{p_0}) \prod_{\substack{p\le\infty \\ p\ne p_0}} \epsilon_p(V_p) = -1,$$

so that such a collection cannot be the set of localizations of any global quadratic space. In this case, the collection $\mathcal{C}$ and the Eisenstein series associated to it will be called *incoherent*. The irreducible admissible representation

$$\Pi(\mathcal{C}) = \Pi_{p_0}(V'_{p_0}) \otimes \Big(\otimes_{\substack{p\le\infty \\ p\ne p_0}} \Pi_p(V_p)\Big)$$

of $G_{\mathbb{A}}$ is also a constituent of $I(0,\chi)$. Then there is a direct sum decomposition

$$I(0,\chi) = \Big(\bigoplus_V \Pi(V)\Big) \oplus \Big(\bigoplus_{\mathcal{C}} \Pi(\mathcal{C})\Big)$$

where V runs over all global quadratic spaces of dimension $n+1$ and character χ, and $\mathcal{C}$ runs over all incoherent collections, where $\epsilon(\mathcal{C}) = -1$, as just described. One then obtains a description of the kernel and image of the map $E(0)$ in terms of the $\Pi(V)$'s and $\Pi(\mathcal{C})$'s, [14], [20], [9].

THEOREM 2.1. — *(i)*

$$\ker(E(0)) = \bigoplus_{\epsilon(\mathcal{C})=-1} \Pi(\mathcal{C}).$$

(ii) Each automorphic representation $\Pi(V)$ *in the image*

$$\mathrm{Im}(E(0)) \simeq \bigoplus_{V} \Pi(V)$$

coincides with space of (regularized) theta integrals

$$I(g,\varphi) = \int_{O(V)(\mathbb{Q})\backslash O(V)(\mathbb{A})} \theta(g,h;\varphi)\,dh, \tag{2.2}$$

attached to the global quadratic space V. *Here, for* $\varphi \in S(V(\mathbb{A})^n)$, $g \in G_{\mathbb{A}}$ *and* $h \in O(V)(\mathbb{A})$,

$$\theta(g,h;\varphi) = \sum_{x\in V(\mathbb{Q})} (\omega(g)\varphi)(h^{-1}x).$$

The integral $I(g,\varphi)$ must be defined by a regularization procedure [20] whenever V is isotropic.

Part (ii) of Theorem 1.2 is essentially the Siegel–Weil formula in the present context, [20], [33]. Note that the theta functions involve global arithmetic, e.g., the number of solutions of diophantine equations of the form $Q(x) = T$ for $T \in Sym_n(\mathbb{Z})$ and $x \in L^n$ for a lattice $L \subset V(\mathbb{Q})$, whereas the Eisenstein series is constructed from local data.

PROBLEM. — What is the arithmetic content of the first derivative $E'(g,0,\Phi)$ when $\Phi \in \Pi(\mathcal{C})$ with $\epsilon(\mathcal{C}) = -1$?

REMARK 2.2. — The results to be discussed in the remainder of the talk suggest an answer to this question, at least for the following particular case:
Let V be a rational quadratic space, as above, with signature $(n-1,2)$, and let $\mathcal{C}$ be the collection of local quadratic spaces obtained from $\{V_p\}$ by replacing V_∞ by the space V'_∞ of signature $(n+1,0)$. Let $\varphi'_\infty \in S((V'_\infty)^n)$ be the Gaussian $\varphi'_\infty(x) = \exp(-\pi Q'(x))$, and let $\Phi_\infty^{(n+1)/2}(s) \in I_\infty(s,\chi_\infty)$ be the corresponding section; it is the unique eigenvector for K_∞ of weight $\frac{n+1}{2}$. For any $\varphi \in S(V(\mathbb{A}_f)^n)$, with corresponding section $\Phi_f(s) \in I_f(s,\chi_f)$,

$$\Phi(0) = \Phi_\infty^{\frac{n+1}{2}}(0) \otimes \Phi_f(0) \in \Pi(\mathcal{C}).$$

Then the central derivative $E'(g,0,\Phi)$ should be related to a generating function for the degrees of 0–cycles on an integral model of the Shimura variety associated to the group $\mathrm{GSpin}(V)$.

Remark 2.3. — For comparison with the generating functions considered below, it is convenient to write the Eisenstein series in more classical language. For $\tau = u + iv \in \mathfrak{H}_n$, the Siegel space of genus n, and for $\varphi \in S(V(\mathbb{A}_f)^n)$, let

$$\begin{aligned} E(\tau, s, \varphi) :&= \det(v)^{-\frac{1}{2}(s+\frac{n+1}{2})} E(g_\tau, s, \Phi) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \det(c\tau + d)^{-\frac{n+1}{2}} |\det(c\tau + d)|^{-s} (\omega_f(\gamma)\varphi)(0), \end{aligned} \tag{2.3}$$

where $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$, $\Gamma_\infty = \Gamma \cap P(\mathbb{Q})$, ω_f denotes the action of $G_{\mathbb{A}_f}$ on $S(V(\mathbb{A}_f)^n)$ via the Weil representation, and $g_\tau = \begin{pmatrix} v^{\frac{1}{2}} & uv^{-\frac{1}{2}} \\ & v^{-\frac{1}{2}} \end{pmatrix}$.

3. GENERATING FUNCTIONS IN THE GEOMETRIC CASE

This section describes the results of [15], [16], [17], and [13] on generating functions for the cohomology classes of special cycles in the case of $O(n-1, 2)$, special cases of which were described in section 1.1. These results suggest what one might hope to prove in the arithmetic case.

For a rational quadratic space V of signature $(n-1, 2)$, let $H = \mathrm{GSpin}(V)$ and let D be the space of oriented negative 2–planes in $V(\mathbb{R})$. The space D is isomorphic to two copies of a bounded domain of type IV in $\mathbb{C}^{n-1}$, [10], [28], and the group $H(\mathbb{R})$ acts on it by holomorphic automorphisms. For a compact open subgroup $K \subset H(\mathbb{A}_f)$, the orbit space

$$X_K(\mathbb{C}) = H(\mathbb{Q}) \backslash D \times H(\mathbb{A}_f)/K$$

is the set of complex points of a quasi–projective variety X_K defined over $\mathbb{Q}$, the Shimura variety attached to H, D and K. The variety X_K is in fact projective if V is anisotropic and smooth if K is sufficiently small.

Fix $r \in \mathbb{Z}$ with $1 \le r \le n-1$. For $x \in V(\mathbb{R})^r$, let

$$D_x = \{z \in D \mid x \perp z\}$$

be the set of z's which are orthogonal to all components of x. If the matrix $T = Q(x) = \frac{1}{2}((x_i, x_j))$ is positive definite, i.e., if the components of x span a positive r–plane, then D_x has complex codimension r in D. If, in addition, $x \in V(\mathbb{Q})^r$, then $x^\perp$ is a rational quadratic space of signature $(n-r-1, 2)$, the stabilizer H_x of x in H is isomorphic to $\mathrm{GSpin}(x^\perp)$, and there is a natural map of Shimura varieties

$$Z(x, K) : H_x(\mathbb{Q}) \backslash D_x \times H_x(\mathbb{A}_f)/H_x(\mathbb{A}_f) \cap K \longrightarrow X_K(\mathbb{C}),$$

giving a cycle of codimension r on $X_K(\mathbb{C})$. Given a function $\varphi \in S(V(\mathbb{A}_f)^r)^K$ and $T \in \mathrm{Sym}_r(\mathbb{Q})_{>0}$, there is a weighted linear combination $Z(T, \varphi)$ of such cycles [13], and the resulting cohomology classes $[Z(T, \varphi)] \in H^{2r}(X_K)$, where $H^*(X_K)$ is the cohomology of $X_K(\mathbb{C})$ with complex coefficients [17]. Examples are given by (1.1) and (1.4) in section 1.1, where φ is the characteristic function of $(\hat{L})^r \subset V(\mathbb{A}_f)^r$, for $\hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. If T is only positive semi–definite with $\mathrm{rank}(T) = r(T)$, the associated cycles have codimension $r(T)$ and their cohomology classes lie in $H^{2r(T)}(X_K)$.

