On a conjecture of Jacquet
by
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For Joe Shalika, with our admiration and appreciation.

Introduction.

Let $\boldsymbol{k}$ be a number field and let π_i, $i = 1, 2, 3$ be cuspidal automorphic representations of $GL_2(\mathbb{A})$ such that the product of their central characters is trivial. Jacquet then conjectured that the central value $L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$ of the triple product L–function is nonzero if and only if there exists a quaternion algebra B over $\boldsymbol{k}$ and automorphic forms $f_i^B \in \pi_i^B$ such that the integral

$$I(f_1^B, f_2^B, f_3^B) = \int_{Z(\mathbb{A})B^\times(\boldsymbol{k})\backslash B^\times(\mathbb{A})} f_1^B(b) f_2^B(b) f_3^B(b)\, d^\times b \neq 0, \tag{0.1}$$

where π_i^B is the representation of $B^\times(\mathbb{A})$ corresponding to π_i via the Jacquet-Langlands correspondence.

In a previous paper [**4**], we proved this conjecture in the special case where $\boldsymbol{k} = \mathbb{Q}$ and the π_i's correspond to a triple of holomorphic newforms. Our method was based on a combination of the Garrett, Piatetski-Shapiro, Rallis integral representation of the triple product L-function with the extended Siegel–Weil formula and the seesaw identity. The restriction to holomorphic newforms over $\mathbb{Q}$ arose from (i) the need to invoke the Ramanujan Conjecture to control the poles of some bad local factors and (ii) the use of a version of the Siegel–Weil formula for similitudes. In this note, we show that, thanks to the recent improvement on the Ramanujan bound due to Kim–Shahidi [**11**], together with a slight variation in the setup of (ii), our method yields Jacquet's conjecture in general.

Since the exposition in [**4**] was specialized from the start to the case of interest for certain arithmetic applications, we will briefly sketch the method in general in the first few sections. We then prove the facts required about the extended Siegel–Weil formula.

Several authors have considered interpretations of the vanishing of the central value $L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$. Here we mention only the work of Dihua Jiang, [**9**], who gave an intriguing relation with a period of an Eisenstein series on G_2 and the recent Princeton thesis of Thomas Watson, [**26**], who applies these central values to problems in 'quantum chaos'.

The authors would like to thank the IHP in Paris where this project was realized during the special program on 'geometric aspects of automorphic forms' in June of 2000. We also thank the referee for helpful comments, and in particular for reminding us that the completion of the proof of Proposition 5.2 made implicit use of recent results of Loke [**16**].

§1. The integral representation of the triple product L–function.

Let $G = GSp_6$ be the group of similitudes of the standard 6 dimensional symplectic vector space over $\boldsymbol{k}$, and let $P = MN$ be the Siegel parabolic subgroup of G. For $a \in GL_3$, $b \in \mathrm{Sym}_3$ and ν a scalar, let

$$(1.1)\qquad m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad \text{and} \quad d(\nu) = \begin{pmatrix} 1 & \\ & \nu \end{pmatrix} \in G.$$

Let $K_G = K_{G,\infty} \cdot K_{G,f}$ be the standard maximal compact subgroup of $G(\mathbb{A})$. For $s \in \mathbb{C}$, let λ_s be the character of $P(\mathbb{A})$ defined by

$$(1.2)\qquad \lambda_s(d(\nu)n(b)m(a)) = |\nu|^{-3s}\,|\det(a)|^{2s}.$$

Let $I(s) = I_P^G(\lambda_s)$ be the normalized induced representation of $G(\mathbb{A})$, consisting of all smooth K_G–finite functions Φ_s on $G(\mathbb{A})$ such that

$$(1.3)\qquad \Phi_s(d(\nu)n(b)m(a)g, s) = |\nu|^{-3s-3}\,|\det(a)|^{2s+2}\,\Phi_s(g).$$

The Eisenstein series associated to a section $\Phi_s \in I(s)$ is defined for $\mathrm{Re}(s)2$ by

$$(1.4)\qquad E(g, s, \Phi_s) = \sum_{\gamma \in P(\boldsymbol{k})\backslash G(\boldsymbol{k})} \Phi_s(\gamma g),$$

and the normalized Eisenstein series is

$$(1.5)\qquad E^*(g, s, \Phi_s) = b_G(s) \cdot E(g, s, \Phi_s),$$

where $b_G(s) = \zeta_{\boldsymbol{k}}(2s+2)\,\zeta_{\boldsymbol{k}}(4s+2)$, as in [**17**]. Note that the central character of $E(g, s, \Phi_s)$ is trivial. These functions have meromorphic analytic continuations to the whole s–plane and have no poles on the unitary axis $\mathrm{Re}(s) = 0$. In particular, the map

$$(1.6)\qquad E^*(0) : I(0) \longrightarrow \mathcal{A}(G), \qquad \Phi_0 \mapsto (g \mapsto E^*(g, 0, \Phi_s))$$

gives a $(\mathfrak{g}_\infty, K_{G,\infty}) \times G(\mathbb{A}_f)$–intertwining map from the induced representation $I(0)$ at $s = 0$ to the space of automorphic forms on G with trivial central character.

Let

$$(1.7)\qquad \begin{aligned} \mathbf{G} &= (GL_2 \times GL_2 \times GL_2)_0 \\ &= \big\{(g_1, g_2, g_3) \in (GL_2)^3 \mid \det(g_1) = \det(g_2) = \det(g_3)\big\}. \end{aligned}$$

This group embeds diagonally in $G = GSp_6$. For automorphic forms $f_i \in \pi_i$, $i = 1$, 2, 3, let $F = f_1 \otimes f_2 \otimes f_3$ be the corresponding function on $\mathbf{G}(\mathbb{A})$. The global zeta integral [**17**] is given by

$$Z(s, F, \Phi_s) = \int_{Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash \mathbf{G}(\mathbb{A})} E^*(\mathbf{g}, s, \Phi_s)\, F(\mathbf{g})\, d\mathbf{g}. \tag{1.8}$$

Suppose that the automorphic forms $f_i \in \pi_i$ have factorizable Whittaker functions $W_i^\psi = \otimes_v W_{i,v}^\psi$ and that the section Φ_s is factorizable. Let S be a finite set of places of $\boldsymbol{k}$, including all archimedean places, such that, for $v \notin S$,

(i) the fixed additive character ψ of $\mathbb{A}/\boldsymbol{k}$ has conductor $\mathcal{O}_{\boldsymbol{k},v}$ at v.
(ii) $\pi_{i,v}$ is unramified, f_i is fixed under $K_v = GL_2(\mathcal{O}_{\boldsymbol{k},v})$, and $W_{i,v}^\psi(e) = 1$.
(iii) $\Phi_{s,v}$ is right invariant under $G(\mathcal{O}_{\boldsymbol{k},v}) = K_{G,v}$ and $\Phi_{s,v}(e) = 1$,

Then

$$Z(s, F, \Phi_s) = L^S(s + \frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \cdot \prod_{v \in S} Z_v(s, W_v^\psi, \Phi_{s,v}), \tag{1.9}$$

for local zeta integrals $Z_v(s, W_v^\psi, \Phi_{s,v})$, where $W_v^\psi = W_{1,v}^\psi \otimes W_{2,v}^\psi \otimes W_{3,v}^\psi$. Here

$$Z(s, W_v^\psi, \Phi_{s,v}) = \int_{Z_G(\boldsymbol{k}_v)\mathbf{M}(\boldsymbol{k}_v)\backslash \mathbf{G}(\boldsymbol{k}_v)} \Phi_{s,v}(\delta\, g)\, W_v^\psi(g)\, dg, \tag{1.10}$$

where $\delta \in G(\boldsymbol{k})$ is a representative for the open orbit of $\mathbf{G}$ in $P\backslash G$, cf. for example [**2**], and

$$\mathbf{M} = \{(\begin{pmatrix} 1 & x_1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_2 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_3 \\ & 1 \end{pmatrix}) \in \mathbf{G} \mid x_1 + x_2 + x_3 = 0\}. \tag{1.11}$$

Here $L^S(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ is the triple product L-functions with the factors for $v \in S$ omitted.

