STRUCTURE OF FUNDAMENTAL GROUPS OF MANIFOLDS WITH RICCI CURVATURE BOUNDED BELOW

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The main result of this paper is the following theorem which settles a conjecture of Gromov.

Theorem 1 (Generalized Margulis Lemma)**.** *In each dimension n there are positive constants $C(n)$, $\varepsilon(n)$ such that the following holds for any complete n-dimensional Riemannian manifold (M, g) with $\mathrm{Ric} > -(n-1)$ on a metric ball $B_1(p) \subset M$.*

The image of the natural homomorphism

$$\pi_1\big(B_\varepsilon(p), p\big) \to \pi_1\big(B_1(p), p\big)$$

contains a nilpotent subgroup N of index $\leq C(n)$. Moreover, N has a nilpotent basis of length at most n.

We call a generator system $b_1, \ldots, b_n$ of a group N a nilpotent basis if the commutator $[b_i, b_j]$ is contained in the subgroup $\langle b_1, \ldots, b_{i-1}\rangle$ for $1 \leq i < j \leq n$. Having a nilpotent basis of length n implies in particular $\mathrm{rank}(\mathsf{N}) \leq n$. We will also show that equality in this inequality can only occur if M is homeomorphic to an infranilmanifold, see Corollary 7.1.

In the case of a sectional curvature bound the theorem is due to Kapovitch, Petrunin and Tuschmann [KPT10], based on an earlier version which was proved by Fukaya and Yamaguchi [FY92].

In the case of Ricci curvature a weaker form of the theorem was stated by Cheeger and Colding [CC96]. The main difference is that – similar to Fukaya and Yamaguchi's theorem – no uniform bound on the index of the subgroup is provided. Cheeger and Colding never wrote up the details of the proof, and relied on some claims in [FY92] stating that their results would carry over from lower sectional curvature bounds to lower Ricci curvature bounds if certain structure results would be obtained. But compare also Remark 6.2 below.

Corollary 2. *Let (M, g) be a compact manifold with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \leq \varepsilon(n)$ then $\pi_1(M)$ contains a nilpotent subgroup N of index $\leq C(n)$. Moreover, N has a nilpotent basis of length $\leq n$.*

One of the tools used to prove these results is

Theorem 3. *Given n and D there exists C such that for any n-manifold with $\mathrm{Ric} \geq -(n-1)$ and $\mathrm{diam}(M, g) \leq D$, the fundamental group $\pi_1(M)$ can be generated by at most C elements.*

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This estimate was previously proven by Gromov [Gro78] under the stronger assumption of a lower sectional curvature bound $K \geq -1$. Recall that a conjecture of Milnor states that the fundamental group of an open manifold with nonnegative Ricci curvature is finitely generated. Although Theorem 3 is far from a solution to that problem, the Margulis Lemma immediately implies the following.

Corollary 4. *Let (M, g) be an open n-manifold with nonnegative Ricci curvature. Then $\pi_1(M)$ contains a nilpotent subgroup N of index $\leq C(n)$ such that any finitely generated subgroup of N has a nilpotent basis of length $\leq n$.*

Of course, by work of Milnor [Mil68] and Gromov [Gro81], the corollary is well known (without uniform bound on the index) in the case of finitely generated fundamental groups. We should mention that by work of Wei [Wei88] (and an extension in [Wil00]) every finitely generated virtually nilpotent group appears as fundamental group of an open manifold with positive Ricci curvature in some dimension. Corollary 4 also implies that the first $\mathbb{Z}_p$-Betti number of M is finite for any prime p, which was previously only known for rational coefficients.

The proofs of our results are based on the structure results of Cheeger and Colding for limit spaces of manifolds with lower Ricci curvature bounds [Col96, Col97, CC96, CC97, CC00a, CC00b]. This is also true for the proof of following new tool which can be considered as a Ricci curvature replacement of Yamaguchi's fibration theorem as well as a replacement of the gradient flow of semi-concave functions used by Kapovitch, Petrunin and Tuschmann.

Theorem 5 (Rescaling theorem)**.** *Suppose a sequence of Riemannian n-manifolds (M_i, p_i) with $\mathrm{Ric} \geq -\frac{1}{i}$ converges in the pointed Gromov–Hausdorff topology to the Euclidean space $(\mathbb{R}^k, 0)$ with $k < n$. Then after passing to a subsequence there is a subset $G_1(p_i) \subset B_1(p_i)$, a rescaling sequence $\lambda_i \to \infty$ and a compact metric space $K \neq \{pt\}$ such that*

- $\mathrm{vol}(G_1(p_i)) \geq (1 - \frac{1}{i})\,\mathrm{vol}(B_1(p_i))$
- *For all $x_i \in G_1(p_i)$ the isometry type of the limit of any convergent subsequence of $(\lambda_i M_i, x_i)$ is given by $K \times \mathbb{R}^k$.*
- *For all $x_i, y_i \in G_1(p_i)$ we can find a diffeomorphism $f_i \colon M_i \to M_i$ such that f_i subconverges in the weakly measured Gromov–Hausdorff sense to an isometry of the Gromov–Hausdorff limits*

$$f_\infty \colon \lim_{G\text{-}H, i\to\infty} (\lambda_i M_i, x_i) \to \lim_{G\text{-}H, i\to\infty} (\lambda_i M_i, y_i).$$

We will prove a technical generalization of this theorem in section 5. It will be important for the proof of the Margulis Lemma that the diffeomorphisms f_i are in a suitable sense close to isometries on all scales. The precise concept is called zooming in property and is defined in section 3. There we also recall the concept of weakly measured Gromov–Hausdorff convergence of maps (Lemma 3.2).

The diffeomorphisms f_i will be composed out of gradient flows of harmonic functions arising in the analysis of Cheeger and Colding. Their L^2-estimates on the Hessian's of these functions play a crucial role.

Like in Fukaya and Yamaguchi's paper the idea of the proof of the Margulis Lemma is to consider a contradicting sequence (M_i, g_i). By a fundamental observation of Gromov, the set of complete n-manifolds with $\mathrm{Ric} \geq -(n-1)$ is precompact in Gromov–Hausdorff topology. Therefore one can assume that the contradicting sequence converges to a (possibly very singular) space X. One then uses various rescalings and normal coverings of (M_i, g_i) in order to find contradicting sequences converging to higher and higher dimensional spaces. One of the differences to Fukaya and Yamaguchi's approach is that if we pass to a normal covering we endow the cover with generators of the deck transformation group in order not to lose information. Since after rescaling the displacements of these deck transformations converge to infinity, this approach seems bound to failure. However, using the rescaling theorem we are able to alter the deck transformations by composing them with a sequence of diffeomorphisms which are isotopic to the identity and which have the zooming in property. With these alterations we are able to keep much more information on the action by conjugation of long homotopy classes on very short ones. Since the altered deck transformations still converge in a weakly measured sense to isometries, this allows one to eventually rule out the existence of a contradicting sequence.

For the proof of the Margulis Lemma we do not need any sophisticated structure results on the isometry group of the limit space except for the relatively elementary Gap Lemma 2.4 guaranteeing that generic orbits of the limit group are locally path connected. However, for the following application of the Margulis Lemma a recent structure result of Colding and Naber [CN10] is key. They showed that the isometry group of a limit space with lower Ricci curvature bound has no small subgroups and thus, by the small subgroup theorem of Gleason, Montgomery and Zippin [MZ55], is a Lie group.

Theorem 6 (Compact Version of the Margulis Lemma). *Given n and D there are positive constants ε_0 and C such that the following holds: If (M, g) is a compact n-manifold M with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \le D$, then there is $\varepsilon \ge \varepsilon_0$ and a normal subgroup $\mathsf{N} \lhd \pi_1(M)$ such that for all $p \in M$:*

- *the image of $\pi_1(B_{\varepsilon/1000}(p), p) \to \pi_1(M, p)$ contains N,*
- *the index of N in the image of $\pi_1(B_\varepsilon(p), p) \to \pi_1(M, p)$ is $\le C$ and*
- *N is a nilpotent group which has a nilpotent basis of length $\le n$.*

If D, n, ε_0 and C are given, there is an effective way to limit the number of possibilities for the quotient group $\pi_1(M, p)/\mathsf{N}$, see Lemma 9.2. In fact we have the following finiteness result. We recall that the torsion elements $\mathrm{Tor}(\mathsf{N})$ of a nilpotent group N form a subgroup.

Theorem 7.

a) *For each $D > 0$ and each dimension n there are finitely many groups $\mathsf{F}_1, \ldots, \mathsf{F}_k$ such that the following holds: If M is a compact n-manifold with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \le D$, then there is a nilpotent normal subgroup $\mathsf{N} \lhd \pi_1(M)$ with a nilpotent basis of length $\le n-1$ and $\mathrm{rank}(\mathsf{N}) \le n-2$ such that $\pi_1(M)/\mathsf{N} \cong \mathsf{F}_i$ for suitable i.*

b) *In addition to a) one can choose a finite collection of irreducible rational representations $\rho_i^j \colon \mathsf{F}_i \to \mathrm{GL}(n_i^j, \mathbb{Q})$ $(j = 1, \ldots, \mu_i, i = 1, \ldots, k)$ such that for a suitable choice of the isomorphism $\pi_1(M)/\mathsf{N} \cong \mathsf{F}_i$ the following holds: There is a chain of subgroups $\mathrm{Tor}(\mathsf{N}) = \mathsf{N}_0 \lhd \cdots \lhd \mathsf{N}_{h_0} = \mathsf{N}$ which are all normal in $\pi_1(M)$ such that $[\mathsf{N}, \mathsf{N}_h] \subset \mathsf{N}_{h-1}$ and $\mathsf{N}_h/\mathsf{N}_{h-1}$ is free abelian. Moroever, the action of $\pi_1(M)$ on N by conjugation induces an action of F_i on $\mathsf{N}_h/\mathsf{N}_{h-1}$ and the induced representation $\rho \colon \mathsf{F}_i \to \mathrm{GL}\big((\mathsf{N}_h/\mathsf{N}_{h-1}) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$ is isomorphic to ρ_i^j for a suitable $j = j(h)$, $h = 1, \ldots, h_0$.*

Part a) of Theorem 7 generalizes a result of Anderson [And90] who proved finiteness of fundamental groups under the additional assumption of a uniform lower bound on volume.

In the preprint [Wil11] a partial converse of Theorem 7 is proved. In particular it is shown there that for a finite collection of finitely presented groups and any finite collection of rational representations one can find D such that each group Γ satisfying the algebraic restrictions described in a) and b) of the above theorem with respect to this data contains a finite nilpotent normal subgroup H such that Γ/H can be realized as a fundamental group of a $n+2$-dimensional manifold with $\mathrm{diam}(M) \leq D$ and sectional curvature $|K| \leq 1$.

We can also extend the diameter ratio theorem of Fukaya and Yamaguchi [FY92].

Theorem 8 (Diameter Ratio Theorem)**.** *For n and D there is a $\widetilde{D}$ such that any compact manifold M with $\mathrm{Ric} \geq -(n-1)$ and $\mathrm{diam}(M) = D$ satisfies: If $\pi_1(M)$ is finite, then the diameter of the universal cover $\tilde{M}$ of M is bounded above by $\widetilde{D}$.*

In the case of nonnegative Ricci curvature the theorem says that the ratio $\mathrm{diam}(\tilde{M})/\mathrm{diam}(M)$ is bounded above. Fukaya and Yamaguchi's theorem covers the case that M has almost nonnegative sectional curvature. The proof of Theorem 8 has some similarities to parts of the proof of Gromov's polynomial growth theorem [Gro81].

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Organization of the Paper

We start in section 1 with prerequisites. We have included a subsection on notational conventions. Next, in section 2 we will prove Theorem 3.

Section 3 is somewhat technical. We define the zooming in property, prove the needed properties, and provide two somewhat similar construction methods. This section serves mainly as a preparation for the proof of the rescaling theorem.

We have added a short section 4 in which we sketch a rough idea of the proof of the Margulis Lemma.

In section 5 the refined rescaling theorem (Theorem 5.1) is stated and proven. In Section 6 we put things together and provide a proof of the Induction Theorem, which has Corollary 2 as its immediate consequence.

The Margulis Lemma follows from the Induction Theorem in two steps which are somewhat similar to Fukaya and Yamaguchi's approach. Nevertheless, we included all details in section 7. We also show that the nilpotent group N in the Margulis

Lemma can only have rank n if the underlying manifold is homeomorphic to an infranilmanifold.

Section 8 uses lower sectional curvature bounds. We give counterexamples to a theorem of Fukaya and Yamaguchi [FY92] stating that almost nonnegatively curved n-manifolds with first $\mathbb{Z}_p$-Betti number equal to n have to be tori, provided p is sufficiently big. We show that instead these manifolds are nilmanifolds and that every nilmanifold covers another nilmanifold with maximal first $\mathbb{Z}_p$ Betti number.

Section 9 contains the proof of Theorem 6 and Theorem 7. Finally, Theorem 8 is proved in section 10.

1. Prerequisites

1.1. **Short basis.** We will use the following construction due to Gromov [Gro82].

Given a manifold (M,p) and a group Γ acting properly discontinuously and isometrically on M one can define a short basis of the action of Γ at p as follows:

For $\gamma \in \Gamma$ we will refer to $|\gamma| = d(p,\gamma(p))$ as the norm or the length of γ. Choose $\gamma_1 \in \Gamma$ with the minimal norm in Γ. Next choose γ_2 to have minimal norm in $\Gamma \backslash \langle \gamma_1 \rangle$. On the n-th step choose γ_n to have minimal norm in $\Gamma\backslash\langle\gamma_1,\gamma_2,...,\gamma_{n-1}\rangle$. The sequence $\{\gamma_1,\gamma_2,...\}$ is called a *short basis* of Γ at p. In general, the number of elements of a short basis can be finite or infinite. In the special case of the action of the fundamental group $\pi_1(M,p)$ on the universal cover $\tilde{M}$ of M one speaks of the short basis of $\pi_1(M,p)$. For any $i > j$ we have $|\gamma_i| \le |\gamma_j^{-1}\gamma_i|$.

While a short basis at p is not unique its length spectrum $\{|\gamma_1|,|\gamma_2|,\ldots\} \subset \mathbb{R}$ is unique. That is the main reason why the non-uniqueness will not matter in any of the proofs in this article and will largely be suppressed.

If M/Γ is a closed manifold, then the short basis is finite and $|\gamma_i| \le 2\,\mathrm{diam}(M/\Gamma)$.

1.2. **Ricci curvature.** Throughout this paper we will use the notation $\fint$ to denote the average integral. We will use the following results of Cheeger and Colding.

Theorem 1.1 (Splitting Theorem, [CC96])**.** *Let* $(M_i^n,p_i)\xrightarrow{G-H}(X,p)$ *with* $\mathrm{Ric}(M_i) \ge -\frac{1}{i}$*. Suppose X has a line. Then X splits isometrically as* $X \cong Y \times \mathbb{R}$*.*

We will also need the following corollary of the stability theorem.

Theorem 1.2. [CC97] *Let* $(M_i^n,p_i)\xrightarrow{G-H}(\mathbb{R}^n,0)$ *with* $\mathrm{Ric}_{M_i} > -1$*. Then* $B_R(p_i)$ *is contractible in* $B_{R+\varepsilon}(p_i)$ *for all* $i \ge i_0(R,\varepsilon)$*.*

Even more important for us is the following theorem which is closely linked with the proof of the splitting theorem for limit spaces.

Theorem 1.3. [CC00a] *Suppose* $(M_i^n,p_i)\xrightarrow{G-H}(\mathbb{R}^k,0)$ *with* $\mathrm{Ric}_{M_i} \ge -1/i$*. Then there exist harmonic functions* $b_1^i,\ldots,b_k^i \colon B_2(p_i) \to \mathbb{R}$ *such that*

$$|\nabla b_j^i| \le C(n) \text{ for all } i \text{ and } j \text{ and} \tag{1}$$

$$\fint_{B_1(p_i)} \sum_{j,l} \left|< \nabla b_j^i, \nabla b_l^i > -\delta_{j,l}\right| + \sum_j \|\mathrm{Hess}_{b_j^i}\|^2 \, d\mu_i \to 0 \text{ as } i \to \infty. \tag{2}$$

Moreover, the maps $\Phi^i = (b_1^i,\ldots,b_k^i)\colon M_i \to \mathbb{R}^k$ *provide* ε_i*-Gromov–Hausdorff approximations between* $B_1(p_i)$ *and* $B_1(0)$ *with* $\varepsilon_i \to 0$*.*

The functions b^i_j in the above theorem are constructed as follows. Approximate Busemann functions f_j in $\mathbb{R}^k$ given by $f_j = d(\cdot, N_i e_j)) - N_i$ are lifted to M_i using Hausdorff approximations to corresponding functions f^i_j. Here e_j is the j-th coordinate vector in the standard basis of $\mathbb{R}^k$ and $N_i \to \infty$ sufficiently slowly so that $d_{G-H}(B_{N_i}(p_i), B_{N_i}(0)) \to 0$ as $i \to \infty$. The functions b^i_j are obtained by solving the Dirichlet problem on $B_2(p_i)$ with $b^i_j|_{\partial B_2(p^i)} = f^i_j|_{\partial B_2(p^i)}$.

We will need a weak type 1-1 estimate for manifolds with lower Ricci curvature bounds which is a well-known consequence of the doubling inequality [Ste93, p. 12].

Lemma 1.4 (Weak type 1-1 inequality)**.** *Suppose* (M^n, g) *has* $\mathrm{Ric} \geq -(n-1)$ *and let* $f\colon M \to \mathbb{R}$ *be a nonnegative function. Define* $\mathrm{Mx}_\rho\, f(p) := \sup_{r \leq \rho} ⨍_{B_r(p)} f$ *for* $\rho \in (0, 4]$ *and put* $\mathrm{Mx}\, f(p) = \mathrm{Mx}_2\, f(p)$*. Then the following holds*
(a) If $f \in L^\alpha(M)$ *with* $\alpha \geq 1$ *then* $\mathrm{Mx}_\rho\, f$ *is finite almost everywhere.*
(b) If $f \in L^1(M)$ *then* $\mathrm{vol}\{x \mid \mathrm{Mx}_\rho\, f(x) > c\} \leq \frac{C(n)}{c} \int_M f$ *for any* $c > 0$*.*
(c) If $f \in L^\alpha(M)$ *with* $\alpha > 1$ *then* $\mathrm{Mx}_\rho\, f \in L^\alpha(M)$ *and* $||\,\mathrm{Mx}_\rho\, f||_\alpha \leq C(n, \alpha)||f||_\alpha$*.*

We will also need the following inequality (for any $\alpha > 1$) which is an immediate consequence of the definition of Mx and Hölder inequality.

$$[\mathrm{Mx}_\rho f(x)]^\alpha \leq \mathrm{Mx}_\rho(f^\alpha)(x). \tag{3}$$

Indeed, for any $r \leq \rho$ we have that $\mathrm{Mx}_\rho(f^\alpha)(x) \geq ⨍_{B_r(x)} f^\alpha \geq \big(⨍_{B_r(x)} f\big)^\alpha$ and the inequality follows by taking the supremum over all $r \leq \rho$.

