Restrictions on collapse with a lower sectional curvature bound

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Abstract

We obtain a partial result towards a conjecture that not every Alexandrov space can be obtained as a limit of a sequence of Riemannian manifolds satisfying a lower sectional curvature bound.

1 Introduction

The study of Alexandrov spaces with curvature bounded from below while being interesting as a subject in itself, has produced a lot of applications to classical Riemannian geometry. (see [BGP92] for the basics of the theory of Alexandrov spaces). One of the major sources of these applications is provided by the combination of the following by now well-known facts [BGP92]:

1. Let $\mathcal{M}_k^{n,D}$ be the class of n-dimensional Riemannian manifolds with sectional curvatures bounded below by k and diameter $\le D$. Then this class is precompact in the Gromov-Hausdorff topology.
2. The property of a metric space to have curvature bounded from below is stable under taking Gromov-Hausdorff limits.

If X is a limit of a sequence of Riemannian manifolds $M_i^n \in \mathcal{M}_k^{n,D}$ then it is not hard to see [BGP92] that the Hausdorff dimension of X can not be greater than n.

If it is equal to n we say that the sequence M_i^n converges without collapse and if it is less than n we say that this sequence collapses.

The first case is understood fairly well at least topologically due to the stability theorem by Perelman [Per91], which says that for sufficiently large indices, Hausdorff approximations $M_i \to X$ are close to homeomorphisms. Unlike the situation in the noncollapsing case, very little is known about the structure of the limit and its relationship to the elements of the sequence when collapse does occur. The main structural result here is the Theorem of Yamagucci [Yam91] which asserts that if the limit is a Riemannian manifold then the Hausdorff approximations into the limit from the elements of the sequence can be chosen to be smooth fibrations.

In particular, the following natural question remains unanswered:

Question 1.1. *Is it true that any Alexandrov space can be obtained as a limit of a sequence of Riemannian manifolds satisfying a lower sectional curvature bound?*

The collapsing phenomenon occurs naturally when one considers a pointed sequence formed by rescaling of a nonnegatively curved open manifold by positive constants approaching 0. The limit in this case is a Euclidean cone over the ideal boundary of M, which we will denote by $M(\infty)$.

From this description it is easy to conclude that $M(\infty)$ is an Aleksandrov space with curvature bounded below by 1 [BGP92]. It was shown in [GK95] that if the ideal boundary is a Riemannian manifold then its topology is severely restricted:

Theorem 1.2. *[GK95] Let M^n be a complete open manifold with $K_{sec} \geq 0$. If $M(\infty)$ is an m-dimensional Riemannian manifold, then it has a finite covering $N \to M(\infty)$ satisfying one of the following conditions:*

- *(i)* N *is homotopy equivalent to* S^m, *or*
- *(ii)* N *is homotopy equivalent to* $\mathbb{CP}^{m/2}$, *or*
- *(iii)* N *has rational cohomology ring of* $\mathbb{HP}^{m/4}$

Let $\bar{\mathcal{M}}_k^n$ be the closure of $\mathcal{M}_k^{n,\infty}$ in the pointed Gromov-Hausdorff topology. In other words, $\bar{\mathcal{M}}_k^n$ consists of pointed limits of sequences of Riemannian manifolds M_i^n satisfying $\sec(M_i^n) \geq k$.

It was also shown in [PWZ95] that if the spherical suspension over a Riemannian manifold N belongs to $\bar{\mathcal{M}}_{1/4}^n$ then N must satisfy one of the conditions $(i) - (iii)$ above.

Based on this and Theorem 1.2, it was conjectured in [GK95] that if M is a positively curved manifold that does not satisfy the conclusions of Theorem 1.2 then the spherical suspension SM does not belong to $\bar{\mathcal{M}}_k^n$ fro any k, n.

The goal of the present paper is to obtain a partial result towards verifying this conjecture.

It is a classical result of Berger that the 24-dimensional Caley flag $\Sigma^{24} = F_4/Spin(8)$ admits a homogeneous metric of $sec \geq 1$ [Ber61].

We prove

Theorem 1.3. *Let X^{25} be an Alexandrov space such there exists a point $x_0 \in X$ such that $\Sigma_{x_0} X$ is a smooth manifold diffeomorphic to Σ^{24}. Then $X \notin \bar{\mathcal{M}}_k^{25+s}$ for any $k \in \mathbb{R}$ and $s < 15$.*

In other words we show that X^{25} can not be a limit of a collapsing sequence of manifolds with a lower curvature bound with the codimension of the collapse less than 15.

Remark 1.4. If we equip Σ^{24} with the Berger metric of $\sec \geq 1$ then the natural suspension metric of $curv \geq 1$ on $X = S\Sigma$ satisfies the assumptions of Theorem 1.3.