To form a generating function, let $\tau = u + iv \in \mathfrak{H}_r$, the Siegel space of genus r, and let $q^T = e(\mathrm{tr}(\tau T))$.

THEOREM 3.1. — [17]. *For $\varphi \in S(V(\mathbb{A}_f)^r)^K$, and for a suitable choice of a Kähler form ω on $X_K(\mathbb{C})$, the generating series*

$$\phi_r(\tau, \varphi) = \sum_{T \in \mathrm{Sym}_r(\mathbb{Q})_{\geq 0}} [Z(T, \varphi)] \cdot [\omega]^{r - r(T)}\, q^T$$

is the q–expansion of a holomorphic Siegel modular form of weight $\frac{n+1}{2}$ and genus r valued in $H^{2r}(X_K)$.

The proof of this result depends on a construction of a theta function taking values in the space of closed $2r$ forms on $X_K(\mathbb{C})$. This method is quite general and applies to the locally symmetric spaces associated to $O(p,q)$, $U(p,q)$ and $Sp(p,q)$, cf. [15], [16], and [17].

Specializing to the case where V is anisotropic and $r = n-1$ and applying $\deg : H^{2(n-1)}(X_K) \to \mathbb{C}$, one obtains a holomorphic Siegel modular form of genus $n-1$ and weight $\frac{n+1}{2}$,

$$\phi_{\deg}(\tau, \varphi) = \deg\big(\phi_{n-1}(\tau, \varphi)\big) = \sum_{T \in\ \mathrm{Sym}_{n-1}(\mathbb{Q})_{\geq 0}} \deg\big(\, [Z(T, \varphi)] \cdot [\omega]^{n-1-r(T)}\,\big)\, q^T. \tag{3.1}$$

By the Siegel–Weil formula [20], this form is, in turn, a value of an Eisenstein series. More precisely, let

$$G'_{\mathbb{A}} = \begin{cases} \mathrm{M}p_{2(n-1)}(\mathbb{A}) & \text{if } n \text{ is even} \\ \mathrm{Sp}_{2(n-1)}(\mathbb{A}) & \text{if } n \text{ is odd.} \end{cases}$$

The machinery described in section 2 carries over, except that the map $\lambda_V : S(V(\mathbb{A})) \longrightarrow I(s_0, \chi)$ now takes values in the induced representation at the point $s_0 = \frac{1}{2}$. Let $E(g, s, \Phi)$ be the Siegel–Eisenstein series associated to the section $\Phi_\infty^{\frac{n+1}{2}}(s) \otimes \Phi_f(s)$, where $\Phi_f(\frac{1}{2}) = \lambda_V(\varphi)$ for the weight function $\varphi \in S(V(\mathbb{A}_f)^{n-1})^K$. The Siegel–Weil formula for the anisotropic space V then implies the following generalization of Corollary 1.5 above [13]:

PROPOSITION 3.2. — *For $\tau = u+iv \in \mathfrak{H}_{n-1}$, and for a weight function $\varphi \in S(V(\mathbb{A}_f)^{n-1})^K$,*

$$\phi_{\deg}(\tau, \varphi) = \mathrm{vol}(X_K(\mathbb{C}))\, \det(v)^{-\frac{n+1}{4}} E(g_\tau, \frac{1}{2}, \Phi).$$

where

$$\mathrm{vol}(X_K(\mathbb{C})) = \int_{X_K(\mathbb{C})} \omega^{n-1}.$$

In particular, the positive definite Fourier coefficients of this Eisenstein series are the degrees of the (weighted) 0–cycles $Z(T, \varphi)$ on X_K.

4. GENERATING FUNCTIONS FOR ARITHMETIC 0–CYCLES: THE CASE OF SHIMURA CURVES

This section describes the generating function for the arithmetic degrees of 0–cycles on the arithmetic surface $\mathfrak{X}$ associated to a Shimura curve X over $\mathbb{Q}$. The series of interest will be analogous to the series $\deg(\phi_2(\tau, L))$ for a complex surface given in Corollary 1.4 and will have the form

$$\hat{\phi}_{\deg}(\tau) = \sum_{T \in \mathrm{Sym}_2(\mathbb{Z})} \widehat{\deg}(\widehat{\mathfrak{Z}}(T, v))\, q^T, \tag{4.1}$$

where $\tau = u + iv \in \mathfrak{H}_2$ and the $\widehat{\mathfrak{Z}}(T, v)$'s are certain classes in $\widehat{CH}^2(\mathfrak{X})$, the top arithmetic Chow group of the arithmetic surface $\mathfrak{X}$, [4]. As in the second example of section 1, the definition of the relevant cycles will depend on a modular interpretation. To give a more detailed explanation, it is convenient to begin with the geometric situation of section 3.

Fix an indefinite division quaternion algebra B over $\mathbb{Q}$. The space $V = \{x \in B \mid \mathrm{tr}(x) = 0\}$, equipped with the restriction of the reduced norm of B, is a three dimensional quadratic space over $\mathbb{Q}$ of signature $(1, 2)$. The group $H = B^\times \simeq \mathrm{GSpin}(V)$ acts on V by conjugation. The Shimura curve X_K associated to a compact open subgroup $K \subset H(\mathbb{A}_f)$, is a moduli space for abelian surfaces with $\mathcal{O}_B$–action and level structure, where $\mathcal{O}_B$ is a maximal order in B. For example, suppose that $K = \left(\mathcal{O}_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}\right)^\times$. Then $X = X_K$ is the (coarse) moduli scheme over $\mathbb{Q}$ for pairs (A, ι) consisting of an abelian surface A together with an action $\iota : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$ of $\mathcal{O}_B$, and $X(\mathbb{C}) \simeq \Gamma_B \backslash \mathfrak{H}$, where $\Gamma_B = (\mathcal{O}_B)_+^\times$ is the group of norm 1 units on $\mathcal{O}_B$.

The 0–cycles on X defined in section 3 can also be defined by specifying additional endomorphisms.

Definition 4.1. — *For an abelian surface (A, ι) with $\mathcal{O}_B$–action, the space of special endomorphisms is*

$$V(A, \iota) = \{x \in \mathrm{End}(A) \mid \iota(b)x = x\iota(b) \quad \textit{and} \quad \mathrm{tr}(x) = 0\}.$$

This space is equipped with a $\mathbb{Z}$–valued quadratic form defined, for a special endomorphism $x \in V(A, \iota)$, by $x^2 = -Q(x) \cdot 1_A$.

Definition 4.2. — *For $t \in \mathbb{Z}_{>0}$, the special cycle $Z(t)$ is the locus of triples (A, ι, x) where $x \in V(A, \iota)$ with $Q(x) = t$.*

In fact, one then has $Z(t) = Z(t, \varphi)$ where $Z(t, \varphi)$ is the 0–cycle on $X = X_K$ defined in section 3 for the weight function $\varphi \in S(V(\mathbb{A}_f))^K$, the characteristic function of the set $V(\mathbb{A}_f) \cap \left(\mathcal{O}_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}\right)$.