As a consequence of Bishop Gromov one has

$$\mathrm{Mx}_\rho f(y) \quad \leq \quad C(n)\mathrm{Mx}_{2\rho} f(x) + \mathrm{Mx}_r f(y) \qquad \text{for } d(x, y) \leq r \leq \rho \leq 2 \tag{4}$$

In fact, if the left hand side is not bounded above by the second summand of the right hand side then $\mathrm{Mx}_\rho f(y) = ⨍_{B_{r_1}(y)} f\, d\mu$ for some $r_1 \in [r, \rho]$ and

$$\mathrm{Mx}_\rho f(y) \leq \tfrac{\mathrm{vol}(B_{2r_1}(x))}{\mathrm{vol}(B_{r_1}(y))} ⨍_{B_{2r_1}(x)} f(p)\, d\mu(p) \leq C(n)\mathrm{Mx}_{2\rho} f(x).$$

Next we claim that if $f \in L^\alpha(M)$ with $\alpha > 1$, then for any $x \in M$ we have the following *pointwise* estimate

$$\mathrm{Mx}_\rho((\mathrm{Mx}_\rho f)^\alpha)(x) \leq C_2(n, \alpha)\mathrm{Mx}_{2\rho}(f^\alpha)(x) \text{ for } \rho \in (0, 2]. \tag{5}$$

In order to show this we may assume that there is an $r \in (0, \rho]$ with

$$\begin{aligned}
\mathrm{Mx}_\rho((\mathrm{Mx}_\rho f)^\alpha)(x) &= ⨍_{B_r(x)} (\mathrm{Mx}_\rho f)^\alpha(y)\, d\mu(y) \\
&\overset{(4)}{\leq} ⨍_{B_r(x)} \big(\mathrm{Mx}_r f(y) + C(n)\mathrm{Mx}_{2\rho} f(x)\big)^\alpha\, d\mu(y) \\
&\leq 2^\alpha C(n)^\alpha (\mathrm{Mx}_{2\rho} f)^\alpha(x) + 2^\alpha ⨍_{B_r(x)} \big(\mathrm{Mx}_r f(y)\big)^\alpha\, d\mu(y) \\
&\leq 2^\alpha C(n)^\alpha (\mathrm{Mx}_{2\rho} f)^\alpha(x) + \tfrac{2^\alpha C(n,\alpha)^\alpha}{\mathrm{vol}(B_r(x))} \int_{B_{2r}(x)} f^\alpha(y)\, d\mu(y) \\
&\overset{(3)}{\leq} C_2(n, \alpha)(\mathrm{Mx}_{2\rho} f^\alpha)(x),
\end{aligned}$$

where in order to deduce the third inequality from Lemma 1.4 c) we used that the integrals on either side only depend on values of f in $B_{2r}(x)$.

By combining (3) and (5) we obtain

$$\mathrm{Mx}_\rho[\mathrm{Mx}_\rho(f)](x) \le \Big(\mathrm{Mx}_\rho[(\mathrm{Mx}_\rho(f))^\alpha](x)\Big)^{1/\alpha} \le \Big(C_2(n,\alpha)\mathrm{Mx}_{2\rho}[f^\alpha](x)\Big)^{1/\alpha} \tag{6}$$

for $\alpha > 1$ and $\rho \in (0,1]$. Finally applying this to the function $g = f^{\frac{\beta}{\alpha}}$ for $\alpha > 1, \beta > 1$ gives

$$\mathrm{Mx}_\rho[(\mathrm{Mx}_\rho(f^{\beta/\alpha}))^\alpha](x) \le C_2(n,\alpha)\mathrm{Mx}_{2\rho}(f^\beta)(x). \tag{7}$$

We will also make use of the so-called segment inequality of Cheeger and Colding which says that in average the integral of a nonnegative function along all geodesics in a ball can be estimated by the L^1 norm of the function.

Theorem 1.5 (Segment inequality, [CC96]). *Given n and r_0 there exists $\tau = \tau(n, r_0)$ such that the following holds. Let $\mathrm{Ric}(M^n) \ge -(n-1)$ and let $g\colon M \to \mathbb{R}^+$ be a nonnegative function. Then for $r \le r_0$*

$$⨍_{B_r(p)\times B_r(p)} \int_0^{d(z_1,z_2)} g(\gamma_{z_1,z_2}(t))\,dt\,d\mu(z_1)d\mu(z_2) \le \tau\cdot r\cdot ⨍_{B_{2r}(p)} g(q)\,d\mu(q),$$

where γ_{z_1,z_2} denotes a minimal geodesic from z_1 to z_2.

Finally we will need the following observation

Lemma 1.6 (Covering Lemma). *There exists a constant $C(n)$ such that the following holds. Suppose (M^n, g) has $\mathrm{Ric}_g \ge -(n-1)$. Let $f\colon M \to \mathbb{R}$ be a nonnegative function, $p \in M$, $\pi\colon \tilde{M} \to M$ be the universal cover of M, $\tilde{p} \in \tilde{M}$ a lift of p, and let $\tilde{f} = f \circ \pi$. Then*

$$⨍_{B_1(\tilde{p})} \tilde{f} \le C(n) ⨍_{B_1(p)} f.$$

Proof. We choose a measurable section $j\colon B_1(p) \to B_1(\tilde{p})$, i.e. $\pi(j(x)) = x$ for any $x \in B_1(p)$. Let $T = j(B_1(p))$. Then we obviously have that $\mathrm{diam}(T) \le 2$ and

$$⨍_{B_1(p)} f = ⨍_T \tilde{f}.$$

Let S be the union of $g(T)$ over all $g \in \pi_1(M)$ such that $g(T) \cap B_1(\tilde{p}) \ne \emptyset$. It is obvious from the triangle inequality that $S \subset B_3(\tilde{p})$. It is also clear that

$$⨍_{B_1(p)} f = ⨍_S \tilde{f}$$

and $B_1(\tilde{p}) \subset S$. Lastly, notice that by Bishop–Gromov relative volume comparison $\mathrm{vol}\, B_3(\tilde{p}) \le C(n)\,\mathrm{vol}\, B_1(\tilde{p})$ for some universal constant $C(n)$. Since $B_1(\tilde{p}) \subset S \subset B_3(\tilde{p})$, we also have $\mathrm{vol}\, S \le C(n)\,\mathrm{vol}\, B_1(\tilde{p})$ and hence

$$⨍_{B_1(\tilde{p})} \tilde{f} \le C(n) ⨍_S \tilde{f} = C(n) ⨍_{B_1(p)} f.$$

□

It is easy to see that the above proof generalizes to an arbitrary cover of M.

1.3. **Equivariant Gromov–Hausdorff convergence.** Let Γ_i be a closed subgroup of the isometry group of a metric space X_i and $\tilde p_i \in X_i$, $i \in \mathbb{N}$. If $(X_i, \tilde p_i) \xrightarrow{G-H} (Y, \tilde p_\infty)$, then after passing to a subsequence one can find a closed subgroup $\mathsf{G} \subset \mathrm{Iso}(Y)$ such that $(X_i, \Gamma_i, \tilde p_i) \to (Y, \mathsf{G}, \tilde p_\infty)$ in the equivariant Gromov–Hausdorff topology. For details we refer the reader to [FY92].

Definition 1.7. A sequence of subgroups $\Upsilon_i \subset \Gamma_i$ is called uniformly open, if there is some $\varepsilon > 0$ such that Υ_i contains the set

$$\big\{g \in \Gamma_i \mid d(gq, q) < \varepsilon \text{ for all } q \in B_{1/\varepsilon}(\tilde p_i)\big\} \text{ for all } i.$$

Γ_i is called uniformly discrete if $\{e\} \subset \Gamma_i$ is uniformly open. We say the sequence Υ_i is boundedly generated if there is some R such that Υ_i is generated by

$$\{g \in \Upsilon_i \mid d(g\tilde p_i, \tilde p_i) < R\}.$$

Lemma 1.8. *Let $\Upsilon_i^j \subset \Gamma_i$ be uniformly open with $\Upsilon_i^j \to \Upsilon_\infty^j \subset \mathsf{G}$, $j = 1, 2$.*

a) $\Upsilon_i^1 \cap \Upsilon_i^2$ is uniformly open and converges to $\Upsilon_\infty^1 \cap \Upsilon_\infty^2$.

b) If $g_i \in \Gamma_i$ converges to $g_\infty \in \mathsf{G}$ then $g_i \Upsilon_i^1 g_i^{-1}$ is uniformly open and converges to $g_\infty \Upsilon_\infty^1 g_\infty^{-1}$.

c) Suppose in addition that Υ_i^j is boundedly generated, $j = 1, 2$. If $\Upsilon_\infty^1 \cap \Upsilon_\infty^2$ has finite index H in Υ_∞^1, then $\Upsilon_i^1 \cap \Upsilon_i^2$ has index H in Υ_i^1 for all large i.

The proof is an easy exercise.

Example 1 ($\mathbb{Z}_l$ actions converging to a $\mathbb{Z}^2$-action.)**.** Consider a 2-torus T_k^2 given by a Riemannian product $\mathsf{S}^1 \times \mathsf{S}^1$ where each of the factors has length k. Put $l = k^2 + 1$ and let $\mathbb{Z}/l\mathbb{Z}$ act on T_k^2 with the generator acting by $\big(e^{\frac{2\pi i}{k^2+1}}, e^{\frac{k 2\pi i}{k^2+1}}\big)$. In the equivariant Gromov–Hausdorff topology the action will converge (for $k \to \infty$) to the standard $\mathbb{Z}^2$ action on $\mathbb{R}^2$.

One can, of course, define a similar sequence of actions on $\mathsf{S}^3 \times \mathsf{S}^3$ but in this case the actions will not be uniformly cocompact.

Remark 1.9.

a) The example shows that it is difficult to relate an open subgroup in the limit to corresponding subgroups in the sequence.

b) If $g_1^i, \ldots, g_{h_i}^i$ is a generator system of some subgroup $\Gamma_i' \subseteq \Gamma_i$, then one can consider – after passing to a subsequence – a different limit construction to get a subgroup $\mathsf{G}' \subset \mathsf{G}$: For some fixed R consider all words w in $g_1^i, \ldots, g_{h_i}^i$ with the property $w \star \tilde p_i \in B_R(\tilde p_i)$. This defines a subset in the Cayley graph of Γ_i'. We enlarge the subset by adding to each vertex all the neighboring edges and consider now all elements in Γ_i' represented by words in its identity component. Passing to the limit gives a closed subset $S_R \subset \mathsf{G}$ of the limit group and one can now take the limit of S_R for $R \to \infty$.

This sort of word limit group can be smaller than the regular limit group as the above example shows. Although it depends on the choice of the generator system, there are occasions where this limit behaves more natural than the usual. However, this idea is used only indirectly in the paper.

1.4. **Notations and conventions.**

- As already mentioned $⨍_S f\, d\mu$ stands for $\frac{1}{\mathrm{vol}(S)} \int_S f\, d\mu$.
- $\mathrm{Mx}_\rho(f)(x)$ denotes the $\rho-$maximum function of f evaluated at x, see Lemma 1.4 for the definition and $\mathrm{Mx}(f)(x) := \mathrm{Mx}_2(f)(x)$.

- A map $\sigma\colon X \to Y$ between inner metric spaces is called a submetry if $\sigma(B_r(p)) = B_r(\sigma(p))$ for all $r > 0$ and $p \in X$.
- If $S \subset \mathsf{F}$ is a subset of a group, then $\langle S\rangle$ denotes the subgroup generated by S.
- If g_1 and g_2 are elements in a group, $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$ denotes the commutator. For subgroups $\mathsf{F}_1, \mathsf{F}_2$ we put $[\mathsf{F}_1, \mathsf{F}_2] := \langle\{[f_1, f_2] \mid f_i \in \mathsf{F}_i\}\rangle$.
- Generators $b_1, \ldots, b_n \in \mathsf{F}$ of a group are called a nilpotent basis if $[b_i, b_j] \in \langle b_1, \ldots, b_{i-1}\rangle$ for $i < j$.
- $\mathsf{N} \lhd \mathsf{F}$ means: N is a normal subgroup of F.
- For a Riemannian manifold (M, g) and $\lambda > 0$ we let λM denote the Riemannian manifold $(M, \lambda^2 g)$.
- For a limit space Y a tangent cone C_pY at p is some Gromov–Hausdorff limit of $(\lambda_i Y, p)$ for some $\lambda_i \to \infty$. Tangent cones are not always metric cones and are not necessarily unique.
- We will sometimes use the concept of measured Gromov–Hausdorff convergence. Recall that for any sequence $(M_i, g_i, p_i) \overset{G-H}{\longrightarrow} (Y, p_\infty)$ with lower Ricci curvature bound the normalized Riemannian measures $\frac{d\mu_{g_i}}{\mathrm{vol}(B_1(p_i))}$ subconverge to a limit measure on Y, see [CC97].
- For a limit space Y of manifolds with lower Ricci curvature bound a regular point $p \in Y$ is a point all of whose tangent cones at p are given by $\mathbb{R}^{k_p}$. By a result of Cheeger and Colding [CC97] these points have full measure (with respect to any limit measure on Y) and are thus dense.
- For two sequences of pointed metric spaces (X_i, p_i), (Y_i, q_i) and maps $f_i\colon X_i \to Y_i$ we use the notation $f_i\colon [X_i, p_i] \to [Y_i, q_i]$ to indicate that f_i subconverges in the weakly measured sense (cf. Lemma 3.2) to a map from the pointed Gromov–Hausdorff limit of (X_i, p_i) to the pointed Gromov–Hausdorff limit of (Y_i, q_i). It usually does not mean that $f_i(p_i) = q_i$. However, if this is the case we write $f_i\colon (X_i, p_i) \to (Y_i, q_i)$.
- If a group G acts on a metric space we say $g \in \mathsf{G}$ displaces $p \in X$ by r if $d(p, gp) = r$. We will denote the orbit of p by $\mathsf{G} \star p$.
- Gromov's short generator system (or short basis for short) of a fundamental group is defined at the beginning of section 1.
- On Riemannian manifolds we always use the Riemannian measure in integrals. It will often be suppressed when variables of integration are clear.

2. Finite generation of fundamental groups

Lemma 2.1 (Product Lemma)**.** *Let M_i be a sequence of manifolds with $\mathrm{Ric}_{M_i} > -\varepsilon_i \to 0$ satisfying*

- *$\overline{B_{r_i}(p_i)}$ is compact for all i with $r_i \to \infty$ and $p_i \in M_i$,*
- *for every i and $j = 1, \ldots, k$ there are harmonic functions $b^i_j\colon B_{r_i}(p_i) \to \mathbb{R}$ which are L-Lipschitz and fulfill*

$$\fint_{B_R(p_i)} \sum_{j,l=1}^{k} | < \nabla b^i_j, \nabla b^i_l > - \delta_{jl}| + \sum_{j=1}^{k} \|\mathrm{Hess}_{b^i_j}\|^2 \, d\mu_i \to 0 \text{ for all } R > 0.$$

Then $(B_{r_i}(p_i), p_i)$ subconverges in the pointed Gromov–Hausdorff topology to a metric product $(\mathbb{R}^k \times X, p_\infty)$ for some metric space X. Moreover, $(b^i_1, \ldots, b^i_k)$ converges to the projection onto the Euclidean factor.

The lemma remains true if one just has a uniform lower Ricci curvature bound and one can also prove a local version of the lemma if $r_i = R$ is a fixed number. However, the above version suffices for our purposes.

Proof. The main problem is to prove this in the case of $k = 1$. Put $b_i = b_1^i$. After passing to a subsequence we may assume that $(B_{r_i}(p_i), p_i)$ converges to some limit space (Y, p_∞). We also may assume that b_i converges to an L-Lipschitz map $b_\infty \colon Y \to \mathbb{R}$.

Step 1. b_∞ is 1-Lipschitz.

This is essentially immediate from the segment inequality. Let $x, y \in Y$ be arbitrary. Choose R so large that $x, y \in B_{R/4}(p_\infty)$ and let $x_i, y_i \in B_{R/2}(p_i)$ be sequences converging to x and y.

For a fixed $\delta << R$ consider all minimal geodesic from points in $B_\delta(x_i)$ to points in $B_\delta(y_i)$. For each geodesic γ_{pq} consider $\int_0^1 ||\nabla b_i| - 1|(\gamma_{pq}(t))\, dt$. Since by Bishop–Gromov the set $B_\delta(x_i) \times B_\delta(x_i)$ fills up a fixed portion of the volume of $B_R(p_i)^2$, the segment inequality (Theorem 1.5) implies that there is a minimal geodesic γ_i from a point in $B_\delta(x_i)$ to a point in $B_\delta(y_i)$ with $\int_0^1 ||\nabla b_i| - 1|(\gamma_i(t))\, dt = h_i \to 0$. Thus

$$|b_i(x_i) - b_i(y_i)| \leq 2L\delta + L(\gamma_i)(1 + h_i)$$

and $|b_\infty(x) - b_\infty(y)| \leq 2(L+1)\delta + d(x, y)$. The claim follows as δ was arbitrary.

Step 2. $b_\infty \colon Y \to \mathbb{R}$ is a submetry.

As the gradient flow ϕ_t^i of b_i is measure preserving and $||\nabla b_i| - 1| \leq ||\nabla b_i|^2 - 1|$,

$$\begin{aligned}
\fint_{B_R(p_i)} \int_0^{t_0} ||\nabla b_i| - 1|(\phi_t^i(q))\, dt\, d\mu_i(q) &= \int_0^{t_0} \fint_{\phi_t^i(B_R(p_i))} ||\nabla b_i| - 1|(q)\, d\mu_i(q)\, dt \\
&\leq C \int_0^{t_0} \fint_{B_{R+t_0 L}(p_i)} ||\nabla b_i| - 1|(q)\, d\mu_i(q)\, dt \\
&\to 0,
\end{aligned}$$

where the inequality holds with some constant C satisfying $\frac{\mathrm{vol}(B_{R+t_0 L}(p_i))}{\mathrm{vol}(B_R(p_i))} \leq C$. Thus for fixed R and t_0 there is $\delta_i \to 0$ such that most gradient curves c_i of b_i defined on $[0, t_0]$ and starting in $B_R(p)$ satisfy $\int_0^{t_0} ||\nabla b_i| - 1|(c_i(t))\, dt \leq \delta_i$. For these curves we have $|L(c_i) - t_0| \leq \delta_i$ and $b_i(c_i(t_0)) - b_i(c_i(0)) \geq t_0 - \delta_i$. These curves subconverge to unit speed curves c in the limit with $b_\infty(c(t_0)) - b_\infty(c(0)) = t_0$. Since b_∞ is 1-Lipschitz, all such limit curves must be geodesics. In the limit these curves go through every point and thus b_∞ is a submetry.