Let us briefly describe the strategy of the proof. Suppose $M_m^n \to X$ as $m \to \infty$. A standard rescaling argument shows that we can assume that X is isometric to the cone over Σ and x_0 is equal to the vertex. In [Per93] (cf. [PP93], [Kap99]). Perelman carried out a construction of strictly convex functions in a neighborhood of a point p of a given Alexandrov space X^n of $curv \geq k$ by using a special kind of averaging procedure for distance functions. Using his technique we construct a strictly convex function f near x_0 and then lift it to the elements of the sequence. Since X is smooth away from x_0, we can use Yamagucci's fibration theorem to conclude that over the regular part of X level sets of the lifts fiber of the level sets of f which are all homeomorphic to Σ.

Our key observation is that the function f can be chosen in such a way that the lifts f_m are partially convex (more precisely they belong to $\mathcal{C}(s+1)$ where $s = n - 25$ is the codimension of collapse) . We will give a careful definition of partial convexity in Section 3 but loosely speaking a function belongs to $\mathcal{C}(s)$ if the sum of any s eignevalues of its hessian at any point is positive. In particular, the hessian of a function from $\mathcal{C}(s+1)$ has at most s nonpositive eigenvalues. Using the approximation result of Wu [Wu87], we can assume that f_m is smooth and belongs to $\mathcal{C}(s+1)$. This puts some obvious restrictions on the cohomology of the level sets of f_m if $s < n/2$. We then use the Serre spectral sequence argument to show that a total space of a bundle over M^{24} can not satisfy these restrictions if the dimension of the fiber is too small.

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2 Notations

Throughout this paper all homology and cohomology groups have $\mathbb{Z}_2$ coefficients.

For an Alexandrov space X we will denote by CX the Euclidean cone over X and by SX the Speherical suspension over X. We will denote the spherical join of X and Y by $X * Y$. For a point x in an Alexandrov space X we will denote the space of directions of X at x by $\Sigma_x X$. The reader is referred to [BGP92] for the definition of a space of directions and other basic notions of Alexandrov geometry.

Let $p.q \in X$ be two points in a finite dimensional Alexandrov space X. We will use the following notation

$p(q)' = \{\xi \in \Sigma_q X |$ there exists a shortest geodesic γ from q to p such that $\gamma'(0) = \xi\}$. Observe that $p(q)'$ is always closed. With this notation the first variation formula takes the following form

$$d(\cdot, p)'(\xi) = -\cos \angle \xi p(q)'$$

for any $\xi \in \Sigma_q X$.

3 Partially concave functions

The notion of partially convex functions was introduced by H. Wu in [Wu87]. In this paper we will work with the dual notion of partially concave functions. For the convenience of the reader we will reproduce the relevant definitions.

Let $f\colon M \to \mathbb{R}$ be a continuous function on a Riemannian manifold M. Let $\gamma\colon (-a,a) \to M$ be a geodesic such that $\gamma(0) = x \in M$ and $\gamma'(0) = X \in T_xM$.

Define

$$Cf(x;X) = \limsup_{r\to 0} \frac{1}{r^2}\{f(\gamma(r)) + f(\gamma(-r)) - 2f(\gamma(0))\} \tag{3.1}$$

We say that a set of s vectors $\{X_1,\dots,X_s\}$ in an inner product space V is ϵ-orthonormal if $|\langle X_i, X_j\rangle - \delta_{i,j}| < \epsilon$ for all i,j.

Let M^n be a Riemannian manifold and let $s \le n$ be a positive integer.

Definition 3.1. *We say that a function $f\colon M \to \mathbb{R}$ belongs to the class $\mathcal{C}(s)$ if f is locally Lipschitz and for each $x_0 \in M$ there exists a neighborhood W of x_0 and constants $\epsilon, \eta > 0$ such that*

$$\sum_{i=1}^{s} Cf(x, X_i) \le -\eta$$

for any $x \in W$ and $X_1,\dots,X_s$ - an ϵ-orthonormal set in T_xM.

Note that $\mathcal{C}(1)$ is equal to the set of all strictly concave functions on M and as it was shown in [Wu87], $\mathcal{C}(n)$ is the set of all locally Lipschitz strictly superharmonic functions on M.

We will make use of the following approximation result proved in [Wu87]:

Theorem 3.2. *Let $f \in \mathcal{C}(s)$ where $1 \le s \le \dim M$ and let $\epsilon\colon M \to \mathbb{R}$ be a positive continuous function.*

Then there exists a C^∞ function $F \in \mathcal{C}(s)$ such that $|F - f| < \epsilon$.

4 Concavity of distance functions on Alexandrov spaces

In [Per93] Perelman introduced the following definition

Definition 4.1. *A function $f\colon U \to \mathbb{R}$ defined on a domain U in an Alexandrov space X is called λ-concave if for any unit speed shortest geodesic $\gamma \subset U$ the function $t \mapsto f(\gamma(t)) + \lambda t^2$ is concave.*

Observe that a lipschitz function on X is λ-concave iff $Cf(x;v) \le -\lambda$ for any $x \in X, v \in \Sigma_x X$.