§**2. Local zeta integrals.**

In this section, we record some consequences of recent results of Kim and Shahidi [**11**] on the Ramanujan estimate for the π_i's. We begin by recalling relevant aspects of the local theory of the triple product, as recently completed by Ikeda and Ramakrishnan. In the following proposition by "local Euler factor" at a finite place v of $\boldsymbol{k}$ we mean a function of the form $P(q_v^{-s})^{-1}$, where P is a polynomial, $P(0) = 1$, and q_v is the order of the residue field; at an archimedean field we mean a finite product of Tate's local Euler factors for $GL(1)$.

Proposition 2.1. *Let v be a place of $\boldsymbol{k}$ and let $\pi_{i,v}$, $i = 1, 2, 3$, be a triple of admissible irreducible representations of $GL(2, \boldsymbol{k}_v)$ that arise as local components at v of cuspidal automorphic representations π_i.*

(i) There exists a local Euler factor $L(s,\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v})$ such that, for any local data $(W_v^{\psi},\Phi_{s,v})$, the quotient

$$\tilde{Z}_v(s,W_v^{\psi},\Phi_{s,v}) = Z_v(s,W_v^{\psi},\Phi_{s,v})\cdot L(s+\frac{1}{2},\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v})^{-1}$$

is entire as a function of s.

(ii) Let $\sigma_{i,v}$, $i=1,2,3$, be the representations of the Weil-Deligne group of $\boldsymbol{k}_v$ associated to $\pi_{i,v}$ by the local Langlands correspondence. Then

$$L(s,\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v}) = L(s,\sigma_{1,v}\otimes\sigma_{2,v}\otimes\sigma_{3,v}).$$

(iii) For any finite place v, there is a local section $\Phi_{s,v}$ and a Whittaker function $W_v^{\psi}=W_{1,v}^{\psi}\otimes W_{2,v}^{\psi}\otimes W_{3,v}^{\psi}$, such that

$$Z(s,\Phi_{s,v},W_v^{\psi})\equiv 1.$$

(iv) For any archimedean place v, there exists a finite collection of Whittaker functions $W_v^{\psi,j}$ and of sections $\Phi_{s,v}^j$, holomorphic in a neighborhood of $s=0$ such that

$$\sum_j Z(0,\Phi_{s,v}^j,W_v^{\psi,j}) = 1.$$

Proof. For v non-archimedean, assertion (i) is proved in §3, Appendix 3, of [**17**]; see [**7**], p. 227 for a concise statement. For v real or complex, (i) and (ii) were proved in several steps by Ikeda, of which the crucial one is [**8**], Theorem 1.10. Assertion (ii) in general is due to Ramakrishnan, [**21**], Theorem 4.4.1. For the moment, the hypothesis that the $\pi_{i,v}$ embed in global cuspidal representations seems to be necessary.

Assertions (iii) and (iv) are contained in Proposition 3.3 of [**17**]. $\square$

Proposition 2.2. *(i) For any triple π_i of cusp forms for GL_2 over $\boldsymbol{k}$, and for any place v, the local Langlands L–factor $L(s,\pi_{1,v}\otimes\pi_{2,v}\otimes\pi_{3,v})$ is holomorphic at $s=\frac{1}{2}$.*
(ii) For any place v, for any triple of Whittaker functions $W_{i,v}^{\psi}$ in the Whittaker spaces of $\pi_{i,v}$, and for any section $\Phi_{s,v}\in I_v(s)$, holomorphic in a neighborhood of $s=0$, the local zeta integral $Z(s,W_v^{\psi},P_{s,v})$ is holomorphic in a neighborhood of $s=0$.

Proof. This follows from the results of Kim and Shahidi. We sketch the simple argument, quoting the proof of Proposition 3.3.2 of [**21**]. Let $\sigma_{i,v}$ correspond to

$\pi_{i,v}$ as in the previous proposition. For the present purposes we can assume each π_i to be unitary. Indeed, this can be arranged by twisting π_i by a (unique) character of the form $|\cdot|^{a_i}$, where $|\cdot|$ is the idèle norm and $a_i \in \mathbb{C}$. Since the product of the central characters of π_i is trivial, we have $a_1 + a_2 + a_3 = 0$, so the triple product L-factor is left unaffected.

To each $\pi_{i,v}$, necessarily generic and now assumed unitary, we can assign an index $\lambda_{i,v}$ which measures the failure of $\pi_{i,v}$ to be tempered; we have $\lambda_{i,v} = t$ if $\pi_{i,v}$ is a complementary series attached to $(\mu|\cdot|^t, \mu|\cdot|^{-t})$ with $t0$ and μ unitary, $\lambda_{i,v} = 0$ otherwise. Then, according to [**21**], (3.3.10),

(2.1) $L(s, \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v})$ is holomophic for $\mathrm{Re}(s)\lambda(\pi_{1,v}) + \lambda(\pi_{2,v}) + \lambda(\pi_{3,v})$.

Now (i) follows from (2.1) and the Kim-Shahidi estimate $\lambda(\pi_{i,v}) < \frac{5}{34}$ for all i and all v [**11**], whereas (ii) follows from (i), and Proposition 2.1 (i) and (ii). $\square$

By (1.8), (1.9), and the holomorphy of $E^*(g, s, \Phi)$ on the unitary axis, the expression

$$L^S(s + \frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \cdot \prod_{v \in S} Z_v(s, F, \Phi_{s,v}) = \int_{Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A})} E^*(\mathbf{g}, s, \Phi_s)\, F(\mathbf{g})\, d\mathbf{g}. \tag{2.2}$$

is holomorphic at $s = 0$ for all choices of data F and Φ_s. By varying the data for places in S and applying (iii) and (iv) of Proposition 2.1, it follows that the partial Euler product $L^S(s + \frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$ is holomorphic at $s = 0$. By (i) of Proposition 2.2, the Euler product $L(s + \frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$ over all finite places is holomorphic at $s = 0$, and we obtain the identity

$$L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \cdot \prod_{v \in S} Z_v^*(0, W_v^\psi, \Phi_{s,v}) = \int_{Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A})} E^*(\mathbf{g}, 0, \Phi_s)\, F(\mathbf{g})\, d\mathbf{g}. \tag{2.3}$$

where

$$Z_v^*(s, W_v^\psi, \Phi_{s,v}) = \begin{cases} \tilde{Z}_v(s, W_v^\psi, \Phi_{s,v}) & \text{if } v \in S_f, \\ Z_v(s, W_v^\psi, \Phi_{s,v}) & \text{if } v \in S_\infty. \end{cases} \tag{2.4}$$

Corollary 2.3. *$L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) = 0$ if and only if*

$$\int_{Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A})} E^*(\mathbf{g}, 0, \Phi_s)\, F(\mathbf{g})\, d\mathbf{g} = 0,$$

for all choices of $F \in \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ and $\Phi_s \in I(s)$.