Notice that the proof of Step 2 also shows that for the submetry b_∞ lines can be lifted to Y. By the splitting theorem of Cheeger and Colding (Theorem 1.1) the result follows for $k = 1$ and the generalization to arbitrary k is an easy exercise. □

Lemma 2.2. *Let $(Y_i, \tilde{p}_i)$ be an inner metric space space endowed with an action of a closed subgroup G_i of its isometry group, $i \in \mathbb{N} \cup \{\infty\}$. Suppose $(Y_i, \mathsf{G}_i, \tilde{p}_i) \to (Y_\infty, \mathsf{G}_\infty, \tilde{p}_\infty)$ in the equivariant Gromov–Hausdorff topology. Let $\mathsf{G}_i(r)$ denote the subgroup generated by those elements that displace $\tilde{p}_i$ by at most r, $i \in \mathbb{N} \cup \{\infty\}$. Suppose there are $0 \leq a < b$ with $\mathsf{G}_\infty(r) = \mathsf{G}_\infty(\frac{a+b}{2})$ for all $r \in (a, b)$.*

Then there is some sequence $\varepsilon_i \to 0$ such that $\mathsf{G}_i(r) = \mathsf{G}_i(\frac{a+b}{2})$ for all $r \in (a + \varepsilon_i, b - \varepsilon_i)$.

Proof. Suppose on the contrary we can find $g_i \in \mathsf{G}_i(r_2) \setminus \mathsf{G}_i(r_1)$ for fixed $r_1 < r_2 \in (a, b)$. Without loss of generality $d(\tilde p_i, g_i\tilde p_i) \le r_2$.

Since $g_i \notin \mathsf{G}_i(r_1)$ it follows that for any finite sequence of orbit points $\tilde p_i = x_1, \dots, x_h = g_i\tilde p_i \in \mathsf{G}_i \star \tilde p_i$ there is one $j \in \{1, \dots, h\}$ with $d(x_j, x_{j+1}) \ge r_1$. Clearly this property carries over to the limit and implies that $g_\infty \in \mathsf{G}_\infty(r_2)$ is not contained in $\mathsf{G}((r_1 + a)/2)$ – a contradiction. □

Lemma 2.3. *Suppose (M_i^n, q_i) converges to $(\mathbb{R}^k \times K, q_\infty)$ where $\mathrm{Ric}_{M_i} \ge -1/i$ and K is compact. Assume the action of $\pi_1(M_i)$ on the universal cover $(\tilde M_i, \tilde q_i)$ converges to a limit action of a group G on some limit space $(Y, \tilde q_\infty)$.*

Then $\mathsf{G}(r) = \mathsf{G}(r')$ for all $r, r' > 2\,\mathrm{diam}(K)$.

Proof. Since Y/G is isometric to $\mathbb{R}^k \times K$, it follows that there is a submetry $\sigma\colon Y \to \mathbb{R}^k$. Hence lines in $\mathbb{R}^k$ can be lifted to lines in Y and it is immediate from the splitting theorem (Theorem 1.1) that this submetry has to be linear, that is, for any geodesic c in Y the curve $\sigma \circ c$ is affine linear. We get a splitting $Y = \mathbb{R}^k \times Z$ such that G acts trivially on $\mathbb{R}^k$ and on Z with compact quotient K. We may think of $\tilde q_\infty$ as a point in Z. For $g \in \mathsf{G}$ consider a mid point $x \in Z$ of $\tilde q_\infty$ and $g\tilde q_\infty$. Because $Z/\mathsf{G} = K$ we can find $g_2 \in \mathsf{G}$ with $d(g_2\tilde q_\infty, x) \le \mathrm{diam}(K)$. Clearly

$$\begin{aligned} d(\tilde q_\infty, g_2\tilde q_\infty) &\le \tfrac12 d(\tilde q_\infty, g\tilde q_\infty) + \mathrm{diam}(K) \\ d(\tilde q_\infty, g_2^{-1}g\tilde q_\infty) &= d(g_2\tilde q_\infty, g\tilde q_\infty) \le \tfrac12 d(\tilde q_\infty, g\tilde q_\infty) + \mathrm{diam}(K). \end{aligned}$$

This proves $\mathsf{G}(r) \subset \mathsf{G}\big(r/2 + \mathrm{diam}(K)\big)$ and the lemma follows. □

Lemma 2.4 (Gap Lemma)**.** *Suppose we have a sequence of manifolds (M_i, p_i) with a lower Ricci curvature bound converging to some limit space (X, p_∞) and suppose that the limit point p_∞ is regular. Then there is a sequence $\varepsilon_i \to 0$ and a number $\delta > 0$ such that the following holds. If $\gamma_1, \dots, \gamma_{l_i}$ is a short basis of $\pi_1(M_i, p_i)$ then either $|\gamma_j| \ge \delta$ or $|\gamma_j| < \varepsilon_i$.*

Moreover, if the action of $\pi_1(M_i)$ on the universal cover $(\tilde M_i, \tilde p_i)$ converges to an action of the limit group G on $(Y, \tilde p_\infty)$, then the orbit $\mathsf{G} \star \tilde p_\infty$ is locally path connected. Here $\tilde p_i$ denotes a lift of p_i.

Proof. That the orbit is locally path connected is a consequence of the following

Claim. There is a $\delta > 0$ such that for all points $x \in \mathsf{G} \star \tilde p_\infty$ with $d(\tilde p_\infty, x) < \delta$ we can find $y \in \mathsf{G} \star \tilde p_\infty$ with $\max\{d(\tilde p_\infty, y), d(y, x)\} \le 0.51 \cdot d(\tilde p_\infty, x)$.

To prove the claim we argue by contradiction and assume that for some $\delta_h > 0$ converging to 0 we can find $g_h \in \mathsf{G}$ with $d(g_h\tilde p_\infty, \tilde p_\infty) = \delta_h$ such that for all $a \in \mathsf{G}$ with $d(a\tilde p_\infty, \tilde p_\infty) \le 0.51\delta_h$ we have $d(a\tilde p_\infty, g_h\tilde p_\infty) > 0.51\delta_h$.

After passing to a subsequence we can assume that $(\frac{1}{\delta_h}Y, \tilde p_\infty)$ converges to a tangent cone $(C_{p_\infty}Y, o)$ and that the action of G on $\frac{1}{\delta_h}Y$ converges to an action of some group K on $C_{p_\infty}Y$. Let $g_\infty \in \mathsf{K}$ be the limit of g_h. Clearly $d(o, g_\infty o) = 1$ and for all $a \in \mathsf{K}$ with $d(o, ao) \le 0.51$ we have $d(go, ao) \ge 0.51$. In particular the orbit $\mathsf{K} \star o$ is not convex.

Because $X = Y/\mathsf{G}$ the quotient $C_{\tilde p_\infty}Y/\mathsf{K}$ is isometric to $\lim_{h\to\infty}(\frac{1}{\varepsilon_h}X, p_\infty)$ which by assumption is isometric to some Euclidean space $\mathbb{R}^k$.

As in the proof of Lemma 2.3 it follows that $C_{\tilde p_\infty}Y$ is isometric to $\mathbb{R}^k \times Z$ and the orbits of the action of K are given by $v \times Z$ where $v \in \mathbb{R}^k$. In particular, the K

orbits are convex – a contradiction.

Clearly, the claim implies that $\mathsf{G} \star \tilde{p}_\infty \cap B_r(\tilde{p}_\infty)$ is path connected for $r < \delta$. In fact, it is now easy to construct a Hölder continuous path from $\tilde{p}_\infty$ to any point in $\mathsf{G} \star \tilde{p}_\infty \cap B_r(\tilde{p}_\infty)$. Of course, the claim also implies $\mathsf{G}(\varepsilon) = \mathsf{G}(\delta)$ for all $\varepsilon \in (0, \delta]$. Therefore the first part of Lemma 2.4 follows from Lemma 2.2. □

Theorem 3 will follow from the following slightly more general result.

Theorem 2.5. *Given n and R there is a constant C such that the following holds. Suppose (M, g) is an n-manifold, $p \in M$, $\overline{B_{2R}(p)}$ is compact and Ric $> -(n-1)$ on $B_{2R}(p)$. Furthermore, we assume that $\pi_1(M, p)$ is generated by loops of length $\leq R$. Then $\pi_1(M, p)$ can be generated by C loops of length $\leq R$.*

Moreover, there is a point $q \in B_{R/2}(p)$ such that any Gromov short generator system of $\pi_1(M, q)$ has at most C elements.

Proof. We first want to prove the second part of the theorem. We consider a Gromov short generator system $\gamma_1, \ldots, \gamma_k$ of $\pi_1(M, q)$. With $|\gamma_i| \leq |\gamma_{i+1}|$ for some $q \in B_{R/2}(p)$. Clearly $|\gamma_i| \leq 2R$ and it easily follows from Bishop–Gromov relative volume comparison that there are effective a priori bounds (depending only on n, R and r) for the number of short generators with length $\geq r$.

We will argue by contradiction. It will be convenient to modify the assumption on π_1 being boundedly generated. We call (M_i, p_i) a contradicting sequence if the following holds

- $\overline{B_3(p_i)}$ is compact and Ric $> -(n-1)$ on $B_3(p_i)$.
- For all $q_i \in B_1(p_i)$ the number of short generators of $\pi_1(M_i, q_i)$ of length ≤ 4 is larger than 2^i.

Clearly it suffices to rule out the existence of a contradicting sequence. We may assume that $(B_3(p_i), p_i)$ converges to some limit space (X, p_∞). We put

$$\dim(X) = \max\big\{k \mid \text{ there is a regular } x \in B_{1/4}(p_\infty) \text{ with } C_x X \cong \mathbb{R}^k\big\}.$$

We argue by reverse induction on $\dim(X)$. We start our induction at $\dim(X) \geq n+1$. It is well known that this can not happen so there is nothing to prove. The induction step is subdivided in two substeps.

Step 1. For any contradicting sequence (M_i, p_i) converging to (X, p_∞) there is a new contradicting sequence converging to $(\mathbb{R}^{\dim(X)}, 0)$.

Choose $q_i \in B_{1/4}(p_i)$ converging to some point $q_\infty \in B_{1/4}(p_\infty)$ with $C_{q_\infty} X = \mathbb{R}^{\dim(X)}$. After passing to a subsequence we can assume that for all $x_i \in B_{1/4}(q_i)$ the number of short generators of $\pi_1(M_i, x_i)$ of length ≤ 1 is at least 3^i.

Since the number of short generators of length $\in [\varepsilon, 4]$ is bounded by some a priori constant, we can find a sequence $\lambda_i \to \infty$ very slowly such that for every $x \in B_{1/\lambda_i}(q_i)$ the number of short generators of $\pi_1(M_i, x)$ of length $\leq 4/\lambda_i$ is at least 2^i and $(\lambda_i M_i, q_i)$ converges to $(C_{q_\infty} X, 0) = (\mathbb{R}^{\dim(X)}, 0)$. Replacing M_i by $\lambda_i M_i$ and p_i by q_i gives a new contradicting sequence, as claimed.

Step 2. If there is a contradicting sequence converging to $(\mathbb{R}^k, 0)$, then we can find a contradicting sequence converging to a space whose dimension is larger than k.

Let (M_i, p_i) be a contradicting sequence converging to $(\mathbb{R}^k, 0)$. We may assume without loss of generality that for some $r_i \to \infty$ and $\varepsilon_i \to 0$ the Ricci curvature on $B_{r_i}(p_i)$ is bounded below by $-\varepsilon_i$ and that $\overline{B_{r_i}(p_i)}$ is compact. In fact, one can run through the arguments of the first step, to see that, after passing to a subsequence, a rescaling by $\lambda_i \to \infty$ (very slowly) is always possible.

By Theorem 1.3 we can find a harmonic map $(b_1^i, \ldots, b_k^i)\colon B_1(q_i) \to \mathbb{R}^k$ with

$$\fint_{B_1(q_i)} \sum_{j,l=1}^{k} \big(|\langle \nabla b_l^i, \nabla b_j^i\rangle - \delta_{lj}| + \|\mathrm{Hess}(b_l^i)\|^2\big)\, d\mu_i = \varepsilon_i \to 0 \quad \text{and}$$

$$|\nabla b_j^i| \le C(n).$$

By the weak (1,1) inequality (Lemma 1.4) we can find $z_i \in B_{1/2}(q_i)$ with

$$\fint_{B_r(z_i)} \sum_{j,l=1}^{k} \big(|\langle \nabla b_l^i, \nabla b_j^i\rangle - \delta_{lj}| + \|\mathrm{Hess}(b_l^i)\|^2\big) d\mu_i \le C\varepsilon_i \to 0$$

for all $r \le 1/4$. By the Product Lemma (2.1), for any sequence $\mu_i \to \infty$ the spaces $(\mu_i B_r(z_i), z_i)$ subconverge to a metric product $(\mathbb{R}^k \times Z, z_\infty)$ for some Z depending on the rescaling.

From Lemmas 2.2 and 2.3 we deduce that there is a sequence $\delta_i \to 0$ such that for all $z_i \in B_2(p_i)$ the short generator system of $\pi_1(M_i, z_i)$ does not contain any elements with length in $[\delta_i, 4]$. Choose $r_i \le 1$ maximal with the property that there is $y_i \in B_{r_i}(z_i)$ such that the short generator system of $\pi_1(M_i, y_i)$ contains one generator of length r_i. We have seen above that $r_i \le \delta_i \to 0$.

Put $N_i = \frac{1}{r_i} M_i$. By construction, $\pi_1(N_i, y_i)$ still has 2^i short generators of length ≤ 1 for all $y_i \in B_1(z_i) \subset N_i$, and there is one with length 1 for a suitable y_i. By the Product Lemma (2.1), (N_i, z_i) subconverges to a product $(\mathbb{R}^k \times Z, z_\infty)$. Lemmas 2.2 and 2.3 imply that Z can not be a point and the claim is proved.

In order to prove the first part of the theorem we consider the subgroup of $\pi_1(M, p)$ generated by loops of length $\le R/10$. By the second part this subgroup can be generated by $C(n, R)$ elements of length $\le 2R/5$. Since the number of short generators of $\pi_1(M, p)$ with length in $[R/10, R]$ is bounded by some a priori constant the theorem follows. □

Finally, let us mention that there is a measured version of Theorem 2.5.

Theorem 2.6. *For any $n > 1$ and any $\varepsilon \in (0, 1)$ there exists $C(n, \varepsilon)$ such that if $\mathrm{Ric}(M^n) \ge -(n-1)$ on $B_3(p)$ with $\overline{B_3(p)}$ being compact, then there is a subset $B_1(p)' \subset B_1(p)$ with $\mathrm{vol}\, B_1(p)' \ge (1-\varepsilon)\,\mathrm{vol}\, B_1(p)$ and any short basis of $\pi_1(M, q)$ has at most $C(n, \varepsilon)$ elements of length ≤ 1 for any $q \in B_1(p)'$.*

Since we do not have any applications of this theorem, we omit its proof.

3. Maps which are on all scales close to isometries.

For a map $f\colon X \to Y$ between metric spaces we define the distance distortion on scale r by

$$\mathrm{dt}_r^f(p, q) = \min\{r, |d(p, q) - d(f(p), f(q))|\} \quad \text{for } p, q \in X. \tag{8}$$

Definition 3.1. Let (M_i^n, p_i^1) and (N_i^n, p_i^2) be two sequences of Riemannian manifolds. We say that a sequence of diffeomorphisms $f_i\colon M_i \to N_i$ has the zooming in property if the following holds: There exist $R_0 > 0$, sequences $r_i \to \infty$, $\varepsilon_i \to 0$ and subsets $B_{2r_i}(p_i^j)' \subset B_{2r_i}(p_i^j)$ $(j = 1, 2)$ satisfying

a) $\overline{B_{4r_i}(p_i^j)}$ is compact and $\mathrm{Ric}_{B_{4r_i}(p_i^j)} > -R_0$.
b) $\mathrm{vol}\big(B_1(q) \cap B_{2r_i}(p_i^j)'\big) \geq (1 - \varepsilon_i)\,\mathrm{vol}(B_1(q))$ for all $q \in B_{r_i}(p_i^j)$.
c) For all $p \in B_{r_i}(p_i^1)'$, all $q \in B_{r_i}(p_i^2)'$ and all $r \in (0, 1]$ we have

$$\begin{aligned} \fint_{B_r(p)\times B_r(p)} \mathrm{dt}_r^{f_i}(x,y)d\mu_i^1(x)d\mu_i^1(y) &\leq r\varepsilon_i \text{ and} \\ \fint_{B_r(q)\times B_r(q)} \mathrm{dt}_r^{f_i^{-1}}(x,y)d\mu_i^2(x)d\mu_i^2(y) &\leq r\varepsilon_i. \end{aligned}$$

d) There are subsets $S_i^j \subset B_1(p_i^j)$ with $\mathrm{vol}(S_i^j) \geq \frac{1}{2}\,\mathrm{vol}\big(B_1(p_i^j)\big)$ $(j = 1, 2)$ and $f(S_i^1) \subset B_{R_0}(p_i^2)$ and $f^{-1}(S_i^2) \subset B_{R_0}(p_i^1)$.

We will call elements of $B_{2r_i}(p_i^1)'$ *good* points and sometimes use the convention $B_r(q)' := B_{2r_i}(p_i^j)' \cap B_r(q)$ for all $B_r(q) \subset B_{2r_i}(p_i^j)$.

In the applications we will always have $N_i = M_i$. However, in some instances $d(p_i^1, p_i^2) \to \infty$. That is why it might be helpful to also think about maps between two unrelated pointed manifolds. If the choice of the base points is not clear we will say that $f_i \colon [M_i, p_i^1] \to [N_i, p_i^2]$ has the zooming in property. Notice that we do not require $f_i(p_i^1) = p_i^2$. However, property d) ensures that the base points are respected in a weaker sense.

Lemma 3.2. *Let (M_i, p_i^1), (N_i, p_i^2) and f_i be as above. Then, after passing to a subsequence, $(M_i, p_i^1) \xrightarrow{G-H} (X_1, p_\infty^1)$, $(N_i, p_i^2) \xrightarrow{G-H} (X_2, p_\infty^2)$ and f_i converges in the weakly measured sense to a measure preserving isometry $f_\infty \colon X_1 \to X_2$, that is for each $r > 0$ there is a sequence $\delta_i \to 0$ and subsets $S_i \subset B_r(p_i^1)$ satisfying*

- *$\mathrm{vol}(S_i) \geq (1 - \delta_i)\,\mathrm{vol}(B_r(p_i^1))$ and*
- *$f_{i|S_i}$ is pointwise close to $f_{\infty|B_r(p_\infty^1)}$.*

Moreover, f_i^{-1} converges in this sense to f_∞^{-1}.