The 0–cycle $Z(t)$ on X is rational over $\mathbb{Q}$ and is analogous to the set of CM points on the modular curve associated to the imaginary quadratic field $\mathbb{Q}(\sqrt{-t})$. For $\tau = u + iv \in \mathfrak{H}$, the upper halfplane and $q = e(\tau)$, the degree generating function (3.1) in the present case

$$\phi_{\deg}(\tau) = \mathrm{vol}(X) + \sum_{t>0} \deg(Z(t))\, q^t \tag{4.2}$$

is the value at $s = \frac{1}{2}$ of an Eisenstein series of weight $\frac{3}{2}$ for a congruence subgroup of $SL_2(\mathbb{Z})$, cf. Proposition 3.2. Here $\mathrm{vol}(X)$ is the volume of $X(\mathbb{C}) \simeq \Gamma_B\backslash\mathfrak{H}$ with respect to $-\frac{1}{2\pi}\, y^{-2}\, dx \wedge dy$.

Now consider the arithmetic case. Let $\mathfrak{X}$ be the coarse moduli scheme over $S = \mathrm{Spec}(\mathbb{Z})$ for abelian surfaces (A, ι) with $\mathcal{O}_B$–action satisfying Drinfeld's 'special' condition [2]. The arithmetic surface $\mathfrak{X}$ has generic fiber $\mathfrak{X}_{\mathbb{Q}} = X$, the canonical model of the Shimura curve; $\mathfrak{X}$ has good reduction at all primes $p \nmid D(B)$, where $D(B)$ is the product of the primes p such that $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra. The points of $\mathfrak{X}$ over an algebraically closed field $\boldsymbol{k}$ correspond to isomorphism classes of (A, ι)'s over $\boldsymbol{k}$.

For an abelian scheme (A, ι) over a connected base, the space of special endomorphisms $V(A, \iota)$ with its $\mathbb{Z}$–valued quadratic form Q is defined as before.

Definition 4.3. — *For $T \in \mathrm{Sym}_2(\mathbb{Z})_{>0}$, the arithmetic special cycle $\mathfrak{Z}(T)$ is the locus of triples (A, ι, x), where $x = [x_1, x_2] \in V(A, \iota)^2$ is a pair of special endomorphisms $x_i \in V(A, \iota)$ with $Q(x) = \frac{1}{2}((x_i, x_j))_{i,j} = T$.*

Proposition 4.4. — [23]. *The cycle $\mathfrak{Z}(T)$ is either empty or is supported in the set of supersingular points of a single fiber $\mathfrak{X}_p$, where p is determined by T, as described in Lemma 4.5 below. If $p \nmid D(B)$, or if $p|D(B)$ but $p \nmid T$, then $\mathfrak{Z}(T)$ is a 0–cycle on $\mathfrak{X}_p$. If $p|D(B)$ and $p|T$, then $\mathfrak{Z}(T)$ is a union, with multiplicities, of components of the fiber $\mathfrak{X}_p$ (and some additional embedded components).*

Sketch of proof. — The proof of this result illustrates the way in which the the basic structure of the cycle $\mathfrak{Z}(T)$ is determined by the space $V(A, \iota)$. Observe that, for a geometric point (A, ι) of $\mathfrak{X}$ and viewing A up to isogeny,

$$\mathrm{End}^0(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \begin{cases} B & \text{if } A \text{ is simple,} \\ M_2(C) & \text{if } A \simeq E \times E \text{ with } E \text{ ordinary,} \\ M_2(\mathbb{B}) & \text{if } A \simeq E \times E \text{ with } E \text{ supersingular,} \end{cases}$$

for an elliptic curve E. In the second case, $\mathrm{End}^0(E) \simeq C$ is an imaginary quadratic field which splits B, and, in the third case, which occurs only in characteristic $p > 0$, $\mathrm{End}^0(E) \simeq \mathbb{B}$ is the quaternion algebra over $\mathbb{Q}$ ramified at ∞ and p. In this case, write $M_2(\mathbb{B}) \simeq B \otimes B^{(p)}$ where $B^{(p)}$ is the definite quaternion algebra over $\mathbb{Q}$ whose local invariants differ from those of B precisely at ∞ and p. Then, in the three cases,

$\mathrm{End}^0(A,\iota) \simeq \mathbb{Q}$, C, and $B^{(p)}$ respectively, and

$$V^0(A,\iota) = V(A,\iota)\otimes_{\mathbb{Z}}\mathbb{Q} \simeq \begin{cases} 0 & \\ C^0 & = \{x\in C \mid \mathrm{tr}(x)=0\}, \\ V^{(p)} & = \{x\in B^{(p)} \mid \mathrm{tr}(x)=0\}. \end{cases}$$

It follows that $\mathfrak{Z}(T)_{\mathbb{Q}} = \emptyset$, i.e., the cycle $\mathfrak{Z}(T)$ has no points in characteristic 0. If $\mathfrak{Z}(T)$ meets the fiber $\mathfrak{X}_p$ at p, then $\mathfrak{Z}(T)\cap\mathfrak{X}_p$ is contained in the supersingular locus of $\mathfrak{X}_p$, and the rational quadratic space $V^{(p)}$ represents T, i.e., there exists a pair of vectors $x=[x_1,x_2]\in V^{(p)}(\mathbb{Q})$ such that $Q^{(p)}(x)=T$. This last condition implies that $\mathfrak{Z}(T)$ is supported in a single fiber, due to the following simple observation.

Lemma 4.5. — (i) The quadratic spaces $V^{(p)}$ have the same determinant as V, i.e., $\det(V^{(p)}) = \det(V) \in \mathbb{Q}^\times/\mathbb{Q}^{\times,2}$.
(ii) For a given $T\in\mathrm{Sym}_2(\mathbb{Q})$ with $\det(T)\neq 0$, there is a unique three dimensional rational quadratic space V_T with $\det(V_T)=\det(V)$ which represents T; the quadratic form on V_T has matrix

$$\begin{pmatrix} T & \\ & \det(V)/\det(T)\end{pmatrix}. \tag{4.3}$$

Thus, if V_T is not isomorphic to any of the $V^{(p)}$'s then $\mathfrak{Z}(T)$ is empty, while if $V_T\simeq V^{(p)}$ for some p, then $\mathfrak{Z}(T)$ is supported in the supersingular locus of $\mathfrak{X}_p$.

Definition 4.6. — *A matrix $T\in\mathrm{Sym}_2(\mathbb{Z})_{>0}$ will be called* irregular *if $V_T\simeq V^{(p)}$ where $p|D(B)$ and $p|T$. Otherwise T will be called* regular.

The assertions about the irregular case require a more detailed argument, using the p-adic uniformization of the formal completion of $\mathfrak{X}_p$, [23]. ☐

The arithmetic Chow group $\widehat{CH}^2(\mathfrak{X})$ is generated by pairs $(\mathfrak{Z},g)$ where $\mathfrak{Z}$ is a 0–cycle on $\mathfrak{X}$ and g is a smooth $(1,1)$–form on $X(\mathbb{C})$, modulo a suitable equivalence, [4], [32]. There is a degree map $\widehat{\deg}:\widehat{CH}^2(\mathfrak{X})\to\mathbb{R}$.