Toponogov triangle comparison implies that distance functions in a space of curvature $\ge k$ are more concave than distance functions in the model space of constant curvature k and therefore it is easy to see that the following property holds:

Let $p,q \in X$ be two points in an Alexandrov space X of *curv* $\ge k$. Let $d = d(p,q)$ and $\epsilon < d/2$. Then $f(\cdot) = d(\cdot, q)$ is λ-concave in $B(p,\epsilon)$ where λ depends *only* on d and the lower curvature bound k.

Remark 4.2. The class of examples of λ-concave functions given by the distance functions can be enlarged using the following simple but important observation from [Per93]: If f is λ-concave and $\phi\colon \mathbb{R} \to \mathbb{R}_+$ is a concave C^2 function satisfying $0 \le \phi' \le 1$ then $\phi(f)$ is again λ-concave. Indeed, it is clearly enough to consider $f\colon \mathbb{R} \to \mathbb{R}$. If f is C^2 then λ-concavity of f is equivalent to the inequality $f'' \le -\lambda$. Computing the second derivative of $\phi(f)$ we observe:

$$\phi(f)'' = \phi''(f)(f')^2 + \phi'(f)f'' \le f'' \le -\lambda$$

The general case immediately follows from this one since any λ-concave function on $\mathbb{R}$ can be approximated by C^∞ λ-concave functions.

5 Proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3 stated in the introdcution.

Recall that we start with a pointed sequence of manifolds (M_m^n, x_m) with sectional curvatures bounded below by k converging to (X, x_0). Our goal is to show that n can not be less than $40 = 25 + 15$.

An easy rescaling argument [Kap99, Lemma 3.1] shows that there exists a sequence of positive numbers $\epsilon_m \underset{m\to\infty}{\longrightarrow} 0$ such that

$$(\frac{1}{\sqrt{\epsilon_m}} M_m, x_m) \underset{m\to\infty}{\overset{G-H}{\longrightarrow}} (T_{x_0}X, o)$$

where o is the cone point of $T_{x_0}X$. Thus, we can assume that to begin with $\sec(M_m) \ge -1, X = C\Sigma$ and

$$(M_m, x_m) \underset{m\to\infty}{\overset{G-H}{\longrightarrow}} (C\Sigma, o) \tag{5.2}$$

The proof of the following Lemma is an elementary exercise in Toponogov angle comparison.

Lemma 5.1. *Let $0 < a < b$ be fixed constants. Then for any sufficiently large m there exists a C^∞ unit vector field V_m on the annulus $\bar{B}(x_m, b)\backslash B(x_m, a)$ satisfying $d(\cdot, x_m)'(V_m) \ge 1 - O(\mu_m)$.* □

To produce f and f_m we use the same construction as in [Per93, Lemma 3.6].

Fix a small $\delta > 0$. Throughout the rest of the proof we will denote by c_i or c various positive constants depending on n, Σ but not on δ.

Choose a collection $\{q^\alpha\}_{\alpha=1}^N$ to be a maximal δ-separated net in Σ.

A standard volume comparison argument shows that N satisfies

$$c_1/\delta^{24} \ge N \ge c_2/\delta^{24} \tag{5.3}$$

Let $\phi_\delta : \mathbb{R} \to \mathbb{R}$ be the continuous function uniquely determined by the following properties:

(1) $\phi_\delta(0) = 0$

(2) $\phi'_\delta(t) = 1$ for $t \le 1 - \delta^3$

(3) $\phi'_\delta(t) = 1/2$ for $t \ge 1 + \delta^3$

(4) $\phi''_\delta(t) = -1/(4\delta^3)$ for $1 - \delta^3 < t < 1 + \delta^3$

Now define f_δ by the following formula:

$$f_\delta(x) = \frac{1}{N}\sum_{\alpha=1}^{N} \phi_\delta(d(x, q^\alpha))$$

Then according to Lemma 3.6 from [Per93], (cf. [PP93, Lemma 4.3]), the function f_δ is strictly c/δ^3-concave in $B(o, \delta^3/2)$ for all sufficiently small δ.

Observe that $f_\delta(o) = \phi_\delta(1)$ since $d(o, q^\alpha) = 1$ for any α. Moreover, we claim that o is a point of a strict local maximum of f_δ. Indeed, let xX be a point sufficiently close to o. Without too much abuse of notations we can write $z = t\xi$ for some $\xi \in \Sigma$. By the diameter rigidity theorem [GG87], $\operatorname{diam}\Sigma \le \pi/2 - \eta$ for some $\eta > 0$. Therefore

$$\angle \xi q_\alpha \le \pi/2 - \eta \tag{5.4}$$

for any α. By the first variation formula this implies that $d(t\xi, q^\alpha) \le 1 - t \sin 2\eta$ for all sufficiently small t and hence , by monotonicity of ϕ_δ, we get

$$f_\delta(t\xi) \le \phi_\delta(1 - t\sin(2\eta)) < \phi_\delta(0) = f_\delta(o)$$