For the proof we will need the following

Sublemma 3.3. *There exists $C_1(n)$ such that for any good point $x \in B_{r_i}(p_i^1)'$ and any $r \leq 1$ there is a subset $B_r(x)'' \subset B_r(x)$ with*

$$\mathrm{vol}\, B_r(x)'' \geq (1 - C_1\varepsilon_i)\,\mathrm{vol}\, B_r(x) \text{ and } f_i(B_r(x)'') \subset B_{2r}(f_i(x)).$$

Proof. The proof proceeds by induction on the size of r as follows. It is clear that at good points $x \in B_{r_i}(p_i^1)'$ the differential of f_i has a bilipschitz constant $e^{C\varepsilon_i}$ with some universal C provided $\varepsilon_i < 1/2$. Therefore it is clear that the statement holds for all small r. Hence it suffices to prove that if it holds for some $r \leq 1/10$, then it holds for $10r$. By assumption

$$\fint_{B_{10r}(x)\times B_{10r}(x)} dt_{10r}^{f_i}(p,q)\, d\mu_i^1(p)d\mu_i^1(q) \leq 10r\varepsilon_i.$$

Furthermore, as long as $C_1\varepsilon_i \leq 1/2$ our induction assumption implies that there is a subset $S \subset B_r(x)$ with $\mathrm{vol}(S) \geq \frac{1}{2}\,\mathrm{vol}(B_r(x))$ and $f_i(S) \subset B_{2r}(f_i(x))$.

By Bishop–Gromov

$$\fint_{S\times B_{10r}(x)} dt_{10r}^{f_i}(p,q)d\mu_i^1(p)d\mu_i^1(q) \leq C_2(n)r\varepsilon_i$$

with some universal constant $C_2(n)$. Therefore there is a subset $B_{10r}(x)'' \subset B_{10r}(x)$ with $\mathrm{vol}(B_{10r}(x)'') \geq (1 - C_2(n)\varepsilon_i/2)\,\mathrm{vol}(B_{10r}(x))$ and

$$\fint_S dt_{10r}^{f_i}(p,q)d\mu_i^1(p) \leq 2r \quad \text{for all } q \in B_{10r}(x)''.$$

Using that $f_i(S) \subset B_{2r}(f_i(x))$ this clearly implies that $f_i(B_{10r}(x)'') \subset B_{20r}(f_i(x))$. Thus the sublemma is valid if we put $C_1(n) = C_2(n)/2$ provided we know in addition that $C_1(n)\varepsilon_i \leq 1/2$. We can remove the upper bound on ε_i by just putting $C_1(n) = C_2(n)$. □

Of course, a similar inequality holds for f_i^{-1}.

Proof of Lemma 3.2. Note that as was observed in the proof of Sublemma 3.3, at good points in $B_{r_i}(p_i^1)'$ $(B_{r_i}(p_i^2)')$ the differential of f_i (f_i^{-1}) has bilipschitz constant $\leq e^{C\varepsilon_i}$ for some universal C provided $\varepsilon_i < 1/2$. Using condition b) from the definition of the zooming in property this will clearly ensure that f_i and f_i^{-1} converge to *measure-preserving* isometries once we establish the following

Claim. Given $\delta \in (0, 1/10)$ there is i_0 such that $|d(f_i(x_i), f_i(y_i)) - d(x_i, y_i)| \leq 10\delta$ holds for all $i \geq i_0$ and all $x_i, y_i \in B_{r_i}(p_i^1)'$ with $d(x_i, y_i) < 1/2$.

Consider subsets $B_\delta(x_i)''$ and $B_\delta(y_i)''$ as in the sublemma. Then $\mathrm{vol}(B_\delta(x_i)'') \geq \frac{1}{2}\mathrm{vol}(B_\delta(x_i))$ for large i and combining with Bishop–Gromov gives

$$\mathrm{vol}(B_1(x_i))^2 \leq C_2(n,\delta)\,\mathrm{vol}(B_\delta(x_i)'')\,\mathrm{vol}(B_\delta(y_i)'').$$

Thus,

$$\fint_{B_\delta(x_i)''\times B_\delta(y_i)''} \mathrm{dt}_1^{f_i}(p,q) \leq C_2 \fint_{B_1(x_i)^2} \mathrm{dt}_1^{f_i}(p,q) \leq C_2\varepsilon_i.$$

Choose i_0 so large that $C_2\varepsilon_i \leq \delta$ for $i \geq i_0$. Then for such i we can find $x_i' \in B_\delta(x_i)''$ and $y_i' \in B_\delta(y_i)''$ with $\mathrm{dt}_1^{f_i}(x_i', y_i') \leq \delta$. Combining with $d(f_i(x_i'), f_i(x_i)) \leq 2\delta$ and $d(f_i(y_i'), f_i(y_i)) \leq 2\delta$ we deduce $\mathrm{dt}_1^{f_i}(x_i, y_i) \leq 7\delta$ as claimed. □

Lemma 3.4. *Consider three pointed Riemannnian manifolds $(M_i, p_i^1), (N_i, p_i^2)$, (P_i, p_i^3) and two sequences of diffeomorphisms $f_i\colon M_i \to N_i$ and $g_i\colon N_i \to P_i$ with the zooming in property. Then $g_i \circ f_i$ also has the zooming in property.*

Proof. Let $R > 10$ be arbitrary. By assumption we can find a sequence $\varepsilon_i \to 0$ and a subset $B_{2R}(p_i^j)' \subset B_{2R}(p_i^j)$ with $\mathrm{vol}(B_{2R}(p_i^j)') \geq (1-\varepsilon_i)\,\mathrm{vol}(B_{2R}(p_i^j))$ $(j = 1, 2, 3)$ such that the following holds

$$\begin{aligned}
\fint_{B_r(q)^2} dt_r^{f_i}(a,b) &\leq \varepsilon_i r \quad \text{for all } r \in (0,2] \text{ and } q \in B_{2R}(p_i^1)',\\
\fint_{B_r(q)^2} dt_r^{f_i^{-1}}(a,b) + dt_r^{g_i}(a,b) &\leq \varepsilon_i r \quad \text{for all } r \in (0,2] \text{ and } q \in B_{2R}(p_i^2)' \text{ and}\\
\fint_{B_r(q)^2} dt_r^{g_i^{-1}}(a,b) &\leq \varepsilon_i r \quad \text{for all } r \in (0,2] \text{ and } q \in B_{2R}(p_i^3)'.
\end{aligned}$$

In order to get the above inequalities for $r \in [1,2]$ we used that f_i and g_i converge in the weakly measured sense to an isometry by Lemma 3.2. Lemma 3.2 also implies that after a possible adjustment of $\varepsilon_i \to 0$ we have

$$\begin{aligned}
S &:= B_R(p_i^2)' \cap f_i(B_R(p_i^1)') \cap g_i^{-1}(B_R(p_i^3)') \text{ satisfies}\\
\mathrm{vol}(S) &\geq (1-\varepsilon_i)\,\mathrm{vol}(B_R(p_i^2))
\end{aligned}$$

Similarly without loss of generality $\mathrm{vol}(f_i^{-1}(S)) \geq (1-\varepsilon_i)\,\mathrm{vol}(B_R(p_i^1))$ and $\mathrm{vol}(g_i(S)) \geq (1-\varepsilon_i)\,\mathrm{vol}(B_R(p_i^3))$. In other words, we may assume

$$f_i(B_R(p_i^1)') = B_R(p_i^2)' \text{ and } g_i(B_R(p_i^2)') = B_R(p_i^3)'.$$

Next we consider the characteristic function χ of the set $B_{2R}(p_i^1) \setminus B_{2R}(p_i^1)'$. Using Lemma 1.4 b) and Bishop–Gromov we can assume that there is a sequence $\delta_i \to 0$ such that $\delta_i > \varepsilon_i$ and

$$G_R(p_i^1) := \{q \in B_R(p_i^1)' \mid \mathrm{Mx}\,\chi(q) \leq \delta_i\} \text{ fulfills}$$

$$\mathrm{vol}\big(B_R(p_i^1) \setminus G_R(p_i^1)\big) \leq \delta_i \min_{q \in B_R(p_i^1)} \mathrm{vol}(B_1(q)). \tag{9}$$

By Sublemma 3.3 for all $q_i \in G_R(p_i^1)$ and $r \leq 1$ there is a subset $B_r(q_i)'' \subset B_r(q_i)$ with $\mathrm{vol}(B_r(q_i)'') \geq (1 - C_1\delta_i)\,\mathrm{vol}\,B_r(q_i)$ and $f_i(B_r(q_i)'') \subset B_{2r}(f(q_i))$. Using $q_i \in G_R(p_i^1)$ we can actually assume $B_r(q_i)'' \subset B_{2R}(q_i)'$ provided we replace C_1 by $C_2 = C_1 + 1$. Thus,

$$\begin{aligned}
\fint_{B_r(q_i)^2} \mathrm{dt}_r^{g_i \circ f_i}(a,b) \;&\leq\; 2C_2\delta_i r + \tfrac{1}{\mathrm{vol}(B_r(q))^2} \int_{(B_r(q_i)'')^2} \mathrm{dt}_r^{g_i}(f_i(a), f_i(b)) + dt_r^{f_i}(a,b)\\
&\leq\; (2C_2+1)\delta_i r + \tfrac{e^{2nC\varepsilon_i}}{\mathrm{vol}(B_r(q_i))^2} \int_{B_{2r}(f_i(q_i))^2} \mathrm{dt}_r^{g_i}(a,b)\\
&\leq\; (2C_2+1)\delta_i r + \tfrac{2e^{2nC\varepsilon_i}\,\mathrm{vol}(B_{2r}(f_i(q_i))^2}{\mathrm{vol}(B_r(q_i))^2}\varepsilon_i r \leq C_3\delta_i r,
\end{aligned}$$

where we used that the differential of f_i at $q \in B_r(q_i)'' \subset B_{2R}(q_i)'$ has a bilipschitz constant $\leq e^{C\varepsilon_i}$ and that the ratio of $\mathrm{vol}(B_{2r}(f_i(q_i)))$ and $\mathrm{vol}(B_r(q_i))$ is for large i bounded by a universal constant. The latter statement follows from Bishop–Gromov and Sublemma 3.3 applied to f_i^{-1} and the ball $B_r(f_i(q_i))$.

The last inequality holds for all $q_i \in G_R(p_i^1)$, where $R \geq 10$ was arbitrary. By the usual diagonal sequence argument one can deduce that there is a sequence $R_i \to \infty$ and an adjusted sequence $\delta_i \to 0$ such that the above inequality holds for all $q_i \in G_{R_i}(p_i^1)$ and in addition we can assume that (9) remains valid.

Since everthing can be carried out for $f_i^{-1} \circ g_i^{-1}$ as well this finishes the proof. □

The next lemma explains the notion zooming in property.

Lemma 3.5. *Let $(M_i, p_i^1), (N_i, p_i^2)$ and f_i be as above. Then there is a $\rho_i \to \infty$, $\delta_i \to 0$ and $T_i^1 \subset B_{\rho_i}(p_i^1)$ such that the following holds*

- $\mathrm{vol}(B_1(q) \cap T_i^1) \geq (1-\delta_i)\,\mathrm{vol}(B_1(q))$ *for all* $q \in B_{\rho_i/2}(p_i^1)$.
- *For any sequence of real numbers $\lambda_i \to \infty$ and any sequence $q_i \in T_i^1$*

  $$f_i\colon (\lambda_i M_i, q_i) \to (\lambda_i N_i, f_i(q_i))$$

  has the zooming in property. We say that f_i is good on all scales at q_i.

Proof. Let $G_i^j = B_{2r_i}(p_i^j)'$ and $B_i^j = B_{2r_i}(p_i^j) \setminus B_{2r_i}(p_i^j)'$. After adjusting $r_i \to \infty$ and $\varepsilon_i \to 0$ we may assume $\mathrm{vol}(B_i^j) \leq \varepsilon_i\,\mathrm{vol}(B_1(q))$ for all $q \in B_{2r_i}(p_i^j)$. Let χ_i^j be the characteristic function of B_i^j. By the weak 1-1 inequality (Lemma 1.4) there exists a universal C such that the set

$$H_i^j := \big\{x \in B_{r_i/2}(p_i^j) \mid \mathrm{Mx}(\chi_i^j)(x) \geq \sqrt{\varepsilon}_i\big\}$$

satisfies

$$\mathrm{vol}(H_i^j) \leq C\sqrt{\varepsilon_i}\,\mathrm{vol}(B_1(q))$$

for all $q \in B_{2r_i}(p_i^j)$. We put $T_i^1 := \big(B_{r_i/2}(p_i^1)\backslash H_i^1\big) \cap f_i^{-1}\big(B_{r_i/2}(p_i^2)\backslash H_i^2\big)$ and $T_i^2 := f_i(T_i^1)$. Using Lemma 3.2 we can find $\rho_i \to \infty$ and $\delta_i \to 0$ such that

$$\mathrm{vol}\big(B_{\rho_i}(p_i^j) \setminus T_i^j\big) \le \delta_i \,\mathrm{vol}(B_1(q)) \text{ for all } q \in B_{\rho_i}(p_i^j),\ j = 1, 2.$$

By definition of T_i^j

$$\frac{\mathrm{vol}(B_r(q) \cap G_i^j)}{\mathrm{vol}(B_r(q))} \ge 1 - \sqrt{\varepsilon_i} \quad \text{for all } q \in T_i^j \text{ and all } r \le 1,\ j = 1, 2.$$

Let $\mathrm{dt}_r^{\lambda_i f}$ denote the distortion on scale r of $f_i\colon \lambda_i M_i \to \lambda_i N_i$. Clearly

$$\mathrm{dt}_r^{\lambda_i f}(p, q) = \lambda_i \,\mathrm{dt}_{r/\lambda_i}^{f_i}(p, q).$$

Thus for all $\lambda_i \to \infty$ and all $q_i \in T_i^j$ the map $f_i\colon (\lambda_i M_i, q_i) \to (\lambda_i N_i, f_i(q_i))$ has the zooming in property.

□

Proposition 3.6 (First main example)**.** *Let $\alpha > 1$. Consider a sequence of n-manifolds (M_i, p_i) with a fixed lower Ricci curvature bound and a sequence of time dependent vector fields X_i^t (piecewise constant in time) with compact support. Let $c_i\colon [0, 1] \to B_{r_i}(p_i)$ be an integral curve of X_i^t with $c_i(0) = p_i$ Assume that X_i^t is divergence free on $B_{r_i+100}(p_i)$ and that $\overline{B_{r_i+100}(p_i)}$ is compact. Put*

$$u_{s,i}(x) := \big(\mathrm{Mx}\, \|\nabla.X_i^s\|^\alpha\big)^{1/\alpha}(x)$$

and suppose

$$\int_0^1 u_{t,i}(c_i(t))\, dt = \varepsilon_i \to 0.$$

Let $f_i = \phi_{i1}$ be the flow of X_i^t evaluated at time 1. Then for all $\lambda_i \to \infty$

$$f_i\colon (\lambda_i M_i, c(0)) \to (\lambda_i M_i, c(1))$$

has the zooming in property. Moreover, for any lift $\tilde f_i\colon \tilde M_i \to \tilde M_i$ of f_i to the universal cover $\tilde M_i$ of M_i and for any lift $\tilde p_i \in \tilde M$ of $c(0) = p_i$ the sequence $\tilde f_i\colon (\lambda_i \tilde M_i, \tilde p_i) \to (\lambda_i \tilde M_i, \tilde f_i(\tilde p_i))$ has the zooming in property as well.

The proposition remains valid if the assumption on X_i^t being divergence free is removed. However, the proof is easier in this case and we do not have any applications of the more general case. We will need the following

Lemma 3.7. *There exists (explicit) $C = C(n)$ such that the following holds. Suppose (M^n, g) has $\mathrm{Ric} \ge -1$ and X^t is a vector field with compact support, which depends on time (piecewise constant). Let $c(t)$ be the integral curve of X^t with $c(0) = p_0 \in M$ and assume that X^t is divergence free on $B_{10}(c(t))$ for all $t \in [0, 1]$.*

Let ϕ_t be the flow of X^t. Define the distortion function $dt_r(t)(p, q)$ of the flow on scale r by the formula

$$\mathrm{dt}_r(t)(p, q) := \min\Big\{r, \max_{0 \le \tau \le t} \big|\, d(p, q) - d(\phi_\tau(p), \phi_\tau(q))\big|\Big\}.$$

Put $\varepsilon := \int_0^1 \mathrm{Mx}_1(\|\nabla.X^t\|)(c(t))\, dt$. Then for any $r \le 1/10$ we have

$$\fint_{B_r(p_0)\times B_r(p_0)} \mathrm{dt}_r(1)(p, q)\, d\mu(p) d\mu(q) \le Cr \cdot \varepsilon$$

and there exists $B_r(p_0)' \subset B_r(p_0)$ *such that*

$$\tfrac{\mathrm{vol}(B_r(p_0)')}{\mathrm{vol}(B_r(p_0))} \geq (1 - C\varepsilon) \quad \textit{and} \quad \phi_t(B_r(p_0)') \subset B_{2r}(c(t)) \textit{ for all } t \in [0,1].$$

Proof. We prove the statement for a constant in time vector field X^t. The general case is completely analogous except for additional notational problems.

Notice that all estimates are trivial if $\varepsilon \geq \frac{2}{C}$. Therefore it suffices to prove the statement with a universal constant $C(n)$ for all $\varepsilon \leq \varepsilon_0$. We put $\varepsilon_0 = 1/2C$ and determine $C \geq 2$ in the process. We again proceed by induction on the size of r.

Notice that the differential of ϕ_s at $c(0)$ is bilipschitz with bilipschitz constant $e^{\int_0^s \|\nabla.X\|(c(t))dt} \leq 1 + 2\varepsilon$. Thus the Lemma holds for very small r.