Of course, relation (2.3) gives a formula for $L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3)$ for a suitable choice of F and Φ_s.

§3. The Weil representation for similitudes.

The material of this section is a slight variation on that of section 5 of [**3**]. We consider only the case of the dual pair $(GO(V), GSp_6)$ where the space V has square discriminant.

Let B be a quaternion algebra over $\boldsymbol{k}$ (including the possibility $B = M_2(\boldsymbol{k})$), and let $V = B$ be a 4 dimensional quadratic space over $\boldsymbol{k}$ where the quadratic form is given by $Q(x) = \alpha\,\nu_B(x)$, where ν_B is the reduced norm on B and $\alpha \in \boldsymbol{k}^\times$. Note that the isomorphism class of V is determined by B and the sign of α at the set $\Sigma_\infty(B)$ of real archimedean places of $\boldsymbol{k}$ at which B is division. Let $H = GO(V)$ and let $H_1 = O(V)$ be the kernel of the scale map $\nu : H \to \mathbb{G}_m$. Let $G = GSp_6$, and let $G_1 = Sp_6$ be the kernel of the scale map $\nu : G \to \mathbb{G}_m$. We want to extend the standard Weil representation $\omega = \omega_\psi$ of $H_1(\mathbb{A}) \times G_1(\mathbb{A})$ on the Schwartz space $S(V(\mathbb{A})^3)$. First, there is a natural action of $H(\mathbb{A})$ on $S(V(\mathbb{A})^3)$ given by

$$L(h)\varphi(x) = |\nu(h)|^{-3}\,\varphi(h^{-1}x). \tag{3.1}$$

For $g_1 \in G_1(\mathbb{A})$ one has

$$L(h)\omega(g_1)L(h)^{-1} = \omega(d(\nu)g_1 d(\nu)^{-1}), \tag{3.2}$$

where $\nu = \nu(h)$, and $d(\nu)$ is as in section 1. Therefore, one obtains a representation of the semidirect product $H(\mathbb{A}) \ltimes G_1(\mathbb{A})$ on $S(V(\mathbb{A})^3)$. Let

$$R = \{(h,g) \in H \times G \mid \nu(h) = \nu(g)\}. \tag{3.3}$$

Then there is an isomorphism

$$R \longrightarrow H \ltimes G_1, \qquad (h,g) \mapsto (h, d(\nu(g))^{-1}g) = (h, g_1), \tag{3.4}$$

(this defines a map $g \mapsto g_1$) and a representation of $R(\mathbb{A})$ on $S(V(\mathbb{A})^3)$ given by

$$\omega(h,g)\varphi(x) = (L(h)\omega(g_1)\varphi)(x) = |\nu(h)|^{-3}(\omega(g_1)\varphi)(h^{-1}x). \tag{3.5}$$

The theta distribution Θ on $S(V(\mathbb{A})^3)$ is invariant under $R(\boldsymbol{k})$, since, for $(h,g) \in R(\boldsymbol{k})$,

$$\begin{aligned} \Theta(\omega(h,g)\varphi) &= \sum_{x \in V(\boldsymbol{k})^3} |\nu(h)|^{-3}(\omega(g_1)\varphi)(h^{-1}x) \\ &= \sum_{x \in V(\boldsymbol{k})^3} (\omega(g_1)\varphi)(x) \\ &= \Theta(\omega(g_1)\varphi) = \Theta(\varphi), \end{aligned} \tag{3.6}$$

since $g_1 \in G_1(\boldsymbol{k})$. The theta kernel, defined for $(h,g) \in R(\mathbb{A})$ by

$$\theta(h,g;\varphi) = \sum_{x \in V(\boldsymbol{k})^3} \omega(h,g)\varphi(x), \tag{3.7}$$

is thus left $R(\boldsymbol{k})$ invariant.

Remark 3.1. Asside from a shift in notation, the convention here is essentially the same as in section 5 of [**3**] and section 3 of [**4**], *except that* we take pairs (h,g) here versus (g,h) there. Compare (3.5) above with (5.1.5) of [**3**]. It turns out that this seemingly slight shift in convention will be crucial for the extension of the Siegel–Weil formula to similitudes, as we will see below.

Note that the set of archimedean places $\Sigma_\infty(B)$ introduced above is the set of all real archimedean places of $\boldsymbol{k}$ at which V is definite (positive or negative). Then,

$$\begin{aligned} G(\mathbb{A})^+ &:= \{g \in G(\mathbb{A}) \mid \nu(g) \in \nu(H(\mathbb{A}))\} \\ &= \{g \in G(\mathbb{A}) \mid \nu(g)_v 0, \ \forall v \in \Sigma_\infty(V)\}. \end{aligned} \tag{3.8}$$

For $g \in G(\mathbb{A})^+$, and $\varphi \in S(V(\mathbb{A})^3)$, and for V anisotropic over $\boldsymbol{k}$, i.e., for $B \neq M_2(\boldsymbol{k})$, the theta integral is defined by

$$I(g,\varphi) = \int_{H_1(\boldsymbol{k})\backslash H_1(\mathbb{A})} \theta(h_1 h, g; \varphi)\, dh_1, \tag{3.9}$$

where $h \in H(\mathbb{A})$ with $\nu(h) = \nu(g)$. It does not depend on the choice of h.

In the case $B = M_2(\boldsymbol{k})$, the theta integral must be defined by regularization. If $\boldsymbol{k}$ has a real place, the procedure outlined on p.621 of [**4**], [**15**], using a certain differential operator to kill support, can be applied. An analogous procedure using an element of the Bernstein center can be applied at a nonarchimedean place, [**25**]. We omit the details.

Lemma 3.2. *(i) (Eichler's norm Theorem) If $\alpha \in \nu(H(\mathbb{A})) \cap \boldsymbol{k}^\times$, then there exists an element $h \in H(\boldsymbol{k})$ with $\nu(h) = \alpha$.*
(ii) The theta integral is left invariant under $G(\mathbb{A})^+ \cap G(\boldsymbol{k})$.
(iii) The theta integral has trivial central character, i.e., for $z \in Z_G(\mathbb{A}) \subset G(\mathbb{A})^+$, $I(zg,\varphi) = I(g,\varphi)$.

Proof. (i) is a standard characterization of $\nu(H(\boldsymbol{k}))$ in the present case. To check (ii), given $\gamma \in G(\mathbb{A})^+ \cap G(\boldsymbol{k})$, choose $\gamma' \in H(\boldsymbol{k})$ with $\nu(\gamma') = \nu(\gamma)$. Then

$$\begin{aligned} I(\gamma g,\varphi) &= \int_{H_1(\boldsymbol{k})\backslash H_1(\mathbb{A})} \theta(h_1\gamma' h, \gamma g; \varphi)\, dh_1 \\ &= \int_{H_1(\boldsymbol{k})\backslash H_1(\mathbb{A})} \theta(\gamma' h_1 h, \gamma g; \varphi)\, dh_1 \\ &= I(g,\varphi), \end{aligned} \tag{3.10}$$

via the left invariance of the theta kernel under $(\gamma',\gamma)\in R(\boldsymbol{k})$. Here, in the next to last step, we have conjugated the domain of integration $H_1(\boldsymbol{k})\backslash H_1(\mathbb{A})$ by the element $\gamma'\in H(\boldsymbol{k})$.