For $T\in\mathrm{Sym}_2(\mathbb{Z})_{>0}$ regular, let

$$\widehat{\mathfrak{Z}}(T,v) = (\mathfrak{Z}(T),0)\in\widehat{CH}^2(\mathfrak{X}). \tag{4.4}$$

Then $\mathfrak{Z}(T)=\mathrm{Spec}(R(T))$ for an Artin ring $R(T)$ in which p is nilpotent, and the corresponding positive definite coefficients of the generating function (4.1) are given by

$$\widehat{\deg}(\widehat{\mathfrak{Z}}(T,v)) = \log|R(T)|. \tag{4.5}$$

For nonsingular $T\in\mathrm{Sym}_2(\mathbb{Z})$ with signature $(1,1)$ or $(0,2)$, set

$$\widehat{\mathfrak{Z}}(T,v) = (0,\mathfrak{g}(T,v))\in\widehat{CH}^2(\mathfrak{X}), \tag{4.6}$$

where $\mathfrak{g}(T,v)$ is a smooth $(1,1)$–form, depending on T and v, which is described in (6.3) of section 6.

Finally, if $T \in \mathrm{Sym}_2(\mathbb{Z})_{\geq 0}$ has rank 1, then, in effect, only one special endomorphism had been imposed, so that there is an associated divisor $\mathfrak{Z}(T)$ on $\mathfrak{X}$. A Green's function $\Xi(T,v)$ for this divisor is constructed in section 6, below. This function continues to make sense when $T \leq 0$. There is a resulting class

$$\widehat{\mathfrak{Z}}(T,v) = (\mathfrak{Z}(T), \Xi(T,v)) \in \widehat{CH}^1(\mathfrak{X}), \tag{4.7}$$

where $\mathfrak{Z}(T)$ is empty when $T \leq 0$. Let $\hat{\boldsymbol{\omega}}_{\mathfrak{X}}$ be the relative dualizing sheaf of $\mathfrak{X}$ over $\mathrm{Spec}(\mathbb{Z})$, with metric coming from the uniformization of $X(\mathbb{C})$ by D, viewed as an element of $\widehat{CH}^1(\mathfrak{X})$ via the isomorphism $\widehat{\mathrm{Pic}}(\mathfrak{X}) \simeq \widehat{CH}^1(\mathfrak{X})$, and let

$$\hat{\boldsymbol{\omega}}_{\mathfrak{X}}(v) = \hat{\boldsymbol{\omega}}_{\mathfrak{X}} + (0, \log\det(v)) \in \widehat{CH}^1(\mathfrak{X}). \tag{4.8}$$

This class plays a role analogous to that of $[\omega]$, the Kähler class, in section 3 above.

Using the arithmetic intersection pairing $\widehat{CH}^1(\mathfrak{X}) \times \widehat{CH}^1(\mathfrak{X}) \to \widehat{CH}^2(\mathfrak{X})$, the full arithmetic generating function (4.1), analogous to $\phi_{\deg}$ given in Corollary 1.4 and (3.1) in the geometric case, is then

$$\hat{\phi}_{\deg}(\tau) = \widehat{\deg}(\hat{\boldsymbol{\omega}}_{\mathfrak{X}}(v)^2) + \sum_{\mathrm{rank}(T)=1} \widehat{\deg}(\ \widehat{\mathfrak{Z}}(T,v) \cdot \hat{\boldsymbol{\omega}}_{\mathfrak{X}}(v)\,)\, q^T + \sum_{\substack{T \\ \det(T)\neq 0}} \widehat{\deg}(\ \widehat{\mathfrak{Z}}(T,v)\ )\, q^T. \tag{4.9}$$

where suitable terms have been added† for irregular T.

In the present situation, the construction of Remark 2.3 yields an incoherent Eisenstein series $E(\tau,s,\varphi)$ of weight $\frac{3}{2}$ associated to $\varphi \in S(V(\mathbb{A}_f)^2)$, the characteristic function of $(\widehat{\mathcal{O}_B} \cap V(\mathbb{A}_f))^2$.

CONJECTURE 4.7. — *The generating function $\hat{\phi}_{\deg}(\tau)$ is a Siegel modular form of weight $\frac{3}{2}$, more precisely*

$$\hat{\phi}_{\deg}(\tau) = \mathrm{vol}(X(\mathbb{C})) \cdot E'(\tau, 0, \varphi). \tag{4.10}$$

The main results in this direction assert that many of the Fourier coefficients of the two series coincide. Recall that $D(B)$ is the product of the primes p at which B_p is division. Also put $V^{(\infty)} = V$, so that there is a rational quadratic space associated to each place of $\mathbb{Q}$. By Lemma 4.5, a given nonsingular $T \in \mathrm{Sym}_2(\mathbb{Q})$ is represented by at most one of these spaces. If T is positive definite, this space, if it exists, must be one of the $V^{(p)}$'s for a finite prime p, while if T has signature $(1,1)$ or $(0,2)$, then this space can only be $V = V^{(\infty)}$.

†by a still not very satisfactory procedure

Theorem 4.8. — [14]. *Suppose that $T \in \mathrm{Sym}_2(\mathbb{Z})$ is nonsingular.*
(i) If T is not represented by any $V^{(p)}$, then the T–th Fourier coefficient of both sides of (4.10) *vanish.*
(ii) If T is represented by $V^{(p)}$ with $p \nmid 2D(B)$, including $p = \infty$, then the T–th Fourier coefficients of the two sides of (4.10) *agree, i.e.,*

$$\widehat{\deg}(\widehat{\mathfrak{Z}}(T, v))\, q^T = \mathrm{vol}(X(\mathbb{C})) \cdot E'_T(\tau, 0, \varphi). \tag{4.11}$$

In fact, work in progress [25] should extend (4.11) to all T of rank 1.

The proof of (ii) is similar to the proof of the main identity at the heart of the Gross–Zagier formula [8]; it amounts to an explicit computation of the two quantites, one from arithmetic geometry, the other from automorphic forms. A sketch of the proof is given in the next two sections.

5. NON–SINGULAR FOURIER COEFFICIENTS

The Fourier coefficients of the central derivative of an incoherent Eisenstein series $E(\tau, s, \varphi)$ associated to a rational quadratic space V of signature $(n-1, 2)$, as defined as in Remark 2.3, have an interesting structure.

For each prime $p \le \infty$, define a quadratic space $V^{(p)}$ of dimension $n+1$ and the same determinant as that of V as follows. Let $V^{(\infty)} = V$. For a finite prime p, $V^{(p)}$ has signature $(n+1, 0)$ and local Hasse invariants at finite primes ℓ determined by

$$\epsilon_\ell(V^{(p)} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) = \begin{cases} \epsilon_\ell(V) & \text{if } \ell \neq p, \text{ and} \\ -\epsilon_p(V_p) & \text{if } \ell = p. \end{cases} \tag{5.1}$$

For each nonsingular $T \in \mathrm{Sym}_n(\mathbb{Q})$, there is a rational quadratic space V_T of dimension $n+1$ and the same determinant as V defined by equation (4.3).

It is well known that if the function $\varphi = \otimes_{p<\infty} \varphi_p \in S(V(\mathbb{A}_f)^n)$ is factorizable and $\mathrm{Re}(s) > \frac{n+1}{2}$, then each nonsingular Fourier coefficient of $E(\tau, s, \varphi)$ is given by a product

$$E_T(\tau, s, \varphi) = W^{\frac{n+1}{2}}_{T,\infty}(\tau, s) \cdot \prod_{p<\infty} W_{T,p}(s, \varphi_p)$$

of local (degenerate) Whittaker functions, [20]. In classical language, the archimedean factor is a confluent hypergeometric function of a matrix argument studied by Shimura [30], while the product over the finite primes is the Siegel series.