Suppose the result holds for some $r/10 \leq 1/100$. It suffices to prove that it then holds for r. By induction assumption we know that for any t there exists $B_{r/10}(c(t))' \subset B_{r/10}(c(t))$ such that for any $s \in [-t, 1-t]$ we have

$$\mathrm{vol}(B_{r/10}(c(t))') \geq (1 - C\varepsilon)\,\mathrm{vol}(B_{r/10}(c(t))) \geq \frac{1}{2}\,\mathrm{vol}(B_{r/10}(c(t)))$$

and

$$\phi_s(B_{r/10}(c(t))') \subset B_{r/5}(c(t+s)),$$

where we used $\varepsilon \leq \frac{1}{2C}$ in the inequality. This easily implies that $\mathrm{vol}(B_{r/10}(c(t)))$ are comparable for all t. More precisely, for any $t_1, t_2 \in [0,1]$ we have that

$$\tfrac{1}{C_0}\,\mathrm{vol}\, B_{r/10}(c(t_1)) \leq \mathrm{vol}\, B_{r/10}(c(t_2)) \leq C_0\,\mathrm{vol}\, B_{r/10}(c(t_1)) \tag{10}$$

with a computable universal $C_0 = C_0(n)$. Put

$$h(s) = \fint_{B_{r/10}(c(0))' \times B_r(c(0))} \mathrm{dt}_r(s)(p,q)\ d\mu(p)\, d\mu(q),$$

$$\begin{aligned} U_s &:= \{(p,q) \in B_{r/10}(c(0))' \times B_r(c(0)) \mid \mathrm{dt}_r(s)(p,q) < r\}, \\ \phi_s(U_s) &:= \{(\phi_s(p), \phi_s(q)) \mid (p,q) \in U_s\}, \quad \text{and} \\ \mathrm{dt}_r'(s)(p,q) &:= \limsup_{h \searrow 0} \tfrac{\mathrm{dt}_r(s+h)(p,q) - \mathrm{dt}_r(s)(p,q)}{h}. \end{aligned}$$

As $\mathrm{dt}_r(t) \leq r$ is monotonously increasing, we deduce that if $\mathrm{dt}_r(s)(p,q) = r$, then $\mathrm{dt}_r'(s)(p,q) = 0$. Since $\mathrm{dt}_r(s+h)(p,q) \leq \mathrm{dt}_r(s)(p,q) + \mathrm{dt}_r(h)(\phi_s(p), \phi_s(q))$ and ϕ_s is measure preserving, it follows

$$\begin{aligned} h'(s) &\leq \fint_{\phi_s(U_s)} \mathrm{dt}_r'(0)(p,q) \\ &\leq \tfrac{4\,\mathrm{vol}\, B_{3r}(c(s))^2}{\mathrm{vol}\, B_{r/10}(c(0))^2} \fint_{B_{3r}(c(s))^2} \mathrm{dt}_r'(0)(p,q), \end{aligned}$$

where we used that $\phi_s(B_{r/10}(p_0)')^2 \subset \phi_s(U_s) \subset B_{3r}(c(s))^2$. If p is not in the cut locus of q and $\gamma_{pq}\colon [0,1] \to M$ is a minimal geodesic between p and q, then

$$dt_r'(0)(p,q) \leq d(p,q) \int_0^1 \|\nabla.X\|(\gamma_{pq}(t))\, dt.$$

Combining the last two inequalities with the segment inequality we deduce

$$\begin{aligned} h'(s) &\leq C_1(n) r \fint_{B_{6r}(c(s))} \|\nabla.X\| \\ &\leq C_1(n) r \,\mathrm{Mx}_1 \|\nabla.X\|(c(s)) \end{aligned}$$

Note that the choice of the constant $C_1(n)$ can be made explicit and *independent* of the induction assumption. We deduce $h(1) \leq C_1(n) r\varepsilon$ and thus the subset

$$B_r(p_0)' := \left\{ p \in B_r(p_0) \;\middle|\; \fint_{B_{r/10}(p_0)'} \mathrm{dt}_r(1)(p,q)\, d\mu(q) \leq r/2 \right\}$$

satisfies

$$\mathrm{vol}(B_r(p_0)') \geq (1 - 2C_1(n)\varepsilon)\,\mathrm{vol}(B_r(p_0)). \tag{11}$$

It is elementary to check that

$$\phi_t(B_r(p_0)') \subset B_{2r}(c(t)) \text{ for all } t \in [0,1].$$

Then arguing as before we estimate that

$$\fint_{B_r(p_0)' \times B_r(p_0)} dt_r(1)(p,q) d\mu(p) d\mu(q) \leq C_2(n) \cdot r \cdot \varepsilon.$$

Using $\mathrm{dt}_r(1) \leq r$ and the volume estimate (11) this gives

$$\fint_{B_r(c(p_0))^2} \mathrm{dt}_r(1)(p,q)\, d\mu(p)\, d\mu(q) \leq C_2(n) \cdot r \cdot \varepsilon + 2rC_1\varepsilon =: C_3 r\varepsilon.$$

This completes the induction step with $C(n) = C_3$ and $\varepsilon_0 = \frac{1}{2C_3}$. In order to remove the restriction $\varepsilon \leq \varepsilon_0$ one can just increase $C(n)$ by the factor 4, as indicated at the beginning. □

Proof of Proposition 3.6. Put $g_{t,i}(x) = \|\nabla.X_i^t\|(x)$. First notice that by (3)

$$\int_0^1 \mathrm{Mx}(g_{t,i})(c_i(t))\, dt \leq \int_0^1 u_{t,i}(c_i(t))\, dt \to 0.$$

Let $\lambda_i \to \infty$ and put $r_i = \frac{R}{\lambda_i}$, where $R > 1$ is arbitrary. By Lemma 3.7 there is $S_i \subset B_{r_i}(c_i(0))$ and $\delta_i \to 0$ with

- $\mathrm{vol}(S_i) \geq (1 - \delta_i)\,\mathrm{vol}(B_{r_i}(c_i(0)))$ and
- $\phi_{it}(S_i) \subset B_{2r_i}(c_i(t))$ for all t.

In the following we assume that i is so large that $\delta_i \leq 1/2$, and $r_i \leq 1/100$. As in the proof of Lemma 3.7 this easily implies that there is a universal constant $C = C(n)$ with

$$\tfrac{\mathrm{vol}(B_{2r_i}(c_i(t)))}{\mathrm{vol}(B_{r_i}(c_i(0)))} \leq \tfrac{C}{2} \text{ for all } t \text{ and all } i.$$

Using that $\phi_{it|S_i}$ is measure preserving, we deduce

$$\begin{aligned} \fint_{S_i}\int_0^1 \mathrm{Mx}_1(g_{t,i})(\phi_{it}(p))\,dt\,d\mu_i(p) &\le C\int_0^1 \fint_{B_{2r_i}(c_i(t))} \mathrm{Mx}_1(g_{t,i})(q)\,d\mu_i\,dt(q)\\ &\le C\int_0^1 \mathrm{Mx}_1(\mathrm{Mx}_1(g_{t,i}))(c_i(t))\,dt\\ &\overset{by\ (6)}{\le} C\cdot C_2\int_0^1 \mathrm{Mx}(g_{t,i}^{\alpha})^{1/\alpha}(c_i(t))\,dt\\ &= C\cdot C_2\int_0^1 u_{t,i}(c_i(t))\;\mathrm{dt}\to 0. \end{aligned}$$

Thus we can find $\tilde{\delta}_i\to 0$ and a subset $B_{r_i}(c_i(0))''\subset S_i$ with

$$\mathrm{vol}(B_{r_i}(c_i(0))) - \mathrm{vol}(B_{r_i}(c_i(0))'') \le \tilde{\delta}_i \min_{q\in B_{r_i}(c_i(0))} \mathrm{vol}(B_{r_i/R}(q))$$

and

$$\int_0^1 \mathrm{Mx}_1(g_{t,i})(\phi_{it}(q))\,dt \le \tilde{\delta}_i \text{ for all } q\in B_{r_i}(c(0))''.$$

Recall that $r_i=\frac{2R}{\lambda_i}$ with an arbitrary R. By a diagonal sequence argument it is easy to deduce that after replacing $\tilde{\delta}_i$ by another sequence converging slowly to 0 we can keep the above estimates for $r_i=\frac{R_i}{\lambda_i}$ with $R_i\to\infty$ sufficiently slowly. Combining this with Lemma 3.7 shows that $f_i=\phi_{i1}\colon(\lambda_iM_i,c_i(0))\to(\lambda_iM_i,c_i(1))$ has the zooming in property.

Let $\tilde{X}_i^t$ be a lift of X_i^t to the universal covering $\tilde{M}_i$ of M_i. Consider the integral curve $\tilde{c}_i\colon[0,1]\to\tilde{M}_i$ of $\tilde{X}_i^t$ with $\tilde{c}(0)=\tilde{p}_i$. Clearly $\tilde{c}_i$ is a lift of c_i and by the Covering Lemma (1.6) we have

$$\int_0^1 \mathrm{Mx}(\|\nabla.\tilde{X}_i^t\|^{\alpha})^{1/\alpha}(\tilde{c}_i(t))\,dt \le C\int_0^1 \mathrm{Mx}(\|\nabla.X_i^t\|^{\alpha})^{1/\alpha}(c_i(t))\,dt$$

with some universal constant C. Thus $\tilde{\phi}_{i1}\colon(\lambda_i\tilde{M}_i,\tilde{p}_i)\to(\lambda_i\tilde{M}_i,\phi_{i1}(\tilde{p}_i))$ has the zooming in property as well. Any other lift $\tilde{f}_i$ of f_i is obtained by composing $\tilde{\phi}_{i1}$ with a deck transformation and thus the result carries over to any lift of f_i. ☐

Proposition 3.8 (Second main example)**.** *Let (M_i,g_i) be a sequence of n-manifolds with* Ric $>-1/i$ *on $B_i(p_i)$ and $\overline{B_i(p_i)}$ compact. Suppose that (M_i,p_i) converges to $(\mathbb{R}^k\times Y,p_\infty)$.*

Then for each $v\in\mathbb{R}^k$ there is a sequence of diffeomorphisms $f_i\colon[M_i,p_i]\to[M_i,p_i]$ with the zooming in property which converges in the weakly measured sense to an isometry f_∞ of $\mathbb{R}^k\times Y$ that acts trivially on Y and by $w\mapsto w+v$ on $\mathbb{R}^k$. Moreover, f_i is isotopic to the identity and there is a lift $\tilde{f}_i\colon[\tilde{M}_i,\tilde{p}_i]\to[\tilde{M}_i,\tilde{p}_i]$ of f_i to the universal cover which has the zooming in property as well.

Proof. Using the splitting $\mathbb{R}^k=\mathbb{R}v\oplus(v)^\perp$ and replacing Y by $Y\times(v)^\perp$ we see that it suffices to prove the statement for $k=1$.

By the work of Cheeger and Colding [CC96] we can find sequences $\rho_i\to\infty$ and $\varepsilon_i\to 0$ and harmonic functions $b_i\colon B_{4\rho_i}(p_i)\to\mathbb{R}$ such that

$$\begin{aligned} |\nabla b_i| &\le L(n) \text{ for all } i \text{ and}\\ \fint_{B_{4R}(p_i)}\big(\big||\nabla b_i|-1\big|+\|\mathrm{Hess}_{b_i}\|\big)^2 &\le \varepsilon_i \quad \text{for any } R\in[1/4,\rho_i]. \end{aligned}$$

Let X_i be a vector field with compact support with $X_i = \nabla b_i$ on $B_{3\rho_i}(p_i)$, and let ϕ_{it} denote the flow of X_i. Clearly for any t we can find $r_i \to \infty$ such that $\phi_{it|B_{r_i}(p_i)}$ is measure preserving.

Put $\psi_i := \big||\nabla b_i| - 1\big| + \|\mathrm{Hess}(b_i)\|$. We deduce from Lemma 1.4 that

$$\fint_{B_{2R}(p_i)} \mathrm{Mx}(\psi_i)^2 \le C(n,R)\varepsilon_i$$

and Cauchy Schwarz gives

$$\fint_{B_R(p_i)} \mathrm{Mx}(\psi_i) \le \sqrt{C(n,R)\varepsilon_i}.$$

Suppose now that $t_0 \le \frac{R}{4L(n)}$. Then $\phi_t(q) \in B_{\frac{3R}{4}}(p_i)$ for all $q \in B_{R/2}(p_i)$ and all $t \in [-t_0, t_0]$. Combining that ϕ_{it} is measure preserving and $\mathrm{vol}(B_R(p_i)) \le C_3(n,R)\,\mathrm{vol}(B_{R/2}(p_i))$, we get that

$$\begin{aligned}\fint_{B_{R/2}(p_i)} \int_0^{t_0} \mathrm{Mx}(\psi_i)(\phi_{it}(p)) &\le C_3(n,R)t_0 \fint_{B_R(p_i)} \mathrm{Mx}(\psi_i)\\ &\le C_4(t_0,R,n)\sqrt{\varepsilon_i}.\end{aligned}$$

It is now easy to find $R_i \to \infty$ and $\delta_i \to 0$ with

$$\fint_{B_R(p_i)} \int_0^{t_0} \mathrm{Mx}(\psi_i)(\phi_{it}(p))\, dt\, d\mu_i(p) \le \delta_i \text{ for all } R \in [1, R_i].$$

After an adjustment of the sequences $\delta_i \to 0$ and $R_i \to \infty$ we can find $B_{R_i}(p_i)' \subset B_{R_i}(p_i)$ with

$$\begin{aligned}\mathrm{vol}(B_{R_i}(p_i)) - \mathrm{vol}(B_{R_i}(p_i)') &\le \delta_i\,\mathrm{vol}(B_1(q)) \quad \text{for all } q \in B_{R_i}(p_i) \text{ and}\\ \int_0^{t_0} \mathrm{Mx}(\psi_i)(\phi_t(q)) &\le \delta_i \qquad\qquad\quad\; \text{for all } q \in B_{R_i}(p_i)'.\end{aligned}$$

Moreover, the displacement of ϕ_{it_0} is globally bounded by $L(n)t_0$ and thus, Lemma 3.7 implies that ϕ_{it_0} has the zooming in property. To be precise the lemma only implies that $\phi_{it_0} \colon \big[10M_i, p_i\big] \to \big[10M_i, p_i\big]$ has the zooming in property and Lemma 3.2 then allows to scale back down by a factor 10 without losing the zooming in property.

As in the proof of the Product Lemma (2.1) we see that b_i converges to the projection $b_\infty \colon \mathbb{R} \times Y \to \mathbb{R}$. In particular b_∞ is a submetry.

For all $q_i \in B_{R_i}(p_i)'$ the length of the integral curve $c_i(t) = \phi_{it}(q_i)$ ($t \in [0,t_0]$) is bounded above by $t_0+\delta_i$. Moreover, $b_i(c_i(t_0))-b_i(c_i(0))$ is bounded below by $t_0-\delta_i$. This implies that $\phi_{\infty t_0}$ satisfies $d(p, \phi_{\infty t_0}(p)) \le t_0$ and $b_\infty(\phi_{\infty t_0}(p)) - b_\infty(p) \ge t_0$. Clearly, equality must hold and $\phi_{\infty t}(q)$ is a horizontal geodesic with respect to the submetry b_∞. This in turn implies that $\phi_{\infty t_0}$ respects the decomposition $Y \times \mathbb{R}$, it acts by the identity on the first factor and by translation on the second as claimed.

Let $\sigma \colon \tilde{M}_i \to M_i$ denote the universal cover, $\tilde{p}_i \in \tilde{M}_i$ a lift of p_i, and let $\tilde{X}_i$ be a lift of X_i. By Lemma 1.6 all estimates for X_i on $B_R(p_i)$ give similar estimates for $\tilde{X}_i$ on $B_R(\tilde{p}_i)$. Thus the flow $\tilde{\phi}_{it}$ of $\tilde{X}_i$ has the zooming in property as well. $\square$

4. A rough idea of the proof of the Margulis Lemma.

The proof of the Margulis Lemma is somewhat indirect and it might not be easy for everyone to grasp immediately how things interplay. In this section we want to give a rough idea of how the arguments would unwrap in a simple but nontrivial case.

Let $p_i \to \infty$ be a sequence of odd primes and let $\Gamma_i := \mathbb{Z} \ltimes \mathbb{Z}_{p_i}$ be the semidirect product where the homomorphism $\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}_{p_i})$ maps $1 \in \mathbb{Z}$ to the automorphism φ_i given by $\varphi_i(z + p_i\mathbb{Z}) = 2z + p_i\mathbb{Z}$.

Suppose, contrary to the Margulis Lemma, we have a sequence of compact n-manifolds M_i with $\mathrm{Ric} > -1/i$ and $\mathrm{diam}(M_i) = 1$ and fundamental group Γ_i.

A typical problem situation would be that M_i converges to a circle and its universal cover $\tilde{M}_i$ converges to $\mathbb{R}$.

We then replace M_i by $B_i = \tilde{M}_i/\mathbb{Z}_{p_i}$ and in order not to lose information we endow B_i with the deck transformation $f_i \colon B_i \to B_i$ representing a generator of $\Gamma_i/\mathbb{Z}_{p_i}$. Then B_i will converge to $\mathbb{R}$ as well.

It is part of the rescaling theorem that one can find $\lambda_i \to \infty$ such that the rescaled sequence $\lambda_i B_i$ converges to $\mathbb{R} \times K$ with K being compact but not equal to a point. Suppose for illustration that $\lambda_i B_i$ converges to $\mathbb{R} \times S^1$ and $\lambda_i \tilde{M}_i$ converges to $\mathbb{R}^2$ and that the action of $\mathbb{Z}_{p_i}$ converges to a discrete action of $\mathbb{Z}$ on $\mathbb{R}^2$.

The maps $f_i \colon \lambda_i B_i \to \lambda_i B_i$ do not converge, because typically f_i would map a base point x_i to some point $y_i = f_i(x_i)$ with $d(x_i, y_i) = \lambda_i \to \infty$ with respect to the rescaled distance.

But the second statement in the rescaling theorem guarantees that we can find a sequence of diffeomorphisms $g_i \colon [\lambda_i B_i, y_i] \to [\lambda_i B_i, x_i]$ with the zooming in property. The composition $f_{new,i} := g_i \circ f_i \colon [\lambda_i B_i, x_i] \to [\lambda_i B_i, x_i]$ also has the zooming in property and thus subconverges to an isometry of the limit.

Moreover, a lift $\tilde{f}_{new,i} \colon \lambda_i \tilde{M}_i \to \lambda_i \tilde{M}_i$ of $f_{new,i}$ has the zooming in property, too. Since g_i can be chosen isotopic to the identity, the action of $\tilde{f}_{new,i}$ on the deck transformation group $\mathbb{Z}_{p_i} = \pi_1(B_i)$ by conjugation remains unchanged.

On the other hand, the $\mathbb{Z}_{p_i}$-action on $\tilde{M}_i$ converges to a discrete $\mathbb{Z}$-action on $\mathbb{R}^2$ and $\tilde{f}_{new,i}$ converges to an isometry $\tilde{f}_{new,\infty}$ of $\mathbb{R}^2$ normalizing the $\mathbb{Z}$-action. This implies that $\tilde{f}^2_{new,\infty}$ commutes with the $\mathbb{Z}$-action and it is then easy to get a contradiction.

Problem. *We suspect that for any given n and D only finitely many of the groups in the above family of groups should occur as fundamental groups of compact n-manifolds with* $\mathrm{Ric} > -(n-1)$ *and diameter* $\le D$. *Note that this is not even known under the stronger assumption of* $K > -1$ *and* $\mathrm{diam} \le D$.

5. The Rescaling Theorem.

Theorem 5.1 (Rescaling Theorem)**.** *Let (M_i, g_i, p_i) be a sequence of n-manifolds satisfying* $\mathrm{Ric}_{B_{r_i}(p_i)} > -\mu_i$ *and* $\overline{B_{r_i}(p_i)}$ *is compact for some* $r_i \to \infty$, $\mu_i \to 0$. *Suppose that* $(M_i, g_i, p_i) \xrightarrow{G-H} (\mathbb{R}^k, 0)$ *for some* $k < n$. *Then after passing to a subsequence we can find a compact metric space K with* $\mathrm{diam}(K) = 10^{-n^2}$, *a sequence of subsets* $G_1(p_i) \subset B_1(p_i)$ *with* $\frac{\mathrm{vol}(G_1(p_i))}{\mathrm{vol}(B_1(p_i))} \to 1$ *and a sequence* $\lambda_i \to \infty$ *such that the following holds*

- *For all $q_i \in G_1(p_i)$ the isometry type of the limit of any convergent subsequence of $(\lambda_i M_i, q_i)$ is given by the metric product $\mathbb{R}^k \times K$.*
- *For all $a_i, b_i \in G_1(p_i)$ we can find a sequence of diffeomorphisms*

$$f_i \colon [\lambda_i M_i, a_i] \to [\lambda_i M_i, b_i]$$

with the zooming in property such that f_i is isotopic to the identity. Moreover, for any lift $\tilde{a}_i, \tilde{b}_i \in \tilde{M}_i$ of a_i and b_i to the universal cover $\tilde{M}_i$ we can find a lift $\tilde{f}_i$ of f_i such that

$$\tilde{f}_i \colon [\lambda_i \tilde{M}_i, \tilde{a}_i] \to [\lambda_i \tilde{M}_i, \tilde{b}_i]$$

has the zooming in property as well.