Finally, the proof of (iii) is just like that of Lemma 5.1.9 (ii) in [**3**]. $\square$

Since $G(\mathbb{A})=G(\boldsymbol{k})G(\mathbb{A})^+$, it follows that $I(g,\varphi)$ has a unique extension to a left $G(\boldsymbol{k})$–invariant function on $G(\mathbb{A})$. Moreover, for any $g_0\in G(\mathbb{A})^+$, we have

$$I(gg_0,\varphi)=I(g,\omega(h_0,g_0)\varphi), \tag{3.11}$$

where $h_0\in H(\mathbb{A})$ with $\nu(h_0)=\nu(g_0)$. In particular, if $h_1\in H_1(\mathbb{A})$, then

$$I(g,\omega(h_1)\varphi)=I(g,\varphi). \tag{3.12}$$

§4. The Siegel–Weil formula for $(GO(V),GSp_6)$.

First we recall the Siegel–Weil formula for $(O(V),Sp_6)$. The results of [**15**] on the regularized Siegel–Weil formula were formulated over a totally real number field, since, at a number of points, we needed facts about degenerate principal series, intertwining operators, etc. which had not been checked for complex places. The proof in the case of the central value of the Siegel–Eisenstein series is simpler than the general case, and the additional facts needed at complex places are easy to check. In the rest of this section, we will state the results for an arbitrary number field $\boldsymbol{k}$. A sketch of the proof of Theorem 4.1 below for such a field $\boldsymbol{k}$ will be given in the Appendix below.

Let $I_1(s)=I_{P_1}^{G_1}(\lambda_s)$ be the global induced representation of $G_1(\mathbb{A})=Sp_6(\mathbb{A})$ induced from the restriction of the character λ_s of $P(\mathbb{A})$ to $P_1(\mathbb{A})=P(\mathbb{A})\cap G_1(\mathbb{A})$.

For a global quadratic space V of dimension 4 over $\boldsymbol{k}$ associated to a quaternion algebra B, as in the previous section, there is a $(\mathfrak{g}_{1,\infty},K_{G_1,\infty})\times G_1(\mathbb{A}_f)$–equivariant map

$$S(V(\mathbb{A})^3)\longrightarrow I_1(0),\qquad \varphi\mapsto[\varphi], \tag{4.1}$$

where

$$[\varphi](g_1)=(\omega(g_1)\varphi)(0). \tag{4.2}$$

The image, $\Pi_1(V)$, is an irreducible summand of the unitarizable induced representation $I_1(0)$. By the results of Rallis [**19**], Kudla–Rallis [**14**], [**13**], and the appendix,

$$\Pi_1(V)\simeq S(V(\mathbb{A})^3)_{O(V)(\mathbb{A})}, \tag{4.3}$$

the space of $H_1(\mathbb{A}) = O(V)(\mathbb{A})$–coinvariants. One then has a decomposition

$$I_1(0) = \Big(\oplus_V \Pi_1(V) \Big) \oplus \Big(\oplus_{\mathcal{V}} \Pi_1(\mathcal{V}) \Big), \tag{4.4}$$

into irreducible representations of $G_1(\mathbb{A})$, as V runs over the isomorphism classes of such spaces and as $\mathcal{V}$ runs over the incoherent collections, obtained by switching one local component of a $\Pi_1(V)$, cf. [**12**].

The Siegel–Weil formula of [**4**], asserts the following in the present case.

Theorem 4.1. *(i) The* $(\mathfrak{g}_{1,\infty}, K_{G_{1,\infty}}) \times G_1(\mathbb{A}_f)$*–intertwining map*

$$E_1(0) : I_1(0) \longrightarrow \mathcal{A}(G_1), \qquad \Phi_0 \mapsto (g_1 \mapsto E(g_1, 0, \Phi_s))$$

has kernel $\oplus_{\mathcal{V}} \Pi_1(\mathcal{V})$.
(ii) For a section $\Phi_s \in I_1(s)$ *with* $\Phi_0 = [\varphi]$ *for some* $\varphi \in S(V(\mathbb{A})^3)$,

$$E(g_1, 0, \Phi_s) = 2\, I(g_1, \varphi), \tag{SW}$$

for the theta integral as defined in §3.

As explained in the previous section, the theta integral can be extended to an automorphic form on $G(\mathbb{A})$. We will see presently that it coincides with an Eisenstein series on $G(\mathbb{A})$.

Restriction of functions from $G(\mathbb{A})$ to $G_1(\mathbb{A})$ yields an isomorphism $I(s) \xrightarrow{\sim} I_1(s)$, which is intertwining for the right action of $G_1(\mathbb{A})$. Here $I(s)$ is the induced representation of $G(\mathbb{A})$ defined in section 1. The inverse map is given by $\Phi_s \mapsto \Phi_s^{\sim}$ where

$$\Phi_s^{\sim}(g) = |\nu(g)|^{-3s-3} \Phi_s(g_1), \tag{4.5}$$

for $g_1 = d(\nu(g))^{-1} g$, as above. The decomposition (4.4) into $G_1(\mathbb{A})$–irreducibles yields a decomposition

$$I(0) = \Big(\oplus_B \Pi(B) \Big) \oplus \Big(\oplus_{\mathcal{B}} \Pi(\mathcal{B}) \Big), \tag{4.6}$$

into irreducible representations of $G(\mathbb{A})$, where, for a global quaternion algebra B over $\boldsymbol{k}$,

$$\Pi(B) = \oplus_V \Pi(V) \tag{4.7}$$

where V runs over the non-isomorphic spaces associated to B (i.e., different multiples of the norm form) and $\Pi(V)$ denotes the image of $\Pi_1(V)$ under the inverse of the restriction isomorphism. Note that there are $2^{|\Sigma_\infty(B)|}$ such V's. In effect, at a real archimedean place v, the local induced representation $I_1(0)_v$ has a decomposition into irreducible $(\mathfrak{g}_{1,v}, K_{G_1,v})$–modules

$$I(0)_v = \Pi(4,0)_v \oplus \Pi(2,2)_v \oplus \Pi(0,4)_v \tag{4.8}$$

according to signatures. The space $\Pi(2,2)_v$ is actually stable under $(\mathfrak{g}_v, K_{G,v})$, as is the sum $\Pi(4,0)_v \oplus \Pi(0,4)_v$, and

$$\Pi(B)_v = \begin{cases} \Pi(2,2)_v & \text{if } B_v \simeq M_2(\mathbb{R}). \\ \Pi(4,0)_v \oplus \Pi(0,4)_v & \text{if } B_v \text{ is division.} \end{cases} \tag{4.9}$$

The sumands $\Pi(\mathcal{B})$ are defined similarly.