Since η is fixed we can from now on assume that $\delta \ll \eta$. By continuity of f_δ,there exists $\nu = \nu(\delta) \ll \delta^3$ such that

$$\inf_{x \in S(o, \nu\delta^3)} f_\delta(x) > \sup_{x \in S(o, \delta^3)} f_\delta(x) \tag{5.5}$$

This implies that there exists a positive constant a such that the level set $\{f_\delta = a\}$ is entirely contained in the open annulus $\{x \in X | \nu\delta^3 < d(x, o) < \delta^3\}$ i.e

$$\{f_\delta = a\} \subset B(o, \delta^3/4) \backslash \bar{B}(o, \nu\delta^3) \tag{5.6}$$

Let us lift f_δ to the elements of the sequence (M_m, x_m) in a natural way. More precisely, according to (5.2) there exists a μ_m-Hausdorff approximation $h_m : B(o, 2) \to B_{M_m}(x_m, 2)$, where $\mu_m \underset{m\to\infty}{\longrightarrow} 0$ and $h_m(o) = x_m$. Let $q_m^\alpha = h_m(q^\alpha)$ and

$$f_\delta^m(y) = \frac{1}{N}\sum_{\alpha=1}^{N} \phi_\delta(d(y, q_m^\alpha))$$

Let us examine this function more carefully. First of all, notice that (5.6) implies that f_δ^m has compact superlevel sets and

$$\{f_\delta^m = a\} \subset B(x_m, \delta^3/4) \backslash \bar{B}(x_m, \nu\delta^3) \tag{5.7}$$

for all sufficiently large m.

By the definition, f_δ^m is Lipschitz and moreover the first variation formula implies that it has directional derivatives everywhere in $B(x_m, 1/2)$. In fact we can even assume that f_δ^m is C^1 by substituting

By lemma 5.1, the distance function $d(\cdot, x_m)$ has no critical points in the annulus $\bar{B}(x_m, \delta^3/4)\backslash B(x_m, \nu\delta^3)$ if m is sufficiently large. The following lemma shows that the same remains true for f_m.

Let V_m be the almost radial smooth vector field on $\bar{B}(x_m, \delta^3/4)\backslash B(x_m, \nu\delta^3)$ whose existence is guaranteed by Lemma 5.1.

Lemma 5.2. *For all sufficiently large m, $(f_\delta^m)'(-V_m(y)) > c$ and $(f_\delta^m)'(V_m(y)) < -c$ for any $y \in \bar{B}(x_m, \delta^3/4)\backslash B(x_m, \nu\delta^3)$.*

Proof. We can assume that m is big enough so that $\mu_m \ll \delta^3$. Let $y_m \in \bar{B}(x_m, \delta^3/4)$. By the chain rule we see that for any $v \in \Sigma_{y_m} M_m$

$$(f_\delta^m)'(v) = \frac{1}{N}\sum_{\alpha=1}^{N} \phi_\delta'(d(y, q_m^\alpha))d(y, q_m^\alpha)'(v) \tag{5.8}$$

Since h_m is a μ_m- Hausdorff approximation, there is $y \in X$ such that $d(y_m, h_m(y)) < \mu_m$. As before we will write y as $y = t\xi$. Let $\xi_m = h_m(\xi)$.

Since $\mu_m \ll \delta$, we see that $|\tilde{\sphericalangle} x_m y_m q_m^\alpha - \tilde{\sphericalangle} oyq^\alpha| \le O(\mu_m)$. On the other hand $\tilde{\sphericalangle} oyq^\alpha = \angle oyq^\alpha = \pi - \angle yoq^\alpha - \angle oq^\alpha y \ge \pi/2 + \eta - O(\delta)$ by (5.4). Since $\delta \ll \eta$ we can conclude that $\tilde{\sphericalangle} oyq^\alpha \ge \pi/2 + \eta/2$ and therefore $\tilde{\sphericalangle} x_m y_m q_m^\alpha \ge \pi/2 + \eta/3$ if m is sufficiently large. By the Toponogov angle comparison this implies that $\angle vu \ge \pi/2 + \eta/3$ for any $v \in x_m(y_m)', u \in q_m^\alpha(y_m)'$

By the first variation formula together with (5.8), this implies that $(f_\delta^m)'(v) > c$ for any $v \in x_m(y_m)'$. By Lemma 5.1, $v = -V_m(y_m) + O(\mu_m)$ which implies that $(f_\delta^m)'(-V_m(y)) > c$ as promised.

A similar argument shows that $(f_\delta^m)'(V_m) < -c$ which concludes the proof of Lemma 5.2. □

Let $s = n - 25$ be the codimension of the collapse. The next lemma is the key ingredient in the proof of Theorem 1.3.