Finally, if $\pi_1(M_i, p_i)$ is generated by loops of length $\leq R$ for all i, then we can find $\varepsilon_i \to 0$ such that $\pi_1(M_i, q_i)$ is generated by loops of length $\leq \frac{1+\varepsilon_i}{\lambda_i}$ for all $q_i \in G_1(p_i)$.

Proof. By Theorem 1.3, after passing to a subsequence we can find a harmonic map $b^i \colon B_i(p_i) \to B_{i+1}(0) \subset \mathbb{R}^k$ that gives a $\frac{1}{2^i}$ Gromov–Hausdorff approximation from $B_i(p_i)$ to $B_i(0) \subset \mathbb{R}^k$. Put

$$h_i = \sum_{j,l=1}^{k} | < \nabla b^i_j, \nabla b^i_l > -\delta_{j,l}| + \sum_j \|\mathrm{Hess}_{b^i_j}\|^2.$$

After passing to a subsequence we also have by Theorem 1.3 that

$$⨍_{B_R(p_i)} h_i \leq \varepsilon_i^4 \text{ for all } R \in [1, i] \text{ with } \varepsilon_i \to 0.$$

Furthermore, we may assume $\mathrm{Ric} > -\varepsilon_i$ on $B_i(p_i)$. Put

$$G_4(p_i) := \big\{ x \in B_4(p_i) \mid \mathrm{Mx}_4 h_i(x) \leq \varepsilon_i^2 \big\}. \tag{12}$$

We will call elements of $G_4(p_i)$ *"good points"* in $B_4(p_i)$. We put $G_r(q) = B_r(q) \cap G_4(p_i)$ for $q \in B_2(p_i)$ and $r \leq 2$. It is immediate from the weak 1-1 inequality (Lemma 1.4) that

$$\tfrac{\mathrm{vol}(G_1(p_i))}{\mathrm{vol}(B_1(p_i))} \to 1.$$

By the Product Lemma (2.1), for any $q_i \in G_4(p_i)$ and any $h \geq 1$ we have that $hB_{1/h}(q_i)$ is $\widetilde{\varepsilon}_i$ close to a ball in a product $\mathbb{R}^k \times Y$ where $\widetilde{\varepsilon}_i \to 0$ can be chosen independently of h and q_i. Note that the space Y may depend on h and q_i.

For any point $q_i \in G_1(p_i)$ let $\rho_i(q_i)$ be the largest number $\rho < 1$ such that the map $b^i \colon B_\rho(q_i) \to \mathbb{R}^k$ has distance distortion exactly equal to $\rho \cdot 10^{-n^2}$. For any choice $q_i \in G_1(p_i)$ we know that $\rho_i(q_i) \to 0$ and by the Product Lemma (2.1) $\big(\frac{1}{\rho_i(q_i)} M_i, q_i\big)$ subconverges to $\mathbb{R}^k \times K$ where K is compact with $\mathrm{diam}(K) = 10^{-n^2}$.

Again note, that a priori, the constants $\rho_i(q_i)$ and the space K depend on the choice of $q_i \in G_1(p_i)$. We will show that, in fact, both are essentially independent of the choice of good points in $B_1(p_i)$.

For each i we define the number ρ_i as the supremum of $\rho_i(q_i)$ with $q_i \in G_1(p_i)$ and put

$$\lambda_i = \tfrac{1}{\rho_i}.$$

The statement of the theorem will follow easily from the following sublemma.

Sublemma 5.2. *There exist constants $\bar{C}_1, \bar{C}_2$ independent of i such that the following holds. For any good point $p \in G_1(p_i)$ and any $r \le 2$ there exists $B_r(p)' \subset B_r(p)$ such that*

$$\operatorname{vol} B_r(p)' \ge (1 - \bar{C}_1 r\varepsilon_i) \operatorname{vol} B_r(p) \tag{13}$$

and such that for any $q \in B_r(p)'$ there exists $p' \in B_{L\rho_i}(p)$ (with $L = 9^n$) and a time dependent (piecewise constant in time) divergence free vector field X^t (dependence on q is suppressed) and its integral curve $c_q(t)\colon [0,1] \to B_2(p_i)$ such that $c(0) = p', c(1) = q$ and

$$\tfrac{1}{\operatorname{vol}(B_r(p))} \int_{B_r(p)'} \int_0^1 \big(\mathrm{Mx}(\|\nabla.X^t\|^{3/2})\big)^{2/3}(c_q(t))\, dt\, d\mu_i(q) \le \bar{C}_2 r\varepsilon_i.$$

Proof. Despite the fact that L can be chosen as $L = 9^n$ we treat it for now as a large constant $L \ge 10$ which needs to be determined. Throughout the proof we will denote by C_j various constants depending on n. Sometimes these constants will also depend on L in which case that dependence will always be explicitly stated.

Although we consider a fixed i for the major part of the proof, we take the liberty of assuming that i is large. This way we can ensure that for any $r \ge \rho_i$ and any good point $p \in G_1(p_i)$ the ball $\frac{1}{Lr} B_{2Lr}(p)$ is by the Product Lemma in the measured Gromov–Hausdorff sense arbitrarily close to a ball in a product $\mathbb{R}^k \times K$ where by our choice of ρ_i, K has diameter $\le 10^{-n^2}$. This implies, for example, that we can assume that

$$\tfrac{1}{2}\big(\tfrac{r_1}{r_2}\big)^k \le \tfrac{\operatorname{vol}(B_{r_1}(q_1))}{\operatorname{vol}(B_{r_2}(q_2))} \le 2\big(\tfrac{r_1}{r_2}\big)^k \text{ for all } q_1, q_2 \in B_{Lr}(p),\ r_1, r_2 \in [r/20, 2Lr]. \tag{14}$$

The sublemma is trivially true for $r \le L\rho_i$ with $X \equiv 0$. By induction it suffices to show that if the sublemma holds for some $r \in (\rho_i, \frac{2}{L}]$ then it holds for Lr.

Let $q_1, \ldots, q_l$ be a maximal $r/2$-separated net in $B_{Lr}(p)$. Note that

$$B_{Lr}(p) \subset \bigcup_{m=1}^{l} B_{r/2}(q_m).$$

By a standard volume comparison argument we can deduce from (14) that

$$\frac{\sum_{m=1}^{l} \operatorname{vol} B_{r/2}(q_m)}{\operatorname{vol} B_{Lr}(p)} \le 3^{k+1} \le 3^n. \tag{15}$$

Fix q_m and consider the following vector field

$$X(x) = \sum_{\alpha=1}^{k} (b^i_\alpha(p) - b^i_\alpha(q_m)) \nabla b^i_\alpha(x).$$

Since b^i_α are Lipschitz with a universal Lipschitz constant by (1), X satisfies

$$|X(x)| \le C(n) \cdot Lr. \tag{16}$$

Also, by construction, X satisfies

$$\mathrm{Mx}_4 \|\nabla.X\|^2(p) \le C_4 \varepsilon_i^2 L^2 r^2. \tag{17}$$

Note that we get an extra L^2r^2 factor as compared to (12) because $|b^i(p) - b^i(q_m)| \le C(n)Lr$. By applying (7) we get

$$\mathrm{Mx}\Big([\mathrm{Mx}(\|\nabla.X\|^{3/2})]^{4/3}\Big)(p) \le C(n)\,\mathrm{Mx}_4(\|\nabla.X\|^2)(p) \le (C_5 L\varepsilon_i r)^2.$$

In particular, for $R_1 = 2C(n)Lr$

$$\fint_{B_{R_1}(p)} [\mathrm{Mx}(\|\nabla.X\|^{3/2})]^{4/3}(p) \le (C_5 L\varepsilon_i r)^2$$

provided $R_1 \le 1$. However, in the case of $R_1 \in [1, 4C(n)]$ the same follows more directly from our initial assumptions combined with Lemma 1.4 for all large i. By Cauchy inequality the last estimate gives

$$\fint_{B_{R_1}(p)} \big(\mathrm{Mx}(\|\nabla.X\|^{3/2})\big)^{2/3}(p) \le C_5 L\varepsilon_i r. \tag{18}$$

Consider the measure preserving flow ϕ of X on $[0,1]$. Because of (16) the flow lines $\phi_t(q)$ with $q \in B_r(p)$ stay in the ball $B_{R_1}(p)$ for $t \in [0,1]$. We choose a universal constant $C_6(L)$ independent of i with $C_5 L \cdot \frac{\mathrm{vol}(B_{2C(n)Lr}(p))}{\mathrm{vol}(B_{r/10}(q_m))} \le C_6(L)$.

Combining that the flow is measure preserving, inequality (18) and our choice of C_6 we see that

$$\frac{1}{\mathrm{vol}(B_{\frac{r}{10}}(q_m))} \int_{B_r(q_m)} \int_0^1 \big(\mathrm{Mx}(\|\nabla.X\|^{3/2})\big)^{2/3}(\phi_t(x))\,dt\,d\mu_i(x) \quad \le \quad C_6(L) r\varepsilon_i. \tag{19}$$

By the Product Lemma (2.1) applied to the rescaled balls $\frac{1}{Lr}B_{Lr}(p)$ we know that $\frac{1}{Lr}B_{Lr}(p)$ is measured Gromov–Hausdorff close to a unit ball in $\mathbb{R}^k \times K_3$ where $\mathrm{diam}\, K_3 \le 10^{-n^2}$. Moreover, ϕ_1 is measured close to a translation by $b^i(p) - b^i(q_m)$ in $\mathbb{R}^k$, see proof of Proposition 3.8. Since we only need to argue for large i, we may assume that by volume 3/4 of the points in $B_{r/10}(q_m)$ are mapped by ϕ_1 to points in $B_{r/9}(p)$.

We choose such a point $q \in B_{r/10}(q_m)$. In view of (19) we may assume in addition that

$$\int_0^1 \mathrm{Mx}(\|\nabla.X\|^{3/2})^{2/3}(\phi_t(q))\,dt \le \tfrac{4}{3} C_6(L) r\varepsilon_i.$$

By Lemma 3.7 this implies

$$\fint_{B_r(q)\times B_r(q)} \mathrm{dt}_r(1)(x,y)\,d\mu_i(x)\,d\mu_i(y) \le C_7(L) r \cdot \varepsilon_i. \tag{20}$$

Note that by definition, $\mathrm{dt}_r(1)(x,y) \ge \min\{r, |d(\phi_1(x),\phi_1(y)) - d(x,y)|\}$. Combining this inequality with the knowledge that 3/4 of the points in $B_{r/10}(q_m)$ end up in $B_{r/9}(p)$, implies that we can find a subset $B_{r/2}(q_m)' \subset B_{r/2}(q_m)$ with

$$\mathrm{vol}(B_{r/2}(q_m')) \quad \ge \quad (1 - C_8(L)\varepsilon_i r)\,\mathrm{vol}(B_{r/2}(q_m)) \text{ and} \tag{21}$$

$$\phi_1(B_{r/2}(q_m)') \quad \subset \quad B_r(p). \tag{22}$$

Set $B_{r/2}(q_m)'' := B_{r/2}(q_m)' \cap \phi_1^{-1}(B_r(p)')$. Then we get

$$\phi_1(B_{r/2}(q_m)'') \subset B_r(p)' \quad \text{and}$$

$$
\begin{aligned}
\operatorname{vol}(B_{r/2}(q_m)'') &\geq \operatorname{vol}(B_{r/2}(q_m)') - (\operatorname{vol} B_r(p) - \operatorname{vol} B_r(p)')\\
&\overset{\text{by (21) and (13)}}{\geq} (1 - C_8(L) r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m) - \bar{C}_1 r\varepsilon_i \operatorname{vol} B_r(p)\\
&\overset{\text{by (14)}}{\geq} (1 - C_8(L) r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m) - 2^{k+1}\bar{C}_1 r\varepsilon_i \operatorname{vol} B_{r/2}(q_m)\\
&\geq (1 - (C_8(L) + 2^n \bar{C}_1) r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m), \qquad (23)
\end{aligned}
$$

where we used that ϕ_1 is volume-preserving in the first inequality.

Using the induction assumption, we can prolong each integral curve from any $q \in B_{r/2}(q_m)''$ to a point $x \in B_r(p)'$ by extending it by a previously constructed integral curve from x to $p' \in B_{L\rho_i}(p)$ of a vector field X^t_{old} (which depends on p'). We set $X^t_{new} = 2X^{2t}_{old}$ for $0 \leq t \leq 1/2$ and $X^t_{new} = -2X$ for $1/2 \leq t \leq 1$. Let $c_q(t)$ be the integral curve of X^t_{new} with $c_q(1) = q$ and $c_q(0) = p'$.

By the induction assumption and (19), we get

$$
\begin{aligned}
\int_{B_{r/2}(q_m)''} \int_0^1 (\operatorname{Mx} \|\nabla . X^t_{new}\|^{3/2})^{2/3} (c_q(t))\, dt\, d\mu_i(q) &\leq\\
&\leq C_6(L)(r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m) + \bar{C}_2 (r\varepsilon_i) \operatorname{vol} B_r(p)\\
&\overset{(14)}{\leq} C_6(L)(r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m) + 2^n \bar{C}_2 (r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m)\\
&= \big(C_9(L) + 2^n \bar{C}_2\big)(r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m). \qquad (24)
\end{aligned}
$$

Recall that the balls $B_{r/2}(q_m)$ cover $B_{Lr}(p)$. We put

$$
B_{Lr}(p)' := B_{Lr}(p) \cap \bigcup_m B_{r/2}(q_m)''.
$$

By construction, for every point in $B_{Lr}(p)'$ there exists a vector field X^t whose integral curve connects it to a point in $B_{L\rho_i}(p)$ satisfying (24). For points covered by several sets $B_{r/2}(q_m)''$ we pick any one. Then we have

$$
\begin{aligned}
\int_{B_{Lr}(p)'} \int_0^1 \Big(\operatorname{Mx}\big(\|\nabla . X^t_{new}\|^{\frac{3}{2}}\big)\Big)^{\frac{2}{3}} (c_q(t))\, dt\, d\mu_i(q) &\leq\\
&\leq \sum_{m=1}^{l} (C_9(L) + 2^n \bar{C}_2)(r\varepsilon_i) \operatorname{vol} B_{r/2}(q_m)\\
&\overset{(15)}{\leq} 3^n (C_9(L) + 2^n \bar{C}_2)(r\varepsilon_i) \operatorname{vol} B_{rL}(p)\\
&\leq \bar{C}_2 \cdot Lr\varepsilon_i \cdot \operatorname{vol} B_{rL}(p).
\end{aligned}
$$

The last inequality holds if $L = 9^n \geq 2 \cdot 2^n \cdot 3^n$ and $\bar{C}_2 \geq 3^n \cdot C_9(L)$.

Recall that by (23)

$$
\operatorname{vol} B_{r/2}(q_m) - \operatorname{vol} B_{r/2}(q_m)'' \geq (C_8(L) + 2^n \bar{C}_1) r\varepsilon_i \operatorname{vol} B_{r/2}(q_m)
$$

for any m and thus

$$
\begin{aligned}
\operatorname{vol} B_{Lr}(p) - \operatorname{vol} B_{Lr}(p)' &\leq \sum_{m=1}^{l} (C_8(L) + 2^n \bar{C}_1) r\varepsilon_i \operatorname{vol} B_{r/2}(q_m)\\
&\overset{(15)}{\leq} 3^n (C_8(L) + 2^n \bar{C}_1) r\varepsilon_i \operatorname{vol}(B_{Lr}(p))\\
&\leq \bar{C}_1 Lr\varepsilon_i \operatorname{vol} B_{Lr}(p)
\end{aligned}
$$

provided that $L = 9^n \geq 2 \cdot 6^n$ and $\bar{C}_1 \geq 3^n C_8(L)$.

This finishes the proof of Sublemma 5.2. $\square$

Observe that for any two good points $x_i, y_i \in G_1(p_i)$ we can deduce from Sublemma 5.2 that $\mathrm{vol}(B_2(x_i)' \cap B_2(y_i)') \geq \mathrm{vol}(B_{1/2}(p_i))$ for all large i.

Then it follows, also by Sublemma 5.2, that there is $q \in B_2(x_i)' \cap B_2(y_i)'$, $x_i' \in B_{L\rho_i}(x_i)$ and $y_i' \in B_{L\rho_i}(y_i)$ such that for the integral curve c connecting x_i' with q on the first half of the interval and q with y_i' on the second half we have for the corresponding time dependent vector field X_i^t

$$\int_0^1 \left(\mathrm{Mx}(\|\nabla . X_i^t\|^{3/2})\right)^{2/3}(c(t))\, dt \leq \bar{C}_3 \varepsilon_i$$

with a constant $\bar{C}_3$ independent of i.

Let $f_i := \phi_{i1}$ be the flow of X_i^t evaluated at time 1. Since $\lambda_i = \frac{1}{\rho_i} \to \infty$, we can employ Proposition 3.6 to see that

$$f_i \colon [\lambda_i M_i, x_i] \to [\lambda_i M_i, y_i]$$

has the zooming in property. Thus, the Gromov–Hausdorff limits of the two sequences are isometric.

For any lifts $\tilde{x}_i$ and $\tilde{y}_i$ of x_i and y_i to the universal cover we can find lifts $\tilde{x}_i'$ and $\tilde{y}_i'$ of x_i' and y_i' in the $L\rho_i$ neighborhoods of $\tilde{x}_i$ and $\tilde{y}_i$. Let $\tilde{f}_i \colon \tilde{M}_i \to \tilde{M}_i$ be the lift f_i with $\tilde{f}_i(\tilde{x}_i') = \tilde{y}_i'$. Proposition 3.6 ensures that $\tilde{f}_i \colon [\lambda_i \tilde{M}_i, \tilde{x}_i] \to [\lambda_i \tilde{M}_i, \tilde{y}_i]$ has the zooming in property as well.