Proposition 5.1. — [14], section 6. *(i) If V_T is not isomorphic to any $V^{(p)}$, then*

$$E'_T(\tau, 0, \varphi) = 0.$$

(ii) If $V_T \simeq V^{(p)}$ for a finite prime, then $W_{T,p}(0, \varphi_p) = 0$ and

$$E'_T(\tau, 0, \varphi) = q^T \cdot W_{T,p}(s, \varphi_p)'\big|_{s=0} \cdot A^{(p)}_T(\varphi),$$

where $A_T^{(p)}(\varphi)$ is, up to a factor at p, the Fourier coefficient of a theta integral (cf. (2.2)*) attached to $V^{(p)}$.*
(iii) If $V_T \simeq V^{(\infty)} = V$,

$$E'_T(\tau, 0, \varphi) = W_{T,\infty}^{\frac{n+1}{2}}(\tau, s)'\big|_{s=0} \cdot A_T^{(\infty)}(\varphi),$$

where $A_T^{(\infty)}(\varphi)$ is the T–th Fourier coefficient of a theta integral attached to V.

Idea of proof of Theorem 4.8 for $p < \infty$. Restricting to the case $n = 2$ as in section 4, suppose that $T \in \mathrm{Sym}_2(\mathbb{Z})$ is nonsingular with $V_T \simeq V^{(p)}$ for a finite prime $p \nmid 2D(B)$. In this case, using Proposition 5.1, the identity to be proved in (ii) of Theorem 4.8 amounts to

$$(5.2) \qquad \widehat{\deg}(\widehat{\mathfrak{Z}}(T, v)) = \widehat{\deg}((\mathfrak{Z}(T), 0)) = \mathrm{vol}(X(\mathbb{C})) \cdot W_{T,p}(s, \varphi_p)'\big|_{s=0} \cdot A_T^{(p)}(\varphi).$$

This identity, which is of the same nature as the identities between heights and Fourier coefficients involved in the Gross-Zagier formula, is proved by computing the two sides explicitly.

On the geometric side, since T is regular, $\mathfrak{Z}(T)$ is a collection of supersingular points on $\mathfrak{X}_p$, each counted with a certain multiplicity. This multiplicity is the length of the local Artin ring associated to the deformations of the triple (A, ι, x), where x is a pair of special endomorphisms. By the Serre–Tate Theorem, one can pass to the deformations of $(A(p), \iota, x)$ where $A(p)$ is corresponding p–divisible group with the action ι of $(\mathcal{O}_B)_p = \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \simeq M_2(\mathbb{Z}_p)$ and a pair of special endomorphisms x. Since A is isogenous to $E \times E$ for a supersingular elliptic curve E, one reduces, via the idempotents in $(\mathcal{O}_B)_p$, to the problem of deforming $(E(p), x)$ for a pair $x = [x_1, x_2]$ of endomorphisms of the p–divisible group of such a curve. Note that $E(p)$ is a p–divisible formal group of dimension 1 and height 2. The length of the associated Artin ring is then obtained by specializing a beautiful result of Gross and Keating [6].

They consider the deformations of a collection $(\mathbb{X}, \mathbb{X}', y)$ where $\mathbb{X}$ and $\mathbb{X}'$ are formal groups of dimension 1 and height 2 over $\bar{\mathbb{F}}_p$ and $y = [y_1, y_2, y_3]$ is a triple of nonzero isogenies $y_i : \mathbb{X} \to \mathbb{X}'$. Note that the universal deformation ring of the pair $(\mathbb{X}, \mathbb{X}')$ is $W[[t, t']]$, where $W = W(\bar{\mathbb{F}}_p)$ is the ring of Witt vectors of $\bar{\mathbb{F}}_p$.

Proposition 5.2. — *(Gross-Keating,* [6]*, Proposition 5.4) Suppose that the matrix Q of inner products of the triple $y = [y_1, y_2, y_3]$ with respect to the degree quadratic form on* $\mathrm{Hom}(\mathbb{X}, \mathbb{X}')$ *has invariants $a_1 \le a_2 \le a_3$. For p odd, this means that $Q \in \mathrm{Sym}_3(\mathbb{Z}_p)$ is $GL_3(\mathbb{Z}_p)$ equivalent to* $\mathrm{diag}(\epsilon_1 p^{a_1}, \epsilon_2 p^{a_2}, \epsilon_3 p^{a_3})$ *for units $\epsilon_i \in \mathbb{Z}_p^\times$. Then the length of the* *deformation ring of* $(\mathbb{X}, \mathbb{X}', y)$ *is given by:*

$$\sum_{i=0}^{a_1-1}(i+1)(a_1+a_2+a_3-3i)\,p^i + \sum_{i=a_1}^{(a_1+a_2-2)/2}(a_1+1)(2a_1+a_2+a_3-4i)\,p^i + \frac{1}{2}(a_1+1)(a_3-a_2+1)\,p^{(a_1+a_2)/2}. \tag{5.3}$$

if a_1+a_2 *is even, and*

$$\sum_{i=0}^{a_1-1}(i+1)(a_1+a_2+a_3-3i)\,p^i + \sum_{i=a_1}^{(a_1+a_2-1)/2}(a_1+1)(2a_1+a_2+a_3-4i)\,p^i$$

if a_1+a_2 *is odd.*

Specialized to the case where y_1 is an isomorphism, so that $a_1 = 0$, one obtains an explicit formula for the multiplicity $\delta_p(T)$ of a point in $\mathfrak{Z}(T)$, and, in particular, observes that this multiplicity depends only on the $\mathrm{GL}_2(\mathbb{Z}_p)$–equivalence class of T, and not on the particular point. Thus, the left hand side of (5.2) has the form $(\delta_p(T)\cdot(\ \#\text{ points in }\mathfrak{Z}(T)))$.

On the other hand, for $p \neq 2$, the quantity $W_{T,p}(s,\varphi_p)'\big|_{s=0}$ on the analytic side of (5.2) can be computed from the result of Kitaoka [12]. More precisely, the quadratic form on the lattice $L_p = (\mathcal{O}_B \otimes \mathbb{Z}_p) \cap V_p$ has matrix $S_0 = \mathrm{diag}(1,1,-1)$ for a suitable basis. Let

$$S_r = \mathrm{diag}(1,1,-1,\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{r})$$

be the quadratic form obtained from S_0 by adding r hyperboic planes. Then

$$W_{T,p}(r,\varphi_p) = \gamma_p \cdot \alpha_p(S_r,T),$$

for a root of unity γ_p, where

$$\alpha_p(S_r,T) = \lim_{t\to\infty} p^{-t(3+4r)}\,|\{x \in M_{3+2r,2}(\mathbb{Z}_p/p^t\mathbb{Z}_p) \mid {}^t x S_r x \equiv T \mod (p^t)\}|$$

is the classical representation density of T by S_r as defined by Siegel, [12]. This quantity has the form $\alpha_p(S_r,T) = A_p(q^{-r},T)$ for a polynomial $A_p(X,T)$, which was first calculated in this case by Kitaoka [12]‡. When T is such that $W_{T,p}(0,\varphi_p) = 0$, and $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$ is $GL_2(\mathbb{Z}_p)$–equivalent to $\mathrm{diag}(\epsilon_1 p^a, \epsilon_2 p^b)$, then Kitaoka's formula yields

$$\begin{aligned} W_{T,p}(s,\varphi_p)'\big|_{s=0} &= -\log(p)\,\gamma_p\,\frac{\partial}{\partial X}\{A_p(X,T)\}\big|_{X=1} \\ &= \log(p)\,\gamma_p\,(1-p^{-2}) \qquad (5.4)\\ &\quad\times \begin{cases} \sum_{j=0}^{(a-1)/2}(a+b-4j)\,p^j & \text{if } a \text{ is odd} \\ \sum_{j=0}^{(a-2)/2}(a+b-4j)\,p^j + \frac{1}{2}(b-a+1)\,p^{a/2} & \text{if } a \text{ is even and } b \text{ is odd} \end{cases} \end{aligned}$$