Lemma 5.3. *The function f_δ^m belongs to $\mathcal{C}(s+1)$ in $\bar{B}(x_m, \delta^3/2)$ for any sufficiently large m.*

Proof. As was explained in section 4, distance functions $d(\cdot, q_m^\alpha)$ are $-\lambda$-concave in $\bar{B}(x_m, \delta^3/2)$ for some universal positive constant λ.

Let $y \in \bar{B}(x_m, \delta^3/2)$.

To prove that $f_\delta^m \in \mathcal{C}(s)$ it is enough to show that for any μ_m-orthonormal frame $v_1, \ldots, v_s \in T_y M_m$ the following estimate holds

$$\sum_{i=1}^{s+1} C f_\delta^m(x, v_i) \le -\lambda \tag{5.9}$$

Suppose that estimate (5.9) is false and that for some μ_m-orthonormal frame $v_1, \dots, v_{s+1} \in T_y M_m$ we have

$$\sum_{i=1}^{s+1} C f_\delta^m(x, v_i) > -\lambda \tag{5.10}$$

By remark 4.2, f_δ^m is $-\lambda$-concave in $\bar{B}(x_m, \delta/2)$ for *any* choice of δ. Therefore $C f_\delta^m(x, v_i) \le \lambda$ for any $i = 1, \dots, s$. Combined with (5.10) this implies that $C f_\delta^m(x, v_i) \ge -(s+1)\lambda$ for every $i = 1, \dots, s+1$.

Applying Toponogov comparison to the triangle $\Delta y q_m^\alpha q_m^\alpha$ we see that $\sphericalangle q_\alpha(y)' q_\beta(y)' \ge c\delta$ for any $\alpha \ne \beta$.

Let $\mathcal{A} = \{1, \dots, N\}$. For each $i = 1, \dots, s+1$, let $\mathcal{A}_i' = \{\alpha \in \mathcal{A} || \cos \sphericalangle \xi v_i| \le c\delta/4$ for any $\xi \in q_m^\alpha(y)'\}$.
Let $\mathcal{A}' = \cap_{i=1}^{s+1} \mathcal{A}_i'$.

Since $v_1, \dots, v_{s+1}$ are μ_m orthonormal this implies that there exists a 23-dimensional totally geodesic sphere $S \subset \Sigma_y M_m$ such that the set $\{q_m^\alpha(y)' | \alpha \in \mathcal{A}'\}$ lies in the $(c\delta/4 + \mu_m)$-neighborhood of S.

For large m we can asssume that $(c\delta/4 + \mu_m) \le c\delta/3$ and since $\sphericalangle q_\alpha(y)' q_\beta(y)' \ge c\delta$ for any $\alpha \ne \beta$, a standard volume comparison argument implies that $|\mathcal{A}'| \le c_3/\delta^{23}$.

Since $N \le c/\delta^{24}$ by (5.3), this implies that

$$\frac{|\mathcal{A}'|}{N} \le c\delta \tag{5.11}$$

We will give separate estimates for $\sum_{i=1}^{s+1} C\phi_d(d(\cdot, q_m^\alpha))(x; v_i)$ for $\alpha \in \mathcal{A}'$ and for $\alpha \notin \mathcal{A}'$.

Claim 1: If $\alpha \in \mathcal{A}'$ then

$$\sum_{i=1}^{s+1} C\phi_d(d(\cdot, q_m^\alpha))(y; v_i) \le (s+1)\lambda \tag{5.12}$$

This follows directly from $-\lambda$-concavity of $\phi_d(d(\cdot, q_m^\alpha))$.

Claim 2: If $\alpha \notin \mathcal{A}'$ then

$$\sum_{i=1}^{s+1} C\phi_d(d(\cdot, q_m^\alpha))(y; v_i) \le -\frac{c}{\delta} \tag{5.13}$$

The proof of Claim 2 is essentially the same as the proof of [Kap99, Lemma 4.2] and thus we will skip some of the technical details.

Let us assume for simplicity that y is not a cut point for any of $d(\cdot, q_m^\alpha)$ so that all the functions involved are actually smooth near y.

Let $v \in T_y M_m$ be a unit vector and $\gamma_v(t)$ be a geodesic through y such that $\gamma_v'(0) = v$. Let $f_m^\alpha(t) = d(\gamma_v(t), q_m^\alpha)$.

Then

$$C\phi_\delta(d(\cdot, q_m^\alpha))(y; v) = \phi_d''(f_m^\alpha(0))(f_m^{\alpha\prime}(0))^2 + \phi_\delta'(f_m^\alpha(0))(f_m^{\alpha\prime\prime}(0))$$

Observe that $1/2 \le \phi_\delta'' \le 1, \phi_\delta'' = -1/2\delta^3$ by construction of ϕ_δ. we also know that $f^{\alpha\prime\prime} \le \lambda$ by the λ-concavity of the distance funtions and therefore

$$C\phi_\delta(d(\cdot, q_m^\alpha))(y; v) \le c_3 - 1/2\delta^3(f_m^{\alpha\prime}(0))^2 \tag{5.14}$$