Theorem 4.2. *(i) The* $(\mathfrak{g}_\infty, K_{G,\infty}) \times G(\mathbb{A}_f)$*–intertwining map*

$$E(0) : I(0) \longrightarrow \mathcal{A}(G), \qquad \Phi_s \mapsto (g \mapsto E(g, 0, \Phi_s))$$

has kernel $\oplus_{\mathcal{B}}\Pi(\mathcal{B})$*.*
(ii) For a section $\Phi_s \in I(s)$ *with* $\Phi_0 \in \Pi(V)$ *so that* $\Phi_0 = [\varphi]^\sim$ *for some* $\varphi \in S(V(\mathbb{A})^3)$*,*

$$E(g, 0, \Phi_s) = 2\, I(g, \varphi), \tag{GSW}$$

for the theta integral as defined in §3.

Proof. For $g_0 \in G(\mathbb{A}_f)$, we have

$$E(gg_0, s, \Phi_s) = E(g, s, r_s(g_0)\Phi_s), \tag{4.10}$$

where r_s denotes the action in the induced representation $I(s)$ by right translation. Taking the value at $s = 0$, we obtain

$$E(gg_0, 0, \Phi_s) = E(g, 0, r_s(g_0)\Phi_s). \tag{4.11}$$

Note that this value depends only on Φ_0 and $r_0(g_0)\Phi_0$.

Lemma 4.3. *For* $\varphi \in S(V(\mathbb{A})^3)$*, let* $[\varphi] \in I_1(0)$ *be defined by (4.2) and let* $[\varphi]^\sim$ *be the corresponding function in* $I(0)$ *under the inverse of the restriction isomorphism.*
(i) For $g \in G(\mathbb{A})^+$*,*

$$[\varphi]^\sim(g) = (\omega(h, g)\varphi)(0),$$

where $h \in GO(V)(\mathbb{A})$ *with* $\nu(h) = \nu(g)$*.*
(ii) For $g_0 \in G(\mathbb{A}_f)$*,*

$$r_0(g_0)[\varphi]^\sim = [\omega(h_0, g_0)\varphi]^\sim,$$

where $h_0 \in GO(V)(\mathbb{A}_f)$ *with* $\nu(h_0) = \nu(g_0)$.

Proof. For (i), we have

$$
\begin{aligned}
[\varphi]^{\sim}(g) &= [\varphi]^{\sim}(d(\nu)g_1) \\
&= |\nu|^{-3}[\varphi](g_1) \\
&= |\nu|^{-3}(\omega(g_1)\varphi)(0) \\
&= (L(h)\omega(g_1)\varphi)(0) \\
&= (\omega(h,g)\varphi)(0).
\end{aligned} \tag{4.12}
$$

For (ii),

$$
\begin{aligned}
(r_0(g_0)[\varphi]^{\sim})(g) &= |\nu|^{-3}[\varphi]^{\sim}(g_1g_0) \\
&= |\nu|^{-3}(\omega(h_0, g_1g_0)\varphi)(0) \\
&= |\nu|^{-3}(\omega(g_1)\omega(h_0,g_0)\varphi)(0) \\
&= |\nu|^{-3}[\omega(h_0,g_0)\varphi](g_1) \\
&= [\omega(h_0,g_0)\varphi]^{\sim}(g).
\end{aligned} \tag{4.13}
$$

□

Thus, if $\Phi_0 = [\varphi]^{\sim}$, then $r_0(g_0)\Phi_0 = [\omega(h_0,g_0)\varphi]^{\sim}$. Since $G(\mathbb{A}) = G(\boldsymbol{k})Z_G(\mathbb{A})G_1(\mathbb{A})G(\mathbb{A}_f)$, we have, by (4.11),

$$
\begin{aligned}
E(g,0,\Phi_s) &= E(\gamma z g_1 g_0, 0, \Phi_s) \\
&= E(g_1, 0, r_s(g_0)\Phi_s),
\end{aligned} \tag{4.14}
$$

the value at g_1 of the Siegel–Eisenstein series attached to $\omega(h_0,g_0)\varphi \in S(V(\mathbb{A})^3)$. On the other hand, by (3.11),

$$
\begin{aligned}
I(g,\varphi) &= I(\gamma z g_1 g_0, \varphi) \\
&= I(g_1, \omega(h_0,g_0)\varphi).
\end{aligned} \tag{4.15}
$$

Thus (GSW) follows from (SW). □

§5. Proof of Jacquet's conjecture.

Applying the Siegel–Weil formula for similitudes to the basic identity (2.3), we obtain

$$
\begin{aligned}
&L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \cdot Z^*(F, \Phi) \\
(5.1) \qquad &= \int_{Z_{\mathbf{G}}(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash \mathbf{G}(\mathbb{A})} E^*(\mathbf{g}, 0, \Phi_s)\, F(\mathbf{g})\, d\mathbf{g} \\
&= 2\zeta_{\boldsymbol{k}}(2)^2 \sum_V \int_{Z_{\mathbf{G}}(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash \mathbf{G}(\mathbb{A})} I(\mathbf{g}, \varphi^V)\, F(\mathbf{g})\, d\mathbf{g}.
\end{aligned}
$$

where

$$
(5.2) \qquad Z^*(F, \Phi) = \prod_{v \in S} Z_v^*(0, W_v^{\psi}, \Phi_{s,v}),
$$

and where $\varphi^V \in S(V(\mathbb{A})^3)$, and, in fact, only a finite set of V's occurs in the sum. More precisely, in the decomposition (4.6),

$$
(5.3) \qquad \Phi_0 = \sum_V\, [\varphi^V]^{\sim} + \text{ terms in the } \Pi(\mathcal{B})\text{'s } \in I(0),
$$

where the quaternion algebras B associated to V's are split outside the set S, due to condition (iii) in the definition of S in section 1. We thus have the following reformulation of Corollary 2.3, generalizing Proposition 5.6 of [**4**]:

Corollary 5.1. $L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) = 0$ *if and only if*

$$
\int_{Z_{\mathbf{G}}(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash \mathbf{G}(\mathbb{A})} I(\mathbf{g}, \varphi^V)\, F(\mathbf{g})\, d\mathbf{g}
$$

vanishes for all choices of $F \in \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$*, all choices of quadratic spaces* V *attached to quaternion algebras* B *over* $\boldsymbol{k}$*, and all choices of* $\varphi^V \in S(V(\mathbb{A})^3)$*.*

Now consider the integral in the last line of (5.1) for a fixed $\varphi = \varphi^V$. To apply the seesaw identity, we set

$$
\begin{aligned}
H &= GO(V) \\
(5.4) \qquad \mathbf{H} &= \{(h_1, h_2, h_3) \in H^3 \mid \nu(h_1) = \nu(h_2) = \nu(h_3)\}, \\
\mathbf{R} &= \{(\mathbf{h}, \mathbf{g}) \in \mathbf{H} \times \mathbf{G} \mid \nu(\mathbf{h}) = \nu(\mathbf{g})\} \\
R_0 &= \{(h, \mathbf{g}) \in H \times \mathbf{G} \mid \nu(h) = \nu(\mathbf{g})\},
\end{aligned}
$$

and hence have the seesaw pair:

$$\begin{array}{cccccccc} I(\cdot,\varphi;F) & (GO(V)^3)_0 = & \mathbf{H} & & & G & = GSp_6 & I(\cdot,\varphi) \\ & & \uparrow & \searrow\!\!\nwarrow & \swarrow\!\!\nearrow & \uparrow & & \\ \mathbb{1} & GO(V) = & H & & & \mathbf{G} & = (GL_2^3)_0 & F \end{array} \tag{5.5}$$

There are representations of both $R(\mathbb{A})$ and $\mathbf{R}(\mathbb{A})$ on $S(V(\mathbb{A})^3)$, and the restriction of these representations to the common subgroup $R_0(\mathbb{A})$ coincide.