‡Recently, it has been calculated in general, for $p \neq 2$, by F. Sato and Y. Hironaka, [29]

Remarkably,

$$W_{T,p}(s,\varphi_p)'\big|_{s=0} = \log(p)\,\gamma_p\,(1-p^{-2})\,\delta_p(T), \tag{5.5}$$

where $\delta_p(T)$ is the multiplicity computed via the Gross–Keating formula! Finally, by a straightforward counting argument, the number of points in $\mathfrak{Z}(T)$ is given by $A_T^{(p)}(\varphi)$, up to simple factors, independent of T, which compensate for the extra p^2-1 and the factor $\mathrm{vol}(X(\mathbb{C}))$. □

6. GREEN'S FUNCTIONS AND WHITTAKER FUNCTIONS

The classes $\widehat{\mathfrak{Z}}(T,v)$ in the arithmetic Chow groups of integral models of Shimura curves involve the Green's functions defined in [14], section 11. This section describes the construction of such functions. First, suppose that V is a rational quadratic space of signature $(n-1,2)$, as in section 3, and recall that for $t\in\mathbb{Q}_{>0}$ and for $\varphi\in S(V(\mathbb{A}_f))^K$, there is a divisor $Z(t,\varphi)$ on X_K, given as a weighted sum of the images in X_K of the divisors D_x in D. A Green's function of logarithmic type for $Z(t,\varphi)$ in the sense of Gillet–Soulé [4] can be constructed by averaging rapidly decreasing Green's functions for D_x's. Next, in the case of a Shimura curve ($n=2$), the smooth $(1,1)$–form $\mathfrak{g}(T,v)$ used in the construction of the terms for indefinite T's in the generating function of arithmetic degrees is defined via the star product. The notation is that of section 3.

For an oriented negative 2–plane $z\in D$, let pr_z be the projection to z with respect to the orthogonal decomposition $V(\mathbb{R})=z+z^{\perp}$. For $x\in V(\mathbb{R})$, $x\neq 0$, the quantity

$$R(x,z) = -(\mathrm{pr}_z(x),\mathrm{pr}_z(x))$$

is nonnegative and vanishes if and only if $z\in D_x$. Let

$$\xi(x,z) = -\mathrm{Ei}(-2\pi R(x,z)),$$

where, for $r>0$,

$$-\mathrm{Ei}(-r) = \int_1^{\infty}\frac{e^{-rt}}{t}\,dt$$

is the exponential integral. Since $-\mathrm{Ei}(-r)$ decays exponentially as r goes to infinity and behaves like $-\log(r)+O(1)$ as r goes to 0, the function $\xi(x)$ has a logarithmic singularity along the (possibly empty) divisor D_x and decays very rapidly away from D_x. In addition, it satisfies the Green's equation:

$$dd^c\xi(x) + \delta_{D_x} = \mu(x)$$

for a smooth $(1,1)$–form $\mu(x)$ on D, and hence defines a Green's form of log type for D_x, in the sense of Gillet-Soulé.

Note that $R(hx,hz) = R(x,z)$ for $h \in \mathrm{GSpin}(V)(\mathbb{R})$, so that $\xi(hx,hz) = \xi(x,z)$ as well. For $v > 0$ and $t \in \mathbb{Q}_{>0}$, and a weight function $\varphi \in S(V(\mathbb{A}_f))^K$, the sum

$$\mathfrak{g}(t,v,\varphi)(z,h) = \sum_{\substack{x \in V(\mathbb{Q}) \\ Q(x)=t}} \varphi(h^{-1}x)\,\xi(x,z) \tag{6.1}$$

depends only on the orbit $H(\mathbb{Q})(z,h)K$ of the point $(z,h) \in D \times H(\mathbb{A}_f)$. Thus, $\mathfrak{g}(t,v,\varphi)$ defines a Green's function of logarithmic type for the divisor $Z(t,\varphi)$, defined in section 3 (for $r = 1$), on X_K.

In the case $n = 2$, for the space V considered in section 4, and a pair of vectors $x = [x_1, x_2] \in V(\mathbb{R})^2$ such that $\det(Q(x)) \neq 0$, the integral

$$\Lambda(x) = \int_D \xi(x_1) * \xi(x_2)$$

of the $*$–product of the associated Green's functions [4] is well defined and satisfies $\Lambda(hx) = \Lambda(x)$ for all $h \in \mathrm{GSpin}(V)(\mathbb{R})$. This implies that $\Lambda(x)$ actually only depends on the matrix of inner products $Q(x)$, i.e.,

$$\Lambda(x) =: \Lambda_0(Q(x)). \tag{6.2}$$

One can view $\Lambda(x)$ as the 'archimedean height pairing' of the 0–cycles D_{x_1} and D_{x_2} in D. In fact this quantity has the following rather surprising additional invariance:

Theorem 6.1. — [14], *section 13. For any $k \in SO(2)$, and any x,*

$$\Lambda(xk) = \Lambda(x).$$

Note that, even when $Q(x_1) > 0$ and $Q(x_2) > 0$, so that one initially has a pair of 0–cycles D_{x_1} and D_{x_2} in D, eventually, after rotation, one encounters a pair of vectors x_1' and x_2' with $Q(x_2') < 0$, so that the cycle $D_{x_2'}$ has vanished (!) and is replaced, in some sense, by the geodesic arc $\{z \in D \mid x_2' \in z\}$. Nonetheless, the 'archimedean height pairing' $\Lambda(x)$ is invariant under such a deformation.

It follows that, writing $v \in \mathrm{Sym}_2(\mathbb{R})_{>0}$ as $v = a^t a$ for $a \in \mathrm{GL}_2(\mathbb{R})^+$, the quantity $\Lambda(xa)$ depends only on v and not on the choice of a. Then, for a nonsingular $T \in \mathrm{Sym}_2(\mathbb{Z})$, one has a smooth $(1,1)$–form

$$\mathfrak{g}(T,v) = \Bigg(\sum_{\substack{x \in V(\mathbb{Q})^2 \\ Q(x) = T \\ \mathrm{mod}\ \Gamma_B}} \varphi(x)\,\Lambda(xa) \Bigg)\,\mu \tag{6.3}$$

on X, where $\mu = y^{-2}\,dx \wedge dy$, φ is the characteristic function of the set $\big(\widehat{\mathcal{O}_B}^{\,2} \cap V(\mathbb{A}_f)^2\big)$, and $\Gamma_B = \mathcal{O}_B^\times$. This is the form used in the definition (4.6) of $\widehat{\mathfrak{Z}}(T,v)$ when T has signature $(1,1)$ or $(0,2)$. Note that the sum is nonempty precisely when V represents T.