If $\alpha \notin \mathcal{A}'$ then by definition of $\mathcal{A}'$ there exists an i such that $|\cos\sphericalangle q_m^\alpha(y)'v_i| \ge c\delta/4$ and therefore by the first variation formula $|f_m^{\alpha\prime}(0)| \ge c\delta/4$ which by (5.14) implies that

$$C\phi_\delta(d(\cdot, q_m^\alpha))(y; v_i) \le c_3 - c_4\delta^3\delta^2 \le -c_5/d \tag{5.15}$$

Because of λ-concavity of the distance functions, for $j \ne i$ we still have that

$$C\phi_\delta(d(\cdot, q_m^\alpha))(y; v_j) \le \lambda \tag{5.16}$$

and therefore

$$\sum_{i=1}^{s+1} C\phi_d(d(\cdot, q_m^\alpha))(y; v_i) \le s\lambda - c_5/\delta \le -c_6/\delta \tag{5.17}$$

which concludes the proof of Claim 2 under the extra assumption that y is not a cut point for any of $d(\cdot, q_m^\alpha)$. The proof of Claim 2 in general is a rather tedious and highly unilluminating exercise in using the discrete approximation for the formula

$$\phi(h)'' = \phi''(h)(h')^2 + \phi'(h)h''$$

and thus is left to the reader (Also see the proof of [Kap99, Lemma 4.2] where this computation is carried out in detail).

Using claim 1, claim 2 and estimate (5.11) we obtain

$$\sum_{i=1}^{s+1} Cf_\delta^m(x, v_i) \le \frac{|\mathcal{A}'|}{N}((s+1)\lambda) - (1 - \frac{|\mathcal{A}'|}{N})\frac{c_6}{\delta} \le c_7\delta - c_8/\delta \le -c_9/\delta \tag{5.18}$$

Finally, since c_9 is independent of δ, we can assume that δ was chosen to be sufficiently small so that $-c_9/\delta < -\lambda$.

This concludes the proof of Lemma 5.3. □

Next let us show that partial concavity imposes some obvious restrictions on the topology of level and superlevel sets of the lifts f_δ^m. We start with the following well-known elementary lemma

Lemma 5.4. *Let M^n be a Riemannian manifold Let $f\colon M^{n+1} \to \mathbb{R}$ be a C^∞ function from $\mathcal{C}(s+1)$ with compact superlevel sets. Let c be a regular value of f. Then $H^i(\{f = c\}) = 0$ for $s < i < n - s$.*

Proof. Since Morse functions are dense in $C\infty$ topology among C^∞ functions we can assume without a loss of generality that f is a Morse function. This means that cirtical points of $-f$ have indices $\le k$. By the main theorem of Morse theory, this implies that the superlevel set $M^c = f^{-1}(c, +\infty)$ is homotopy equivalent to a finite CW complex with cells of dimension at most s. Therefore $H^i(M^c) = H_i(M^c) = 0$ for any $i > s$. By Poincare duality, this implies that $H^{n+1-i}(M^c, \partial M^c) = 0$ for $i > s$. let us look at the cohomology long exact sequence of the pair $(M^c, \partial M^c)$:

$$\to H^i(M^c) \to H^i(\partial M^c) \to H^{i+1}(M^c, \partial M^c) \to \tag{5.19}$$

If $s < i < n - s$ then by above $H^i(M^c) = 0$ and $H^{i+1}(M^c, \partial M^c) = 0$ and therefore by the exactness of (5.19) , $H^i(\partial M^c) = 0$. □

Next we are going to show this Lemma remains true for arbitrary function from $\mathcal{C}(s+1)$ once the notion of a regular value is properly understood.

Lemma 5.5. *Let $h\colon M^{n+1} \to \mathbb{R}$ be a function from $\mathcal{C}(s+1)$ with compact superlevel sets. Let $[c_1, c_2] \subset Im(h)$. Suppose that the following two conditions are satisfied*

(i) h has directional derivatives everywhere in $h^{-1}([c_1, c_2])$ and moreover there exists an $L > 0$ such that the derivative h'_x is L-Lipschitz on T_xM for any $x \in h^{-1}([c_1, c_2])$

(ii) there exists a gradient-like smooth vector field X for h on $h^{-1}([c_1, c_2])$.

Then $H^i(h^{-1}(c)) = 0$ for $s < i < n - s$ and any $c \in (c_1, c_2)$.

Remark 5.6. The first variation formula shows that f_δ^m satisfies condition (i) of Lemma 5.5

Proof of Lemma 5.5. The proof is essentially an application of the Smoothing Theorem of Wu mentioned in the in Section 3. Unfortunately the result we want follows from the proof rather than the statement of that theorem. Therefore we will briefly outline the construction involved in its proof.