It remains to check the last part of the Rescaling Theorem concerning the fundamental group. Assume that $\pi_1(M_i, p_i)$ is generated by loops of length $\leq R$. For every good point $q \in B_1(p_i)$ let $r_i(q)$ denote the minimal number such that $\pi_1(M_i, q)$ can be generated by loops of length $\leq r_i(q)$. Let r_i denote the supremum of $r_i(q)$ over all good points q and choose a good point q_i with $r_i(q_i) \geq \frac{1}{2} r_i$. It suffices to check that $\limsup_{i\to\infty} \lambda_i r_i \leq 1$. Suppose we can find a subsequence with $\lambda_i r_i \geq 1/4$ for all i. By the Product Lemma (2.1), the sequence $(\frac{1}{r_i} M_i, q_i)$ subconverges to $(\mathbb{R}^k \times K', q_\infty)$ with $\mathrm{diam}(K') \leq 4 \cdot 10^{-n^2}$. Combining Lemma 2.2 with Lemma 2.3 we see that $\pi_1(\frac{1}{r_i} M_i, q_i)$ can be generated by loops of length $\leq 2\,\mathrm{diam}(K') + c_i$ with $c_i \to 0$ – a contradiction. $\square$

6. The Induction Theorem for C-Nilpotency

C-nilpotency of fundamental groups of manifolds with almost nonnegative Ricci curvature (Corollary 2) will follow from the following technical result.

Theorem 6.1 (Induction Theorem)**.** *Suppose (M_i^n, p_i) is a sequence of pointed n-dimensional Riemannian manifolds (not necessarily complete) satisfying*

(1) $\mathrm{Ric}_{M_i} \geq -1/i$;
(2) $\overline{B_i(p_i)}$ is compact for any i;
(3) There is some $R > 0$ such that $\pi_1(B_R(p_i)) \to \pi_1(M_i)$ is surjective for all i;
(4) $(M_i, p_i) \xrightarrow{G-H} (\mathbb{R}^k \times K, (0, p_\infty))$ where K is compact.

Suppose in addition that we have k sequences $f_i^j \colon [\tilde{M}_i, \tilde{p}_i] \to [\tilde{M}_i, \tilde{p}_i]$ of diffeomorphisms of the universal covers $\tilde{M}_i$ of M_i which have the zooming in property, where $\tilde{p}_i$ is a lift of p_i, and which normalize the deck transformation group acting on $\tilde{M}_i$, $j = 1, \ldots k$.

Then there exists a positive integer C such that for all sufficiently large i, $\pi_1(M_i)$ contains a nilpotent subgroup $\mathsf{N} \triangleleft \pi_1(M_i)$ of index at most C such that N has an $(f_i^j)^{C!}$-invariant ($j = 1, \ldots, k$) cyclic nilpotent chain of length $\le n - k$, that is:

We can find $\{e\} = \mathsf{N}_0 \triangleleft \cdots \triangleleft \mathsf{N}_{n-k} = \mathsf{N}$ such that $[\mathsf{N}, \mathsf{N}_h] \subset \mathsf{N}_{h-1}$ and each factor group $\mathsf{N}_{h+1}/\mathsf{N}_h$ is cyclic. Furthermore, each N_h is invariant under the action of $(f_i^j)^{C!}$ by conjugation and the induced automorphism of $\mathsf{N}_h/\mathsf{N}_{h+1}$ is the identity.

Although we will have $f_i^j = \mathrm{id}$ in the applications, it is crucial for the proof of the theorem by induction to establish it in the stated generality.

Proof. The proof proceeds by reverse induction on k. For $k = n$ the result follows immediately from Theorem 1.2. We assume that the statement holds for all $k' > k$ and we plan to prove it for k.

We argue by contradiction. After passing to a subsequence we can assume that any subgroup $\mathsf{N} \subset \pi_1(M_i)$ of index $\le i$ does not have a nilpotent cyclic chain of length $\le n - k$ which is invariant under $(f_i^j)^{i!}$ ($j = 1, \ldots, k$).

After passing to a subsequence, $(\tilde{M}_i, \tilde{p}_i)$ converges to $(\mathbb{R}^{\bar{k}} \times \widetilde{K}, (0, \tilde{p}_\infty))$, where $\widetilde{K}$ contains no lines. Moreover, we can assume that the action of $\pi_1(M_i)$ converges with respect to the pointed equivariant Gromov–Hausdorff topology to an isometric action of a closed subgroup $\mathsf{G} \subset \mathrm{Iso}(\mathbb{R}^{\bar{k}} \times \widetilde{K})$.

It is clear that the metric quotient $(\mathbb{R}^{\bar{k}} \times \widetilde{K})/\mathsf{G}$ is isometric to $\mathbb{R}^k \times K$. Using that lines in $\mathbb{R}^k \times K$ can be lifted to lines in $\mathbb{R}^{\bar{k}} \times \widetilde{K}$, we deduce that $\mathbb{R}^{\bar{k}} = \mathbb{R}^k \times \mathbb{R}^l$ and the action of G on the first Euclidean factor is trivial, (cf. proof of Lemma 2.3). Since the action of G on $\mathbb{R}^l \times \widetilde{K}$ and hence on $\widetilde{K}$ is cocompact, we can use an observation of Cheeger and Gromoll [CG72] to deduce that $\widetilde{K}$ is compact for there are no lines in $\widetilde{K}$.

By passing once more to a subsequence, we can assume that f_i^j converges in the weakly measured sense to an isometry f_∞^j on the limit space, $j = 1, \ldots, k$, see Lemma 3.2. The overall most difficult step is essentially a consequence of the Rescaling Theorem:

Step 1. Without loss of generality we can assume that K is not a point.

We assume $K = pt$. The strategy is to find a new contradicting sequence converging to $\mathbb{R}^k \times K'$ where $K' \neq pt$.

After rescaling every manifold down by a fixed factor we may assume that the limit isometry f_∞^j displaces $(0, p_\infty)$ by less than $1/100$, $j = 1, \ldots, k$. We choose the set of "good" points $G_1(p_i)$ as in Rescaling Theorem 5.1. We let $G_1(\tilde{p}_i)$ denote those points in $B_1(\tilde{p}_i)$ projecting to $G_1(p_i)$. By Lemma 1.6, $\frac{\mathrm{vol}(G_1(\tilde{p}_i))}{\mathrm{vol}(B_1(\tilde{p}_i))} \to 1$.

By Lemma 3.5, we can remove small subsets $H_i \subset G_1(p_i)$ (and the corresponding subsets from $G_1(\tilde{p}_i)$) such that $\frac{\mathrm{vol}\, H_i}{\mathrm{vol}\, G_1(p_i)} \le \delta_i \to 0$ and for any choice of points $\tilde{q}_i \in G_1(\tilde{p}_i) \backslash H_i$ the sequence f_i^j and $(f_i^j)^{-1}$ ($i \in \mathbb{N}$) is good on all scales at $\tilde{q}_i$, $j = 1, \ldots, k$. To simplify notations we will assume that it is already true for all choices of $\tilde{q}_i \in G_1(\tilde{p}_i)$.

For all large i we can choose a point $\tilde{q}_i \in G_{1/2}(\tilde{p}_i)$ with $f_i^j(\tilde{q}_i) \in G_1(\tilde{p}_i)$, $j = 1, \ldots, k$. Let q_i be the image of $\tilde{q}_i$ in M_i and $\bar{f}_i^j \colon M_i \to M_i$ be the induced diffeomorphism.

By the Rescaling Theorem (5.1), there is a sequence $\lambda_i \to \infty$ and a sequence of diffeomorphisms g_i^j of M_i which are isotopic to identity such that

$$g_i^j \colon [\lambda_i M_i, \bar{f}_i^j(q_i)] \to [\lambda_i M_i, q_i]$$

has the zooming in property and we can find lifts $\tilde{g}_i^j$ of g_i^j such that

$$\tilde{g}_i^j \colon [\lambda_i \tilde{M}_i, f_i^j(\tilde{q}_i)] \to [\lambda_i \tilde{M}_i, \tilde{q}_i]$$

has the zooming in property. Using Lemma 3.4 we see that

$$f_{i,new}^j := f_i^j \circ \tilde{g}_i^j \colon [\lambda_i \tilde{M}_i, \tilde{q}_i] \to [\lambda_i \tilde{M}_i, \tilde{q}_i]$$

has the zooming in property as well for $j = 1, \dots, k$. Since g_i^j is isotopic to the identity, it follows that conjugation by $\tilde{g}_i^j$ induces an inner automorphism of $\pi_1(M_i)$. Therefore $f_{i,new}^j$ produces the same element $\alpha_i^j \in \mathrm{Out}(\pi_1(M_i))$ as f_i^j.

The Rescaling Theorem also ensures that $\pi_1(\lambda_i M_i, q_i)$ remains boundedly generated. Finally, it states that $(\lambda_i M_i, q_i)$ converges to $(\mathbb{R}^k \times K, (0, q_\infty))$ with K being a compact space with $\mathrm{diam}(K) = 10^{-n^2}$.

Thus, $(\lambda_i M_i, q_i)$ and the maps $f_{i,new}^j$ on the universal covers give a new contradicting sequence with the limit satisfying $K \neq pt$.

From now on we will assume that it is true for the original contradicting sequence.

Step 2. Without loss of generality we can assume that f_i^j converges in the weakly measured sense to the identity map of the limit space $\mathbb{R}^k \times \mathbb{R}^l \times \widetilde{K}$, $j = 1, \dots, k$.

We prove this by finite induction on j. Suppose we already found a contradicting sequence where $f_i^1, \dots, f_i^{j-1}$ converge to the identity. We have to construct one where in addition $f_i := f_i^j$ converges to the identity.

We first consider the induced diffeomorphisms $\bar{f}_i \colon M_i \to M_i$ which converge to an isometry $\bar{f}_\infty$ of $\mathbb{R}^k \times K$ in the weakly measured sense. Thus, there is $A \in \mathrm{O}(k)$ and $v \in \mathbb{R}^k$ such that the induced isometry of the Euclidean factor $\mathrm{pr}(\bar{f}_\infty) \colon \mathbb{R}^k \to \mathbb{R}^k$ is given by $(w \mapsto Aw + v)$.

We claim that we can assume without loss of generality that $v = 0$. In fact, by Proposition 3.8, we can find a sequence of diffeomorphisms $g_i \colon M_i \to M_i$ with the zooming in property such that g_i converges to an isometry g_∞ which induces translation by $-v$ on the Euclidean factor. Moreover, there is a lift $\tilde{g}_i \colon \tilde{M}_i \to \tilde{M}_i$ which also has the zooming in property if we endow $\tilde{M}_i$ with the base point $\tilde{p}_i$. By Lemma 3.4, we are free to replace f_i by $\tilde{g}_i \circ f_i$ and hence $v = 0$ without loss of generality.

Since $\bar{f}_\infty$ fixes the origin of the Euclidean factor, we obtain that for the limit f_∞ of f_i the following property holds. Given any $m > 0$ we can find $g_m \in \mathsf{G}$ such that $d(f_\infty^m(g_m(0, \tilde{p}_\infty)), (0, \tilde{p}_\infty)) \leq \mathrm{diam}(K)$ in $\mathbb{R}^k \times \mathbb{R}^l \times \widetilde{K}$. It is now an easy exercise to find a sequence ν_b of natural numbers and a sequence of $g_b \in \mathsf{G}$ for which $f_\infty^{\nu_b} \circ g_b$ converges to the identity:

We consider a finite ε-dense set $\{a_1, \dots, a_N\}$ in the ball of radius $\frac{1}{\varepsilon}$ around $(0, \tilde{p}_\infty) \in \mathbb{R}^{k+l} \times \widetilde{K}$. For each integer m choose a $g_m \in \mathsf{G}$ such that

$$d\big(f_\infty^m\big(g_m(0, \tilde{p}_\infty)\big), (0, \tilde{p}_\infty)\big) \leq \mathrm{diam}(K).$$

The elements $(f_\infty^m(g_m a_1), \cdots, f_\infty^m(g_m a_N))$ are contained in $B_{1/\varepsilon + \mathrm{diam}(K)}(0, \tilde{p}_\infty)$. Thus, there are $m_1 \neq m_2$ such that $d(f_\infty^{m_1}(g_{m_1} a_j), f_\infty^{m_2}(g_{m_2} a_j)) < \varepsilon$ for $j =$

$1,\dots,N$. Consequently, $d(f_\infty^{m_1-m_2}(ga_j),a_j)<\varepsilon$ for $j=1,\dots,N$ with

$$g=f_\infty^{m_2-m_1}g_{m_1}^{-1}f_\infty^{m_1-m_2}g_{m_2}\in\mathsf{G}.$$

In summary, $f_\infty^{m_1-m_2}\circ g$ displaces any point in the ball of radius $\frac{1}{\varepsilon}$ around $(0,p_\infty)$ by at most 4ε. Since ε was arbitrary, this clearly proves our claim.

Let (ν_b,g_b) be as above. For each b we choose a large $i=i(b)\geq 2\nu_b$ and $g_b\in\pi_1(M_i)$ such that $f_i^{\nu_b}\circ g_b$ is in the weakly measured sense close to $f_\infty^{\nu_b}\circ g_b$. We also choose i so large that $(f_{i(b)}^{\nu_b}\circ g_b)_{b\in\mathbb{N}}$ still has the zooming in property and converges to the identity. This finishes the proof of Step 2.

Further Notations. We may replace p_i with any other point $p\in B_{1/2}(p_i)$. Thus we may assume that p_∞ is a regular point in K. Let $\pi_1(M_i,p_i,r)=\pi_1(M_i,p_i)(r)$ denote the subgroup of $\pi_1(M,p_i)$ generated by loops of length $\leq r$. Similarly, recall that $\mathsf{G}(r)$ is the group generated by elements which displace $(0,\tilde p_\infty)$ by at most r. By the Gap Lemma (2.4) there is an $\varepsilon>0$ and $\varepsilon_i\to 0$ such that $\pi_1(M_i,p_i,\varepsilon)=\pi_1(M_i,p_i,\varepsilon_i)$. Moreover, $\mathsf{G}(r)=\mathsf{G}(\varepsilon)$ for all $r\in(0,2\varepsilon]$. We put $\Gamma_i:=\pi_1(M_i,p_i)$ and $\Gamma_{i\varepsilon}=\pi_1(M_i,p_i,\varepsilon)$.

Step 3. The desired contradiction arises if the index $[\Gamma_i:\Gamma_{i\varepsilon}]$ is bounded by some constant C for all large i.

If C denotes a bound on the index and $d=C!$, then $(f_i^j)^d$ leaves the subgroup $\Gamma_{i\varepsilon}$ invariant and we can find a new contradicting sequence for the manifolds $\tilde M_i/\Gamma_{i\varepsilon}$. Thus we may assume that $\Gamma_i=\Gamma_{i\varepsilon}$.

As we have observed above, $\pi_1(M_i,p_i)=\pi_1(M_i,p_i,\varepsilon_i)$ for some $\varepsilon_i\to 0$. We now choose a sequence $\lambda_i\to\infty$ very slowly such that

- $(\lambda_iM_i,p_i)\xrightarrow{G-H}(\mathbb{R}^k\times C_{p_\infty}K,o)=(\mathbb{R}^{k'},0)$ with $k'>k$.
- $\pi_1(\lambda_iM_i,p_i)$ is generated by loops of length ≤ 1.
- $f_i^j\colon\lambda_i\tilde M_i\to\lambda_i\tilde M_i$ still has the zooming in property and still converges to the identity of the limit space.

We put $f_i^{k+1}=\dots=f_i^{k'}=\mathrm{id}$ and obtain a new sequence contradicting our induction assumption.

Step 4. There is $H>0$ and a sequence of uniformly open subgroups (see Definition 1.7) $\Upsilon_i\subset\Gamma_{i\varepsilon}$ such that the index $[\Gamma_{i\varepsilon}:\Upsilon_i]\leq H$ and such that Υ_i is normalized by a subgroup of Γ_i with index $\leq H$ for all large i.

After passing to a subsequence we may assume that $\Gamma_{i\varepsilon}$ converges to the limit action of a closed subgroup $\mathsf{G}_\varepsilon\subset\mathsf{G}$. Clearly G_ε contains the open subgroup $\mathsf{G}(\varepsilon)\subset\mathsf{G}$. Since the kernel of pr is compact (as a closed subgroup of $\mathrm{Iso}(\widetilde K)$), the images $\mathrm{pr}(\mathsf{G})$ and $\mathrm{pr}(\mathsf{G}_\varepsilon)$ are closed subgroups. Moreover, $\mathrm{pr}(\mathsf{G}_\varepsilon)$ is an open subgroup of $\mathrm{pr}(\mathsf{G})$ since it contains $\mathrm{pr}(\mathsf{G}(\varepsilon))\supset\mathrm{pr}(\mathsf{G})_0$. Using Fukaya and Yamaguchi [FY92, Theorem 4.1] or that the component group of $\mathrm{pr}(\mathsf{G})$ is the fundamental group of the nonnegatively curved manifold $\mathrm{pr}(\mathsf{G})\backslash\mathrm{Iso}(\mathbb{R}^l)$, we see that $\mathrm{pr}(\mathsf{G})/\mathrm{pr}(\mathsf{G})_0$ is virtually abelian.

Choose a subgroup $\mathsf{G}'\lhd\mathsf{G}$ of finite index such that $\mathrm{pr}(\mathsf{G}')/\mathrm{pr}(\mathsf{G})_0$ is free abelian and $\mathsf{G}'=\mathrm{pr}^{-1}(\mathrm{pr}(\mathsf{G}'))$. Let $\mathsf{G}'_\varepsilon=\mathsf{G}'\cap\mathsf{G}_\varepsilon$. Then $\mathrm{pr}(\mathsf{G})_0\subset\mathrm{pr}(\mathsf{G}'_\varepsilon)\subset\mathrm{pr}(\mathsf{G}')$. Moreover, $\mathrm{pr}(\mathsf{G}'_\varepsilon)$ is normal in $\mathrm{pr}(\mathsf{G}')$ and $\mathrm{pr}(\mathsf{G}')/\mathrm{pr}(\mathsf{G}'_\varepsilon)$ is a finitely generated abelian group.

The inverse image $\hat{\mathsf{G}}_\varepsilon := \mathrm{pr}^{-1}(\mathrm{pr}(\mathsf{G}_\varepsilon)) \cap \mathsf{G}'$ is a normal subgroup of G' with the same quotient, i.e. $\mathsf{G}'/\hat{\mathsf{G}}_\varepsilon = \mathrm{pr}(\mathsf{G}')/\mathrm{pr}(\mathsf{G}'_\varepsilon)$. Since the kernel of pr is compact, the open subgroup G'_ε is cocompact in $\hat{\mathsf{G}}_\varepsilon$ and thus its index $[\hat{\mathsf{G}}_\varepsilon : \mathsf{G}'_\varepsilon]$ is finite.

In particular, we can say that there is some integer $H_1 > 0$ and a subgroup G' with $[\mathsf{G} : \mathsf{G}'] \le H_1$ such that $[\mathsf{G}_\varepsilon : (g\mathsf{G}_\varepsilon g^{-1} \cap \mathsf{G}_\varepsilon)] < H_1$ for all $g \in \mathsf{G}'$. We want to check that there are only finitely many possibilities for $g\mathsf{G}_\varepsilon g^{-1} \cap \mathsf{G}_\varepsilon$ where $g \in \mathsf{G}'$.