Idea of proof of Theorem 4.8 for $p=\infty$. Again using Proposition 5.1, the identity to be proved in this case is

$$\widehat{\deg}(\widehat{\mathfrak{Z}}(T,v))\,q^T = \mathrm{vol}(X(\mathbb{C}))\cdot W_{T,\infty}^{\frac{n+1}{2}}(\tau,s)'\big|_{s=0}\cdot A_T^{(\infty)}(\varphi).$$

The left hand side is simply

$$q^T\cdot\Lambda_0({}^t aTa)\cdot\sum_{\substack{x\in V(\mathbb{Q})^2\\ Q(x)=T\\ \mathrm{mod}\ \Gamma_B}}\varphi(x), \tag{6.4}$$

where $v=a^ta$, as before and Λ_0 is given by (6.2). Then some transformations of the integral representations of the matrix argument confluent hypergeometric function of Shimura's paper [30] together with a manipulation of the integral defining Λ shows that

$$W_{T,\infty}^{\frac{n+1}{2}}(\tau,s)'\big|_{s=0}=c_\infty\cdot\Lambda_0({}^t aTa)\cdot q^T,$$

where c_∞ is an innocuous constant. Again, the sum in (6.4) counts pairs of lattice vectors modulo Γ_B and coincides with $A_T^{(\infty)}$, the Fourier coefficient of the theta integral, up to a constant which absorbs c_∞ and provides the required $\mathrm{vol}(X(\mathbb{C}))$ factor. □

7. FURTHER RESULTS

One would like to identify the central derivative $E'(\tau,0,\varphi)$ of the incoherent Eisenstein series (2.3) as a generating function for arithmetic degrees in the general case. In the series of papers [24], [22], [21] the cases $n=1$, 3 and 4 are considered. In each of these cases (and also for $n=5$), the Shimura variety X associated to $H=\mathrm{GSpin}(V)$ is of PEL type, i.e., can be interpreted as a moduli space for abelian varieties (A,ι) with a specified endomorphism ring, due to the existence of an accidental isomorphism. This allows one to give a modular definition of an integral model $\mathfrak{X}_K$ of X_K, at least over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$ for a suitable N depending on the compact open subgroup K. For each abelian scheme (A,ι), there is a space of special endomorphisms $V(A,\iota)$, equipped with a $\mathbb{Z}$–valued quadratic form. Special cycles on $\mathfrak{X}$ are then defined by imposing collections of such endomorphisms, as in section 4 and the example of section 1.3 above.

The case $n=1$. This case is considered in [24] and is described in classical language in section 1.3 above. From the point of view now developed it amounts to the following. An imaginary quadratic field $\boldsymbol{k}=\mathbb{Q}(\sqrt{-d})$ with quadratic form $Q(x)=-N_{\boldsymbol{k}/\mathbb{Q}}(x)$ gives a rational quadratic space (V,Q) of signature $(0,2)$. The group $\mathrm{GSpin}(V)$ is then the torus over $\mathbb{Q}$ with $\mathrm{GSpin}(V)(\mathbb{Q})\simeq\boldsymbol{k}^\times$, and $\mathfrak{X}\simeq\mathrm{Spec}(\mathcal{O}_H)$ is the restriction of scalars to $\mathrm{Spec}(\mathbb{Z})$ of the coarse moduli space over $\mathcal{O}_{\boldsymbol{k}}$ of elliptic curves (A,ι) with complex multiplication by $\mathcal{O}_{\boldsymbol{k}}$. In this case the space of special endomorphisms (1.6) is

$$V(A,\iota) = \{x \in \mathrm{End}(A) \mid x\iota(b) = \iota(\bar{b})x\}.$$

This space is zero unless A is a supersingular elliptic curve in characteristic p, where p is not split in $\boldsymbol{k}$, in which case, $V(A,\iota)\otimes_{\mathbb{Z}}\mathbb{Q} \simeq V^{(p)}$, the rational quadratic space given by $V^{(p)} = \boldsymbol{k}$ with quadratic form $Q^{(p)}(x) = p\,N_{\boldsymbol{k}/\mathbb{Q}}(x)$. For $\tau = u + iv \in \mathfrak{H}$, the generating function is then given by

$$\hat{\phi}_{\mathrm{deg}}(\tau) = \sum_{t\in\mathbb{Z}} \widehat{\deg}(\widehat{\mathfrak{Z}}(t,v))\, q^t. \tag{7.1}$$

Here, for $t > 0$, $\widehat{\mathfrak{Z}}(t,v) = (\mathfrak{Z}(t), 0) \in \widehat{CH}^1(\mathfrak{X})$ where $\mathfrak{Z}(t)$ is the locus of (A,ι,x)'s where $x \in V(A,\iota)$ with $Q(x) = -x^2 = t$. For $t < 0$, $\widehat{\mathfrak{Z}}(t,v) = (0, \mathfrak{g}(t,v)) \in \widehat{CH}^1(\mathfrak{X})$, where $\mathfrak{g}(t,v)$ is the function on $X(\mathbb{C}) \simeq H(\mathbb{Q})\backslash D \times H(\mathbb{A}_f)/K$ given by (6.1). Here $K \simeq \widehat{\mathcal{O}}_{\boldsymbol{k}}^{\times}$. Note that in this case, D consists of two points (via the two possible orientations of $V(\mathbb{R})$), and $R(x,z) = 2N_{\boldsymbol{k}/\mathbb{Q}}(x)$, so that $\xi(xa,z) = -\mathrm{Ei}(-4\pi v N_{\boldsymbol{k}/\mathbb{Q}}(x))$, precisely as in [24]. This gives an improved version of Theorem 1.7:

Theorem 7.1. — [24]. *When $d \equiv 3 \mod (4)$ is a prime and for a suitable definition of the constant term $\widehat{\deg}(\widehat{\mathfrak{Z}}(0,v))$,*

$$\hat{\phi}_{\mathrm{deg}}(\tau) = \mathrm{vol}(X(\mathbb{C}))\cdot E'(\tau,0,\varphi),$$

where φ is the characteristic function of $\widehat{\mathcal{O}_{\boldsymbol{k}}} \subset V(\mathbb{A}_f)$ and $\mathrm{vol}(X(\mathbb{C})) = h_{\boldsymbol{k}}$ is the class number of $\boldsymbol{k}$.

The result is proved by a direct calculation of both sides, using the results of Gross [5] to compute multiplicities on the moduli space. The restriction to prime discriminant is only made to streamline the calculations.

The higher dimensional cases treated so far exhibit some new phenomena.

The case $n = 3$. This case is considered in [22]. The incoherent Eisenstein series (2.2) associated to a rational quadratic space of signature $(2,2)$ will have weight 2 and genus 3. To define the relevant moduli problem and generating function for arithmetic degrees, let $C(V) = C^+(V) \oplus C^-(V)$ be the Clifford algebra of V with its $\mathbb{Z}_2$–grading. Then the center of $C^+(V)$ is a real quadratic field $\boldsymbol{k}$ (possibly $\boldsymbol{k} = \mathbb{Q}\oplus\mathbb{Q}$), and $C^+(V)$ has the form $B_0 \otimes_{\mathbb{Q}} \boldsymbol{k}$ for an indefinite quaternion algebra B_0 over $\mathbb{Q}$. The associated Shimura variety X is a surface whose complex points parametrize polarized abelian varieties (A,ι) of dimension 8 with an action of a maximal order in $C(V)\otimes\boldsymbol{k}$. Included among the X's are products of modular curves ($\boldsymbol{k} = \mathbb{Q}\oplus\mathbb{Q}$, $B_0 = M_2(\mathbb{Q})$), products of Shimura curves ($\boldsymbol{k} = \mathbb{Q}\oplus\mathbb{Q}$, B_0 division), Hilbert-Blumenthal surfaces ($\boldsymbol{k}$ a field, $B_0 = M_2(\mathbb{Q})$) and their twisted (quaternionic) analogues ($\boldsymbol{k}$ a field, B_0 division). An integral model $\mathfrak{X}$ of X over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$, is defined as the moduli space of polarized abelian schemes with such an action, level structure, etc. The space of special endomorphisms of a given (A, ι) is then