For F a cuspidal automorphic form on $\mathbf{G}(\mathbb{A})$ and for $\mathbf{h} \in \mathbf{H}(\mathbb{A})$, let

$$I(\mathbf{h},\varphi;F) = \int_{\mathbf{G}_1(\boldsymbol{k})\backslash\mathbf{G}_1(\mathbb{A})} \theta(\mathbf{h},\mathbf{g}_1\mathbf{g};\varphi)\, F(\mathbf{g}_1\mathbf{g})\, d\mathbf{g}_1, \tag{5.6}$$

where $\mathbf{g} \in \mathbf{G}(\mathbb{A})$ with $\nu(\mathbf{g}) = \nu(\mathbf{h})$.

Lemma 5.2. (Seesaw identity)

$$\int_{Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A})} I(\mathbf{g},\varphi)\, F(\mathbf{g})\, d\mathbf{g} = \int_{Z_H(\mathbb{A})H(\boldsymbol{k})\backslash H(\mathbb{A})} I(h,\varphi;F)\, dh.$$

Proof. Note that $Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A}) \simeq Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})^+\backslash\mathbf{G}(\mathbb{A})^+$, and that

$$Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})^+\mathbf{G}_1(\mathbb{A})\backslash\mathbf{G}(\mathbb{A})^+ \simeq Z_H(\mathbb{A})H(\boldsymbol{k})\backslash H(\mathbb{A}) \simeq \mathbb{A}^{\times,2}\boldsymbol{k}^{\times,+}\backslash\mathbb{A}^{\times,+} =: C, \tag{5.7}$$

is compact, where $\mathbb{A}^{\times,+} = \nu(H(\mathbb{A}))$ and $\boldsymbol{k}^{\times,+} = \nu(H(\boldsymbol{k}))$. Fixing a Haar measure dc giving C volume 1, we have

$$\begin{aligned} &\int_{Z_{\mathbf{G}}(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A})} I(\mathbf{g},\varphi)\, F(\mathbf{g})\, d\mathbf{g} \\ &= \int_C \int_{\mathbf{G}_1(\boldsymbol{k})\backslash\mathbf{G}_1(\mathbb{A})} \int_{H_1(\boldsymbol{k})\backslash H_1(\mathbb{A})} \theta(h_1 h(c),\mathbf{g}_1\mathbf{g}(c);\varphi)\, F(\mathbf{g}_1\mathbf{g}(c))\, dh_1\, d\mathbf{g}_1\, dc \\ &= \int_{Z_H(\mathbb{A})H(\boldsymbol{k})\backslash H(\mathbb{A})} I(h,\varphi;F)\, dh, \end{aligned} \tag{5.8}$$

generalizing the proof of Proposition 7.1.4 of [**3**]. $\square$

To apply the seesaw identity to the restriction to $\mathbf{G}(\mathbb{A})$ of a function $F \in \Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$, we recall the description, from sections 7 and 8 of [**4**], of the corresponding space of functions $\Theta(\Pi)$ on $\mathbf{H}(\mathbb{A})$ spanned by the $I(\mathbf{h},\varphi;F)$'s for $F \in \Pi$ and $\varphi \in$

$S(V(\mathbb{A})^3)$. Note that one obtains the same space by fixing a nonzero F and only varying φ [**5**].

The action of $B^\times \times B^\times$ on $V = B$, $\rho(b_1, b_2)x = b_1 x b_2^{-1}$ determines an extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \big(B^\times \times B^\times\big) \rtimes \langle \mathbf{t} \rangle \longrightarrow H = GO(V) \longrightarrow 1 \tag{5.9}$$

where the involution $\mathbf{t}$ acts on V by $\rho(\mathbf{t})(x) = x^\iota$ and on $B^\times \times B^\times$ by $(b_1, b_2) \mapsto (b_2^\iota, b_1^\iota)^{-1}$. Write

$$\tilde{H} = \big(B^\times \times B^\times\big) \rtimes \langle \mathbf{t} \rangle \qquad \text{and} \qquad \tilde{H}^0 = B^\times \times B^\times, \tag{5.10}$$

and let $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}^0$ be the analogous groups for $\mathbf{H} = (GO(V)^3)_0$. Thus, we have the diagram

$$\begin{array}{ccccc} & & \tilde{\Theta}(\Pi) & & \Theta(\Pi) \\ \tilde{\mathbf{H}}^0 & \hookrightarrow & \tilde{\mathbf{H}} & \longrightarrow & \mathbf{H} \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{H}^0 & \hookrightarrow & \tilde{H} & \longrightarrow & H \end{array} \tag{5.11}$$

For an irreducible cuspidal automorphic representation π of $GL_2(\mathbb{A})$, let π^B be the associated automorphic representation of $B^\times(\mathbb{A})$ under the Jacquet-Langlands correspondence. We take π^B to be zero if π does not correspond to a representation of $B^\times(\mathbb{A})$. Similarly, let $\Pi^B = \pi_1^B \otimes \pi_2^B \otimes \pi_3^B$ be the corresponding representation of $B^\times(\mathbb{A})^3$, or zero if some factor does not exist. Note that the central character of Π^B is trivial, and so, $(\Pi^B)^\vee \simeq \Pi^B$, where $(\Pi^B)^\vee$ is the contragradient of Π^B. Thus we can view the space of of functions Π^B on $B^\times(\mathbb{A})^3$ as the automorphic realization of both Π^B and its contragredient.

The following result is proved in [**4**], sections 7 and 8, based on the work of Shimizu and Prasad.

Proposition 5.3. *(i)* $\Theta(\Pi)$ *is either zero or a cuspidal automorphic representation of* $\mathbf{H}(\mathbb{A})$ *and is nonzero if and only if* Π^B *is nonzero.*
(ii) As spaces of functions on $\tilde{\mathbf{H}}^0(\mathbb{A})$*,*

$$\tilde{\Theta}(\Pi)\big|_{\tilde{\mathbf{H}}^0(\mathbb{A})} = \Big(\Pi^B \otimes (\Pi^B)^\vee\Big)\Big|_{\tilde{\mathbf{H}}^0(\mathbb{A})}.$$