Let $\kappa\colon R \to [0, 1]$ be a C^∞ function with support in $[-1, 1]$ such that $\kappa = const$ near 0 and

$$\int_{\mathbb{R}^n} \kappa(|v|) dv = 1$$

Define $h_\rho\colon M \to \mathbb{R}$ by the formula

$$h_\rho(x) = \frac{1}{\rho^n} \int_{T_xM} f(exp_x v)\kappa(\rho v) d\mu_v$$

where $d\mu_v$ stands for the lebesgue measure on T_xM.

Then according to the proof of [Wu87, Lemma 2], h_ρ is smooth and belongs to $\mathcal{C}(s+1)$ on $h^{-1}([c_1, +\infty))$ if ρ is sufficiently small. Fix a $c \in (c_1, c_2)$ We are going to show that if ρ is sufficiently small then the level sets of h_ρ and h are homeomorphic.

First of all let us show that c is a regular value of h_ρ.

It is not hard to compute the differential of h_ρ. According to [GS77, Proposition 2.1], the differential of h_ρ can be computed as follows:

Let $u \in T_xM$.

Construct a vector field U on $B_\rho(x)$ as follows.

Let γ be the unique geodesic with $\gamma'(0) = u$ and define for each $y \in B_\rho(x)$ a smooth curve γ_y by the formula

$$\gamma_y(t) = \exp_{\gamma(t)}(P_{\gamma(t)}(\exp^{-1}_{\gamma(0)}(y)))$$

Observe that U is well defined and smooth if $\rho < \mathrm{injrad}M$.

Then $dh_\rho(u)$ is given by the following formula

$$dh_\rho(u) = \frac{1}{\rho^n} \int_{T_xM} h'_{exp(v)}(U)\kappa(\rho v) d\mu_v \tag{5.20}$$

Let $u = X(x)$. By construction of U we see that U is close to X on $B_\rho(x)$ if ρ is sufficiently small, which by the Lipschitz condition (i) on h' implies that $dh_\rho(X(x))$ is uniformely close in $h^{-1}([c_1, +\infty))$ to

$$\frac{1}{\rho^n} \int_{T_xM} h'_{exp(v))}(X(\exp(v))\kappa(\rho v) d\mu_v$$

Now condition (ii) on h implies that X is a gradient-like vector field for h_ρ on $h^{-1}([c_1, c_2])$ for all sufficiently small ρ.

Since X is gradient-like for both h and h_ρ, a standard argument using the flow of X implies that $h^{-1}(c)$ and $h_\rho^{-1}(c)$ are homeomorphic for all sufficiently small ρ.

Applying Lemma 5.4 to h_ρ we finally obtain that $H^i(\{h = c\}) = 0$ for $k < i < n - k$ as claimed. □

By (5.7), we can find an $a > 0$ such that the level set $\{f_\delta^m = a\}$ is totally conatined in $\bar{B}(x_m, \delta^3/4) \backslash B(x_m, \nu\delta^3)$.

Since V_m is gradient-like for both f_δ^m and $d(\cdot, x_m)$, the same flow argument for V_m implies that $\{f_\delta^m = a\}$ is homeomorphic to $S(x_m, a)$ for all sufficiently large m.

Next notice that X is a smooth Riemannian manifold of sec ≥ 0 away from the vertex o. Therefore by Yamaguchi's fibration theorem for any sufficiently large m there exists an almost Riemannian submersion $\pi_m \colon U_m \to B(o, \delta^3/4) \backslash B(o, \delta^3/8)$ where U_m is an open subset of M_m satisfying

$$B(x_m, \delta^3/4 - c\mu_m) \backslash B(x_m, \delta^3/8 + c\mu_m) \subset U \subset B(x_m, \delta^3/4 + c\mu_m) \backslash B(x_m, \delta^3/8 - c\mu_m)$$

Let V be the gradient field of $d(\cdot, o)$ on $X \backslash \{o\}$.

By [Yam91, Lemma 2.8] and Lemma 5.1,

$$\frac{|d\pi_m(V_m(y)) - V(\pi_m(y))|}{|V(\pi_m(y))|} \leq O(\mu_m)$$

for all $y \in U_m$ Therefore V_m is almost perpendicular to the level sets $\pi_m^{-1}(S(o,a))$. Hence, using the same flow argument once more we obtain that $\pi_m^{-1}(S(o,a))$ is homeomorphic to $S(x_m, a)$ which by above is homeomorphic to $\{f_\delta^m = a\}$.

Since $S(o,a)$ is obviously homeomorphic to Σ we can finally conclude that for all sufficiently large m there exists a fiber bundle $\pi_m \colon \{f_\delta^m = a\} \to \Sigma$.

On the other hand, h_δ^m belongs to $\mathcal{C}(s+1)$ by Lemma 5.3 and therefore by Lemma 5.5, $H^i((h_\delta^m)^{-1}(a)) = 0$ for $s < i < n - s$.

We will show that these two conditions can not be simultaneously true if $s < 15$.