$$V(A,\iota) = \{x \in \mathrm{End}(A) \mid x\,\iota(c\otimes a) = \iota(c\otimes \bar{a})\,x,\ \text{and } x^* = x\}$$

where $*$ denotes the Rosati involution of A. As before, this space has a $\mathbb{Z}$–valued quadratic form define by $x^2 = Q(x)\cdot 1_A$. Again, a key point is that, for (A,ι) supersingular, the space $V(A,\iota)\otimes\mathbb{Q} \simeq V^{(p)}$ is the 4 dimensional rational quadratic space defined by (5.1) above. For $T \in \mathrm{Sym}_3(\mathbb{Z})_{>0}$, the special cycle $\mathfrak{Z}(T)$ is the locus of (A,ι,x)'s where $x \in V(A,\iota)^3$ with $Q(x) = T$. This cycle is either empty or is supported in the supersingular locus $\mathfrak{X}_p^{\text{s.s.}}$ of the fiber at p for the unique prime p for which $V_T \simeq V^{(p)}$. Here one assumes that $p \nmid N$, so that p is a prime of good reduction; in particular, p is not ramified in $\boldsymbol{k}$.

If a prime $p \nmid N$ splits in $\boldsymbol{k}$, then the supersingular locus $\mathfrak{X}_p^{\text{s.s.}}$ consists of a finite set of points. If p is inert in $\boldsymbol{k}$, then $\mathfrak{X}_p^{\text{s.s.}}$ has dimension 1 and is, in fact, is a union of $\mathbb{P}^1$'s [31], [22], section 4, crossing at $\mathbb{F}_{p^2}$ rational points.

Let $\varphi \in S(V(\mathbb{A}_f)^3)$ be the characteristic function of $L\otimes\hat{\mathbb{Z}}$ for a lattice $L \subset V(\mathbb{Q})$ such that L_p is self dual for all $p \nmid N$, and let $E(\tau, s, \varphi)$ be the associated incoherent Eisenstein series (2.3) of weight 2.

Theorem 7.2. — [22]. *Suppose that $T \in \mathrm{Sym}_3(\mathbb{Z})_{>0}$.*
(i) If V_T is not isomorphic to any $V^{(p)}$, then $\mathfrak{Z}(T)$ is empty and $E'_T(\tau,0,\varphi) = 0$.
(ii) If $V_T \simeq V^{(p)}$ for a prime $p \nmid N$ which splits in $\boldsymbol{k}$, then there is a class $\widehat{\mathfrak{Z}}(T) = (\mathfrak{Z}(T), 0) \in \widehat{CH}^3(\mathfrak{X})$ and

$$\widehat{\deg}(\widehat{\mathfrak{Z}}(T))\cdot q^T = C\cdot E'_T(\tau,0,\varphi), \tag{7.2}$$

for a constant C independent of T.
(iii) Suppose that $V_T \simeq V^{(p)}$ for a prime $p \nmid N$ which is inert in $\boldsymbol{k}$. If $p \nmid T$, then $\mathfrak{Z}(T)$ consists of a finite number of points, there is a class $\widehat{\mathfrak{Z}}(T) = (\mathfrak{Z}(T),0) \in \widehat{CH}^3(\mathfrak{X})$ and, again,

$$\widehat{\deg}(\widehat{\mathfrak{Z}}(T))\cdot q^T = C\cdot E'_T(\tau,0,\varphi). \tag{7.3}$$

If $p \mid T$, then $\mathfrak{Z}(T)$ is a union of components of the supersingular locus $\mathfrak{X}_p^{\text{s.s.}}$.

Here, $T \in \mathrm{Sym}_3(\mathbb{Z})_{>0}$ will be called irregular if $V_T \simeq V^{(p)}$ with $p \nmid N$ inert in $\boldsymbol{k}$ and $p \mid T$. The situation for $p \mid N$, e.g., p ramified in $\boldsymbol{k}$, has not yet been studied.

Here, as in (4.5), $\widehat{\deg}(\widehat{\mathfrak{Z}}(T)) = \log|R(T)|$ for the Artin ring $R(T)$ defining $\mathfrak{Z}(T)$, so that the fact that $\mathfrak{X}$ is only defined over $\mathrm{Spec}(\mathbb{Z}[N^{-1}])$ and need not be proper will not be important. These issues will be essential, however, if one wants to define the full generating function $\hat{\phi}_{\deg}(\tau)$ and compare it to $E'(\tau,0,\varphi)$. The proof of (7.2) and (7.3), where $\mathfrak{Z}(T)$ is a 0–cycle, again comes down to a relation between derivatives of representation densities for quadratic forms and the result of Gross and Keating described in Proposition 5.2 above, now in its full generality. Indeed, that result was obtained in connection with the study of the derivative of a Siegel–Eisenstein series of weight 2, due to the connection of such a series with the triple product L–function [3], [27], [9], [7].

The case $n = 4$. This case is considered in [21]. Here the Shimura varieties are (twisted) Siegel 3–folds, and the pattern is similar to that for $n = 3$. The one new point is that the 'regularity' condition on $T \in \mathrm{Sym}_4(\mathbb{Z})$ required to obtain a 0–cycle in the supersingular locus (which is again a curve with $\mathbb{P}^1$ components) becomes: $V_T \simeq V^{(p)}$ and T represents 1 over $\mathbb{Z}_p$. One then obtains an analogue of Theorem 6.2, with the comparison again based on [6].

8. FINAL REMARKS

Beyond the range of the accidental isomorphisms, i.e., for $n \geq 6$, the Shimura varieties associated to $\mathrm{GSpin}(V)$ for rational quadratic spaces V of signature $(n-1, 2)$ are no longer of PEL type, so that the modular interpretation of points, special endomorphisms, and other tools used before are no longer available. Instead, it will be necessary to work with integral models defined by suitable types of Hodge classes, etc., [26]. Presumably there is a good notion of special endomorphisms or special Hodge classes in this situation which cut out the required cycles. This theory remains to be established.

Even in the range $2 \leq n \leq 5$, many difficulties lie in the way of a full treatment of the generating function for $\hat{\phi}_{\mathrm{deg}}(\tau)$. For example, there is the problem of the contribution of irregular T's, which occur even in the case of good reduction. The contribution of the singular T's is 'global' and will presumably involve models over $\mathbb{Z}$, detailed information about the fibers of bad reduction, etc. Much work remains to be done.

In addition, there is the question of defining modular arithmetic generating functions $\hat{\phi}_r(\tau)$, analogous to the series $\phi_r(\tau)$ of Theorem 3.1 above, valued in arithmetic Chow groups $\widehat{CH}^r(\mathfrak{X})$ for arbitrary codimension. Of course the image of such series under the cycle class map should be the generating series discussed in section 3. Recent work of Borcherds [1], which gives a generating function involving the classes of the divisors $Z(t, \varphi)$ on $X_K(\mathbb{C})$ in the *Chow group* $CH^1(X_K(\mathbb{C}))$, rather than in cohomology, will be relevant here.

In all cases, it remains to work out the modifications required in the case of non-compact quotient.

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