For fixed $F \in \Pi$ and $\varphi \in S(V(\mathbb{A})^3)$, we let $\tilde{I}(\cdot, \varphi; F)$ denote the pullback of $I(\cdot, \varphi; F)$ to $\tilde{\mathbf{H}}(\mathbb{A})$, and, via (ii) of Proposition 5.3, we write the restriction of this function to $\tilde{\mathbf{H}}^0(A)$ as

$$\tilde{I}((\mathbf{b}_1, \mathbf{b}_2), \varphi; F) = \sum_r I^{1,r}(\mathbf{b}_1, \varphi; F)\, I^{2,r}(\mathbf{b}_2, \varphi; F) \tag{5.12}$$

for functions $I^{i,r}(\cdot,\varphi;F)\in\Pi^B$ and $\mathbf{b}_i\in B^\times(\mathbb{A})^3$. The seesaw then gives

$$
\begin{aligned}
&\int_{Z_G(\mathbb{A})\mathbf{G}(\boldsymbol{k})\backslash\mathbf{G}(\mathbb{A})} I(\mathbf{g},\varphi)\,F(\mathbf{g})\,d\mathbf{g}\\
&=\int_{Z_H(\mathbb{A})H(\boldsymbol{k})\backslash H(\mathbb{A})} I(h,\varphi;F)\,dh\\
&=\int_{Z_{\tilde H^0}(\mathbb{A})\tilde H(\boldsymbol{k})\backslash\tilde H(\mathbb{A})} \tilde I(h,\varphi;F)\,dh\\
&=\int_{Z_{\tilde H^0}(\mathbb{A})\tilde H^0(\boldsymbol{k})\backslash\tilde H^0(\mathbb{A})} \tilde I(h,\varphi;F)\,dh\\
&=\sum_r\int_{\mathbb{A}^\times B^\times(\boldsymbol{k})\backslash B^\times(\mathbb{A})} I^{1,r}(b_1,\varphi;F)\,db_1\cdot\int_{\mathbb{A}^\times B^\times(\boldsymbol{k})\backslash B^\times(\mathbb{A})} I^{2,r}(b_2,\varphi;F)\,db_2
\end{aligned}
\tag{5.13}
$$

The fact that the integral over $Z_{\tilde H^0}(\mathbb{A})\tilde H(\boldsymbol{k})\backslash\tilde H(\mathbb{A})$ in the third line can be replaced by the integral over $Z_{\tilde H^0}(\mathbb{A})\tilde H^0(\boldsymbol{k})\backslash\tilde H^0(\mathbb{A})$ is (7.3.2), p.632 of [**4**]. Its proof in section 8.6, p.636 of [**4**] depends on Prasad's uniqueness theorem [**18**] for invariant trilinear forms. This theorem was recently completed by H. Y. Loke [**16**], who treated general triples of admissible irreducible representations of $GL(2,\mathbb{R})$ and $GL(2,\mathbb{C})$.[1] Thus the calculation in (5.13) is valid for all number fields and for all triples of cuspidal automorphic representations.

Finally, we observe that the integrals in the last line of (5.13) are finite linear combinations of the quantities $I(f_1^B,f_2^B,f_3^B)$ of (0.1). By (ii) of Proposition 5.3 , every such quantity can be obtained as an integral $\int_{\mathbb{A}^\times B^\times(\boldsymbol{k})\backslash B^\times(\mathbb{A})} I^{1,r}(b_1,\varphi;F)\,db_1$ for some φ, F and r.

Jacquet's conjecture now follows upon combining this observation with Corollary 5.1 (compare the proof of Theorem 7.4 in [**4**]).

Remark 5.4. In fact, by Prasad's uniqueness theorem, if the root number

$$\epsilon(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3)=1, \tag{5.14}$$

then there is a unique B for which $\Pi^B\neq 0$ and for which the space of global invariant trilinear forms on Π^B has dimension 1. The *automorphic* trilinear form is given by integration over $\mathbb{A}^\times B^\times(\boldsymbol{k})\backslash B^\times(\mathbb{A})$ is then non-zero if and only if $L(\frac{1}{2},\pi_1\otimes\pi_2\otimes\pi_3)\neq 0$. Choose $f_i^B\in\pi_i^B$, $i=1, 2, 3$, such that

$$I(f_1^B,f_2^B,f_3^B)\neq 0. \tag{5.15}$$

[1] This is the only place in [**4**] where Prasad's uniqueness theorem is used, although it was an important motivation for the article as well as for Jacquet's conjecture.

For any nonzero $F \in \Pi$, we can choose $\varphi \in S(V(\mathbb{A}))^3$ such that

$$\tilde{I}((\mathbf{b}, \mathbf{b}'), \varphi; F) = f_1^B(b_1) f_2^B(b_2) f_3^B(b_3)\, f_1^B(b_1') f_2^B(b_2') f_3^B(b_3'), \tag{5.16}$$

where $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{b}' = (b_1', b_2', b_3')$. We then obtain

$$L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) \cdot Z^*(F, \Phi) = 2\zeta_{\boldsymbol{k}}(2)^2\, I(f_1^B, f_2^B, f_3^B)^2. \tag{5.17}$$

where, Φ is determined by φ, and $Z^*(F, \Phi) \neq 0$. Of course, this identity is only useful when one has sufficient information about the function φ and the product of local zeta integrals $Z^*(F, \Phi)$. This was a main concern in [**4**].

On the other hand, when the root number $\epsilon(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3) = -1$, then there is no Π^B which supports an invariant trilinear form, and the central value of the triple product L-function vanishes due to the sign in the functional equation.

Appendix: The Siegel–Weil formula for general $\boldsymbol{k}$.

In this appendix, we will sketch the proof of Theorem 4.1 for an arbitrary number field $\boldsymbol{k}$, indicating the additional facts which are needed when $\boldsymbol{k}$ has complex places.

First, suppose that v is a complex place of $\boldsymbol{k}$ and consider the local degenerate principal series representation $I_{1,v}(0)$ of $G_{1,v} = Sp_3(\mathbb{C})$ and the Weil representation of $G_{1,v}$ on $S(V_v^3)$, where $V_v \simeq M_2(\mathbb{C})$ with $Q(x) = \det(x)$.

Lemma A.1. *(i) $I_{1,v}(0)$ is an irreducible unitarizable representation of $G_{1,v}$.*
(ii) (Coinvariants) The map $S(V_v^3) \to I_{1,v}(0)$, $\varphi \mapsto [\varphi]$, analogous to (4.1) induces an isomorphism

$$S(V_v^3)_{H_{1,v}} \xrightarrow{\sim} I_{1,v}(0).$$

Here $H_1 = O(V)$.

Remark: Statement (i) is in Sahi's paper, [**23**], Theorem 3A. The proof of (ii) was explained to us by Chen-bo Zhu[2] [**28**], and is based on the method of [**27**].

In the case $B = M_2(\boldsymbol{k})$, the theta integral must be defined by regularization, – cf. the remarks before Lemma 3.2 above. be applied. Alternatively, formula for certain We write

$$I_{\mathrm{reg}}(g_1, \varphi) = \begin{cases} I(g_1, \varphi) & \text{if } V \text{ is anisotropic,} \\ B_{-1}(g_1, \varphi) & \text{if } V \text{ is isotropic,} \end{cases} \tag{A.1}$$

where B_{-1} is as in (5.5.24) of [**15**], except that we normalize the auxillary Eisenstein series $E(h, s)$ to have residue 1 at s_0'. The key facts which we need are the following.

[2]He also directed us to [**23**]. We wish to thank him for his help on these points.

Lemma A.2. *(i) The map* $I_{\mathrm{reg}} : S(V(\mathbb{A})^3) \to \mathcal{A}(G_1)$ *factors through the space of coinvariants* $S(V(\mathbb{A})^3)_{H_1(\mathbb{A})} = \Pi(V)$.
(ii) For all $\beta \in \mathrm{Sym}_3(\boldsymbol{k})$,

$$I_{\mathrm{reg},\beta}(g_1, \varphi) = \frac{1}{2} \cdot \int_{H_1(\mathbb{A})} \omega(g_1)\varphi(h^{-1}x)\, dh,$$

where $x \in V(\boldsymbol{k})^3$ *with* $Q(x) = \beta$.