Lemma 5.7. *Let $\pi \colon P^{24+s} \to \Sigma$ be a topological fiber bundle such that its fiber F and the total space P are closed topological manifolds. If $s < 15$ then $H^i(P) = 0$ for some i satisfying $s < i < n - s$.*

Proof. Without loss of generality we can assume that F is connected.

Let us look at the Serre spectral sequence of the fibration $F \hookrightarrow P \to \Sigma$. It is elementary to check that the nontrivial Betti numbers of Σ are as follows: $b_0 = b_{24} = 1$ and $b_8 = b_{16} = 2$.

If $s < 7$ then the spectral sequence collapses on the E_2 term for degree reasons and therefore $H^8(N) \neq 0$.

Now let $s = 7$. Since $H^7(F) \cong \mathbb{Z}_2$, we see that at most one generator of $E_2^{8,0} \cong \mathbb{Z}_2^2$ can get killed by some differential in the spectral sequence and therefore $E_\infty^{8,0} \neq 0$. Hence we once again conclude that $H^8(P) \neq 0$.

Next look at the case $8 \le s \le 13$. For degree reasons we have that $d_r = 0$ for $1 < r < 7$. Consider $d_7|_{E_7^{0,s}} \colon E_7^{0,s} \cong H^0(\Sigma) \otimes H^s(F) \to E_7^{8,s-7} \cong H^8(\Sigma) \otimes H^{s-7}(F)$.

Since $b_8(\Sigma) = 2$ and $b_s(F) = 1$ we see that this map is either identically zero (if $b_{s-7} = 0$) or not onto (if $b_{s-7} > 0$).

In the former case we conclude by the multiplicativity of the spectral sequence, that $d_r|_{E_r^{8,s}} = 0$ for any $r > 1$. Hence $E_\infty^{8,s} \cong Z_2^2$ and therefore $H^{s+8}(P) \neq 0$.

If $d_7|_{E_7^{0,s}} \colon E_7^{0,s} \to E_7^{8,s-7}$ is not onto then $E_8^{8,s-7} \cong E_7^{8,s-7}/d_7(E_7^{0,s}) \neq 0$. For degree reasons we have that $E_8^{8,s-7} \cong E_\infty^{8,s-7}$ and therefore $H^{s+1}(P) \neq 0$.

Finally let us consider the case $s = 14$. Then as in the previous case $d_7|_{E_7^{0,s}} \colon E_7^{0,s} \to E_7^{8,s-7}$ is either zero or not onto. If it is zero then the same argument as before shows that $H^{s+8}(P) \neq 0$.

If $d_7|_{E_7^{0,14}} \colon E_7^{0,14} \to E_7^{8,7}$ is not onto then $E_7^{8,7} \cong H^8(\Sigma) \otimes H^7(F) \neq 0$. Therefore $b_7(F) \neq 0$. By Poincare duality, the cuproduct on $H^7(F)$ is nondegenerate; therefore $b_7(F)$ is even and hence $\dim_{\mathbb{Z}_2}(E_7^{8,7}) \ge 4$.

Since $\dim_{\mathbb{Z}_2}(E_7^{0,14}) = 1$ and $\dim_{\mathbb{Z}_2}(E_7^{16,0}) = 2$ we see that $\dim_{\mathbb{Z}_2}(E_8^{8,7}) \ge 4 - 2 - 1 = 1$. For degree reasons $E_8^{8,7} = E_\infty^{8,7}$ and therefore $H^{15}(P) \neq 0$.

□

Applying Lemma 5.7 to $\pi_m \colon \{h_\delta^m = a\} \to \Sigma$ we conclude that the codimension of the collapse s must satisfy $s \ge 15$. This concludes the proof of Theorem 1.3. □

6 Concluding remarks

Examining the proof of Theorem 1.3, we see that explicit topology of Σ was essentially only used in Lemma 5.7. Therefore it is easy to derive analogues of Theorem 1.3 with most other known positively curved manifolds in place of Σ. For example, when applied to HP^n, the same proof shows that if $S(HP^n) \notin \bar{\mathcal{M}}_k^{\dim(S(HP^n))+s}$ for any $k \in \mathbb{R}$ and $s < 3$. Note that this estimate is actually *sharp* because it is easy to construct a sequence of Riemannian metrics on S^{4n+4} with $sec \geq 0$ Gromov-Hausdorff converging to $S(HP^n)$.

Observe that due to the lack of examples of positively curved manifolds, Theorem 1.3 does not produce examples of Alexandrov spaces with arbitrary large minimal collapsing codimension. However, the author suspects that it should be possible to show that the minimal collapsing codimension of $\underbrace{\Sigma * \Sigma * \ldots * \Sigma}_{l}$ grows linearly l.

Using the same ideas as in the proof of Theorem 1.3, it should also be possible to show that Theorem 1.3 remains true for $X \times M$ where M is any Riemannian manifold (i.e that $X \times M \notin \bar{\mathcal{M}}_k^{\dim(X\times M)+s}$ for any $k \in \mathbb{R}$ and $s < 15$).

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