The second statement here is Corollary 6.11 of [**15**]; it asserts that the nonsingular Fourier coefficients behave as though no regularization were involved.

Next we have the analogue of Lemma 4.2, p.111 of [**20**]; the main point of the proof is the local uniqueness, and the 'submersive set' argument for the archimedean places carries over for a complex place.

Lemma A.3. *For* $\beta \in \mathrm{Sym}_3(\boldsymbol{k})$ *with* $\det(\beta) \neq 0$, *let* $\mathcal{T}_\beta$ *be the space of distributions* $T \in S(V(\mathbb{A})^3)'$ *such that*
(i) T *is* $H(\mathbb{A})$*–invariant.*
(ii) For all $b \in \mathrm{Sym}_3(\mathbb{A}_f)$,

$$T(\omega(n(b)\varphi) = \psi_\beta(b)\, T(\varphi),$$

where $\psi_\beta(b) = \psi(\mathrm{tr}(\beta b))$.
(iii) For an archimedean place v *of* $\boldsymbol{k}$ *and for all* $X \in \mathfrak{n} = \mathrm{Lie}(N)$,

$$T(\omega(X)\varphi) = d\psi_\beta(X) \cdot T(\varphi).$$

Then $\mathcal{T}_\beta$ *has dimension at most* 1 *and is spanned by the orbital integral*

$$T(\varphi) = \int_{H_1(\mathbb{A})} \varphi(h^{-1}x)\, dh,$$

where $x \in V(\boldsymbol{k})^3$ *with* $Q(x) = \beta$. *In particular,* $\mathcal{T}_\beta = 0$ *if an only if there is no such* x.

sketch of the Proof of Theorem 4.1. First consider a global space V associated to a quaternion algebra B. We have two intertwining maps

$$E_1(0) : \Pi(V) \longrightarrow \mathcal{A}(G_1) \qquad \text{and} \qquad I_{\mathrm{reg}} : \Pi(V) \longrightarrow \mathcal{A}(G_1) \tag{A.2}$$

from the irreducible representation $\Pi(V) \simeq S(V(\mathbb{A})^3)_{H(\mathbb{A})}$ of $G_1(\mathbb{A})$ to the space of automorphic forms. For a nonsingular $\beta \in \mathrm{Sym}_3(\boldsymbol{k})$, the distributions obtained by taking the βth Fourier coefficient of the composition of the projection $S(V(\mathbb{A})^3) \to$

$\Pi(V)$ with each of the embeddings in (A.2) satisfy the conditions of Lemma A.3 and hence are proportional. In particular, the β-th Fourier of the Eisenstein series vanishes unless β is represented by V. By the argument of pp. 111–115 of [**20**], the constant of proportionality is independent of β and so there is a constant c such that $E_1(g, 0, [\varphi]) - c \cdot I_{\text{reg}}(g, \varphi)$ has vanishing nonsingular Fourier coefficients. But then the argument at the top of p.28 of [**15**], cf. also, [**20**], implies that this difference is identically zero.

In the case of a component $\Pi(\mathcal{V}) \subset I_1(0)$, the nonsingular Fourier coefficients of $E(g, 0, \Phi)$ vanish by the argument on p. 28 of [**15**], so, again by 'nonsingularity' the map $E_1(0)$ must vanish on $\Pi(\mathcal{V})$. $\square$

References

[**1**] P. Garrett, *Decomposition of Eisenstein series: Rankin triple products*, Annals of Math. **125** (1987), 209–235.

[**2**] B. Gross and S. Kudla, *Heights and the central critical values of triple product L-functions*, Compositio Math. **81** (1982), 143–209.

[**3**] M. Harris and S. Kudla, *Arithmetic automorphic forms for the nonholomorphic discrete series of GSp(2)*, Duke Math. J. **66** (1992), 59–121.

[**4**] ———, *The central critical value of a triple product L-function*, Annals of Math. **133** (1991), 605–672.

[**5**] R. Howe and I.I. Piatetski–Shapiro, *Some examples of automorphic forms on Sp_4*, Duke Math. J. **50** (1983), 55–106.

[**6**] T. Ikeda, *On the functional equations of triple L-functions*, J. Math. Kyoto Univ. **29** (1989), 175–219.

[**7**] ———, *On the location of poles of the triple L-functions*, Compositio Math. **83** (1992), 187–237.

[**8**] ———, *On the gamma factor of the triple L-function I*, Duke Math. J. **97** (1999), 301–318.

[**9**] Dihua Jiang, *Nonvanishing of the central critical value of the triple product L-functions*, Internat. Math. Res. Notices 2 (1998), 73–84.

[**10**] H. H. Kim and F. Shahidi, *Holomorphy of Rankin triple L-functions; special values and root numbers for symmetric cube L-functions*, Israel J. Math. **120** (2000), 449-466.

[**11**] ———, *Functorial products for $GL_2 \times GL_3$ and functorial symmetric cube for GL_2*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 599–604.

[**12**] S. Kudla, *Central derivatives of Eisenstein series and height pairings*, Ann. of Math. **146** (1997), 545-646.

[**13**] S. Kudla and S. Rallis, *Degenerate principal series and invariant distributions*, Israel J. Math. **69** (1990), 25–45.

[**14**] ______, *Ramified degenerate principal series*, Israel J. Math. **78** (1992), 209–256.

[**15**] ______, *A regularized Siegel–Weil formula: The first term identity*, Annals of Math. **140** (1994), 1–80.

[**16**] H. Y. Loke, *Trilinear forms of gl_2.*, Pacific J. Math. **197** (2001), 119–144.

[**17**] I.I. Piatetski-Shapiro and S. Rallis, *Rankin triple L-functions*, Compositio Math. **64** (1987), 31–115.

[**18**] D. Prasad, *Trilinear forms for representations of $GL(2)$ and local (epsilon)-factors*, Compositio Math. **75** (1990), 1-46.

[**19**] S. Rallis, *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.

[**20**] ______, *L-functions and the oscillator representation*, Lecture Notes in Math. **1245**, Springer–Verlag, New York, 1987.

[**21**] D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for $SL(2)$*, Annals of Math. **152** (2000), 45–111..

[**22**] B. Roberts, *The theta correspondence for similitudes*, Israel J. Math. **94** (1996), 285–317.

[**23**] S. Sahi, *Jordan algebras and degenerate principal series*, Crelle's Jour **462** (1995), 1-18.

[**24**] H. Shimizu, *Theta series and automorphic forms on GL_2*, Jour. Math. Soc. Japan **24** (1972), 638–683.

[**25**] V. Tan, *A regularized Siegel–Weil formula on U(2,2) and U(3)*, Duke Math. J. **94** (1998), 341–378.

[**26**] T. Watson, *Rankin triple products and quantum chaos*, Thesis, Princeton Univ. (2002).

[**27**] Chen-bo Zhu, *Invariant distributions of classical groups*, Duke Math. Jour. **65** (1992).

[**28**] ______, *private communication.*