Let $g \in \mathsf{G}'$. Notice that $\Gamma_{i\varepsilon}$ is a uniformly open sequence of subgroups. Choose $g_i \in \Gamma_i$ with $g_i \to g$. By Lemma 1.8, $\Upsilon_i := g_i\Gamma_{i\varepsilon}g_i^{-1} \cap \Gamma_{i\varepsilon}$ converges to $g\mathsf{G}_\varepsilon g^{-1} \cap \mathsf{G}_\varepsilon$ and the index of Υ_i in $\Gamma_{i\varepsilon}$ is $\le H_1$ for all large i. By Theorem 2.5, the number of generators of $\Gamma_{i\varepsilon}$ is bounded by some a priori constant. Therefore, $\Gamma_{i\varepsilon}$ contains at most $h_0 = h_0(n, H_1)$ subgroups of index $\le H_1$.

Combining these statements we see that

$$\{g\mathsf{G}_\varepsilon g^{-1} \cap \mathsf{G}_\varepsilon \mid g \in \mathsf{G}'\} = \{g_h\mathsf{G}_\varepsilon g_h^{-1} \cap \mathsf{G}_\varepsilon \mid h = 1, \dots, h_0\}$$

for suitably chosen elements $g_1, \dots, g_{h_0} \in \mathsf{G}'$. We choose $g_{hi} \in \Gamma_i$ converging to $g_h \in \mathsf{G}$. By Lemma 1.8, the sequence of subgroups

$$\Upsilon_i := \bigcap_{h=1}^{h_0} g_{hi}\Gamma_{i\varepsilon}g_{hi}^{-1}$$

is uniformly open and converges to

$$\Upsilon_\infty := \bigcap_{i=1}^{h_0} g_h\mathsf{G}_\varepsilon g_h^{-1} = \bigcap_{g\in\mathsf{G}'} g\mathsf{G}_\varepsilon g^{-1}.$$

Clearly, G' normalizes Υ_∞. It is not hard to see that we can find elements $c_i^1, \dots, c_i^\tau \in \Gamma_i$ that generate a subgroup of index $\le [\mathsf{G} : \mathsf{G}']$ such that each c_i^j converges to an element $c_\infty^j \in \mathsf{G}'$. Since $c_i^j\Upsilon_i(c_i^j)^{-1}$ is uniformly open and converges to $c_\infty^j\Upsilon_\infty(c_\infty^j)^{-1} = \Upsilon_\infty$, one can now apply Lemma 1.8 c) to see that c_i^j normalizes Υ_i for large i.

Thus, the normalizer of Υ_i has finite index $\le [\mathsf{G} : \mathsf{G}']$ in Γ_i for all large i and Step 4 is established.

Step 5. The desired contradiction arises if, after passing to a subsequence, the indices $[\Gamma_i : \Gamma_{i\varepsilon}] \in \mathbb{N} \cup \{\infty\}$ converge to ∞.

By Step 4 there is a subgroup $\Upsilon_i \subset \Gamma_{i\varepsilon}$ of index $\le H$ which is normalized by a subgroup of Γ_i of index $\le H$. Similarly to the beginning of the proof of Step 3, one can reduce the situation to the case of $\Upsilon_i \lhd \Gamma_i$.

Moreover, $\Upsilon_i \lhd \Gamma_i$ is uniformly open and hence the action of Γ_i/Υ_i on M_i/Υ_i is uniformly discrete and converges to a properly discontinuous action of the virtually abelian group $\mathsf{G}/\Upsilon_\infty$ on the space $\mathbb{R}^k \times (\mathbb{R}^l \times \widetilde{K})/\Upsilon_\infty$. It is now easy to see that $\Gamma_{i\varepsilon}/\Upsilon_i$ contains an abelian subgroup of controlled finite index for large i.

After replacing Γ_i once more by a subgroup of controlled finite index we may assume that Γ_i/Υ_i is abelian. This also shows that without loss of generality $\Upsilon_i = \Gamma_{i\varepsilon}$.

Recall that by Step 2 we have assumed that f_i^j converges to the identity in the weakly measured sense for every $j = 1, \dots, k$. Therefore, the same is true for the commutator $[f_i^j, \gamma_i] \in \Gamma_i$ if $\gamma_i \in \Gamma_i$ has bounded displacement. This in turn implies that $[f_i^j, \gamma_i] \in \Gamma_{i\varepsilon}$ for all large i and $j = 1, \dots, k$.

By Theorem 2.5, we can find for some large σ elements $d_1^i, \dots, d_\sigma^i \in \Gamma_i$ with bounded displacement generating Γ_i. We can also assume that the first τ elements generate $\Gamma_{i\varepsilon}$ for some $\tau < \sigma$.

Let $\hat{\Gamma}_i := \langle d_1^i, \dots, d_{\sigma-1}^i \rangle$. Note that $\hat{\Gamma}_i \lhd \Gamma_i$ since $\hat{\Gamma}_i$ contains $\Gamma_{i\varepsilon} = \Upsilon_i \lhd \Gamma_i$ and $\Gamma_i / \Gamma_{i\varepsilon}$ is abelian.

If for some subsequence and some integer H the group $\hat{\Gamma}_i$ has index $\leq H$ in Γ_i then we may replace Γ_i by $\hat{\Gamma}_i$. Thus, without loss of generality, the order of the cyclic group $\Gamma_i / \hat{\Gamma}_i$ tends to infinity. The element $f_i^{k+1} := d_\sigma^i \in \Gamma_i$ represents a generator of this factor group which has bounded displacement.

We have seen above that f_i^j normalizes $\hat{\Gamma}_i$ and $[f_i^{k+1}, f_i^j] \in \Gamma_{i\varepsilon} \subset \hat{\Gamma}_i$, $j = 1, \dots, k$. Clearly f_i^{k+1}, being an isometry with bounded displacement, has the zooming in property.

We replace M_i by $B_i := \tilde{M}_i / \hat{\Gamma}_i$. Notice that $(B_i, \hat{p}_i)$ converges to $(\mathbb{R}^k \times (\mathbb{R}^l \times \widetilde{K}) / \hat{\mathsf{G}}, \hat{p}_\infty)$ where $\hat{\mathsf{G}}$ is the limit group of $\hat{\Gamma}_i$ and $\hat{p}_i$ is the image of $\tilde{p}_i$ under the projection $\tilde{M}_i \to B_i$. Since $\mathsf{G}/\hat{\mathsf{G}}$ is a noncompact group acting co-compactly on $(\mathbb{R}^l \times \widetilde{K})/\hat{\mathsf{G}}$, we see that $(\mathbb{R}^l \times \widetilde{K})/\hat{\mathsf{G}}$ splits as $\mathbb{R}^p \times K'$ with K' compact and $p > 0$. If $p > 1$, we put $f_i^j = \mathrm{id}$ for $j = k+2, \dots, k+p$.

Thus the induction assumption applies: There is a positive integer C such that the following holds for all large i:

We can find a subgroup N^i of $\pi_1(B_i)$ of index $\leq C$ and a nilpotent chain of normal subgroups $\{e\} = \mathsf{N}_0 \lhd \ldots \lhd \mathsf{N}_{n-k-1} = \mathsf{N}^i$ such that the quotients are cyclic. Moreover, the groups are invariant under the automorphism induced by conjugation of $(f_i^j)^{C!}$ and the induced automorphism of $\mathsf{N}_{h+1}/\mathsf{N}_h$ is the identity, $j = 1, \dots, k+1$.

We put $d = C!$, and consider the group $\bar{\mathsf{N}}^i \subset \pi_1(M_i)$ generated by $(f_i^{k+1})^d$ and N^i. Clearly, the index of $\bar{\mathsf{N}}^i$ in $\pi_1(M_i)$ is bounded by $C \cdot d$.

Moreover, the chain $\{e\} = \mathsf{N}_0 \lhd \ldots \lhd \mathsf{N}_{n-k-1} \lhd \mathsf{N}_{n-k} := \bar{\mathsf{N}}^i$ is normalized by the elements $(f_i^j)^d$ $(j = 1, \dots, k)$ and the action on the cyclic quotients is trivial.

In other words, the sequence fulfills the conclusion of our theorem − a contradiction. □

Remark 6.2. If one was only interested in proving that the fundamental group contains a polycyclic subgroup of controlled index, the proof would simplify considerably. First of all, one would not need any diffeomorphisms f_i^j to make the induction work. In the proof of Step 1 the use of the rescaling theorem could be replaced with the more elementary Product Lemma 2.1. Step 2 would become unnecessary. The core of the remaining arguments would be the same although they would simplify somewhat.

7. Margulis Lemma

Proof of Theorem 1. Let $B_1(p)$ be a metric ball in complete n-manifold with $\mathrm{Ric} > -(n-1)$ on $B_1(p)$, $\tilde{N}$ the universal cover of $B_1(p)$ and let $\tilde{p} \in \tilde{N}$ be a lift of p.

Step 1. There are universal positive constants $\varepsilon_1(n)$ and $C_1(n)$ such that the group

$$\Gamma := \Big\langle \big\{ g \in \pi_1(B_1(p), p) \;\big|\; d(q, gq) \leq \varepsilon_1(n) \text{ for all } q \in B_{1/2}(\tilde{p}) \big\} \Big\rangle$$

has a subgroup of index $\leq C_1(n)$ which has a nilpotent basis of length at most n.

Put $N = \tilde{N}/\Gamma$ and let $\hat{p}$ denote the image of $\tilde{p}$. By the definition of Γ, for each point $q \in B_{1/2}(\hat{p})$, the fundamental group $\pi_1(N, q)$ is generated by loops of length $\leq \varepsilon_1(n)$. Moreover, $\overline{B_{3/4}(\hat{p})}$ is compact.

Assume, on the contrary, that the statement is false. Then we can find a sequence of pointed n-dimensional manifolds (N_i, p_i) satisfying

- $\mathrm{Ric}_{N_i} \geq -(n-1)$,
- $\overline{B_{3/4}(p_i)}$ is compact,
- for each point $q \in B_{1/2}(p_i)$ the fundamental group $\pi_1(N_i, q)$ is generated by loops of length $\leq 2^{-i}$, and
- $\pi_1(N_i, p_i)$ does not contain a subgroup of index $\leq 2^i$ which has a nilpotent basis of length $\leq n$.

After passing to a subsequence we can assume that (N_i, p_i) converges to (X, p_∞). We choose $q_i \in B_{1/4}(p_i)$ such that q_i converges to a regular point $q \in X$.

Choose $\lambda_i \to \infty$ very slowly such that $\pi_1(\lambda_i N_i, q_i)$ is still generated by loops of length ≤ 1 and such that $(\lambda_i N_i, q_i)$ converges to $C_q X \cong \mathbb{R}^k$ for some $k \geq 0$.

Notice that the rescaled sequence $M_i = \lambda_i N_i$ satisfies $\mathrm{Ric}_{M_i} \geq -\frac{n-1}{\lambda_i^2} \to 0$ and $\overline{B_{r_i}(q_i)}$ is compact with $r_i = \frac{\lambda_i}{2} \to \infty$. Thus the existence of this sequence contradicts the Induction Theorem 6.1 with $f_i^1 = \cdots = f_i^k = \mathrm{id}$.

We can now finish the proof of the theorem by establishing.

Step 2. Consider $\varepsilon_1(n) > 0$ and Γ from Step 1. Then there are $\varepsilon_2(n), C_2(n) > 0$ such that the group

$$\mathsf{H} := \Big\langle \big\{ g \in \pi_1(B_1(p), p) \mid d(\tilde{p}, g\tilde{p}) \leq \varepsilon_2(n) \big\} \Big\rangle$$

satisfies: $\Gamma \cap \mathsf{H}$ has index at most $C_2(n)$ in H.

We will provide (in principle) effective bounds on ε_2 and C_2 depending on n and the (ineffective) bound $\varepsilon_1(n)$. Put

$$\Gamma' := \Big\langle \big\{ g \in \pi_1(B_1(p), p) \mid d(q, gq) \leq \varepsilon_1(n) \text{ for all } q \in B_{2/3}(\tilde{p}) \big\} \Big\rangle \subset \Gamma.$$

By Theorem 2.5, there exists $h = h(n)$ such that H can be generated by some $b_1, \ldots, b_h \in \mathsf{H}$ satisfying $d(\tilde{p}, b_i\tilde{p}) \leq 4\varepsilon_2(n)$ for any $i = 1, \ldots h$. We can obviously assume that the generating set $\{b_1, \ldots, b_h\}$ contains inverses of all its elements. We proceed in three substeps.

Claim 1. There is a positive integer $L(n, \varepsilon_1(n))$ such that for all $\varepsilon_2(n) < \frac{1}{100L}$ and any choice of $g_i \in \{b_1, \ldots, b_h\}$, $i = 1, \ldots, L$ we can find $l, k \in \{1, \ldots, L\}$ with $l \leq k$ and $g_l \cdot g_{l+1} \cdots g_k \in \Gamma'$.

We assume $\varepsilon_2(n) < \frac{1}{100L}$ and we will show that there is an a priori estimate for L. Notice that $d(\tilde{p}, g_1 \cdots g_l \tilde{p}) < 4\varepsilon_2(n) l \leq \frac{1}{25}$ holds for $l = 1, \ldots, L$. Thus $g_1 \cdots g_l$ maps the ball $B_{2/3}(\tilde{p})$ into the ball $B_{3/4}(\tilde{p})$.

We choose a maximal collection of points $\{a_1, \ldots, a_m\}$ in $B_{2/3}(\tilde{p})$ with pairwise distances $\geq \varepsilon_1(n)/4$. It is immediate from Bishop–Gromov that m can be estimated just in terms of ε_1 and n.

Assume we can choose $g_i \in \{b_1, \ldots, b_h\}$, $i = 1, \ldots, L$, such that $g_l \cdots g_k \notin \Gamma'$ for all $1 \leq l \leq k \leq L$. This implies that $d(g_l \cdots g_k a_u, a_u) \geq \varepsilon_1(n)/4$ for some $u = u(l, k) \in \{1, \ldots, m\}$. Hence,

$$d\big(g_1 \cdots g_k a_{u(l,k)},\, g_1 \cdots g_{l-1} a_{u(l,k)}\big) \geq \varepsilon_1(n)/4, \quad \text{for } 1 < l \leq k < L.$$

We consider the diagonal action of Γ on the m-fold product $\tilde N_i^m$ and the point $a = (a_1, \dots, a_m)$. We just showed $d(g_1 \cdots g_k a, g_1 \cdots g_l a) \geq \varepsilon_1(n)/4$ for all $1 \leq k < l \leq L$. Thus there are L points in the m-fold product $B_{3/4}(\tilde p)^m$ which are $\varepsilon_1(n)/4$ separated. Now the Bishop–Gromov inequality provides an a priori bound for L.

Claim 2. Choose L as in Claim 1 and assume $\varepsilon_2(n) \leq \frac{1}{100L}$. Then for every word $w = b_{\nu_1} \dots b_{\nu_l}$ of length $l \leq L$ we have $w\Gamma' w^{-1} \subset \Gamma$.

Let $\gamma' \in \pi_1(B_1(p))$ be an element that displaces every point in $B_{2/3}(\tilde p)$ by at most $\varepsilon_1(n)$. By the definition of Γ' it suffices to show that $w\gamma' w^{-1} \in \Gamma$. Let $q \in B_{1/2}(\tilde p)$. Since $\varepsilon_2(n) < \frac{1}{100L}$, it follows that $w^{-1}\tilde p \in B_{1/25}(\tilde p)$ and hence $w^{-1}q \in B_{2/3}(\tilde p)$. Therefore

$$d(q, w\gamma' w^{-1} q) = d(w^{-1}q, \gamma' w^{-1} q) < \varepsilon_1(n) \text{ for all } q \in B_{1/2}(\tilde p).$$

Hence, $w\gamma' w^{-1} q \in \Gamma$ by the definition of Γ and Claim 2 is proven. Step 2 now follows from

Claim 3. Every element $h \in \mathsf{H}$ is of the form $h = w\gamma$ with $\gamma \in \Gamma$ and $w = b_{\nu_1} \cdots b_{\nu_l}$ with $l \leq L$.

We write $h = w \cdot \gamma$, where $\gamma \in \Gamma$, w is a word of length l and l is chosen minimal. It suffices to prove that $l \leq L$. Suppose $l > L$. Then we can apply Claim 1 to the tail of w and obtain $w = w_1 \gamma' w_2$ with $\gamma' \in \Gamma'$, w_2 a word of length $< L$ and the word-length of $w_1 \cdot w_2$ is smaller than the length of w.

By Claim 2, $w_2^{-1}\gamma' w_2 \in \Gamma$. Putting $\gamma_2 = w_2^{-1}\gamma' w_2 \gamma$ gives $h = w_1 \cdot w_2 \gamma_2$ – a contradiction, as the length of $w_1 w_2$ is smaller than l. □

Corollary 7.1. *In each dimension n there exists $\varepsilon > 0$ such that the following holds for any complete n-manifold M and $p \in M$ with $\mathrm{Ric} > -(n-1)$ on $B_1(p)$.*

If the image of $\pi_1(B_\varepsilon(p)) \to \pi_1(B_1(p))$ contains a nilpotent group of rank n, then M is homeomorphic to a compact infranilmanifold.

Actually, we will only show that M is homotopically equivalent to an infranilmanifold. By work of Farrell and Hsiang [FH83] this determines the homeomorphism type in dimensions above 4. The 4 -dimensional case follows from work of Freedman–Quinn [FQ90]. Lastly, the 3-dimensional case follows from Perelman's solution of the geometrization conjecture.

Proof of Corollary 7.1. Notice that it suffices to prove that M is aspherical: In fact, since the cohomological dimension of a rank n nilpotent group is n, the group $\pi_1(M)$ must then be a torsion free virtually nilpotent group of rank n and thus, by a result of Lee and Raymond [LR85], it is isomorphic to the fundamental group of an infranilmanifold. Therefore M is homotopically equivalent to an infranilmanifold and as explained above this then gives the result.

We argue by contradiction and assume that we can find $\varepsilon_i \to 0$ and a sequence of complete manifolds M_i with $\mathrm{Ric} > -1$ on $B_1(p_i)$ such that the image of $\pi_1(B_{\varepsilon_i}(p_i), p_i) \to \pi_1(B_1(p_i), p_i)$ contains a nilpotent group of rank n and M_i is not aspherical.

By the Margulis Lemma (Theorem 1), the nilpotent group can be chosen to have index $\leq C(n)$ in the image. Arguing on the universal cover of $B_1(p_i)$ it is not hard to deduce that there is $\delta_i \to 0$ such that for all $q_i \in B_{1/10}(p_i)$ the image of $\pi_1(B_{\delta_i}(q_i), q_i) \to \pi_1(B_1(p_i), p_i)$ also contains a nilpotent subgroup of rank n.

Next we observe that $\mathrm{diam}(M_i, p_i) \to 0$. In fact, otherwise we could, after passing to a subsequence, assume that (M_i, p_i) converges in the Gromov–Hausdorff sense to (Y, p_∞) with $Y \neq pt$. Choose $q_i \in B_{1/10}(p_i)$ converging to a regular point q_∞ in the limit Y. Similarly, choose $\lambda_i \leq \frac{1}{\sqrt{\delta_i}}$ slowly converging to infinity such that $(\lambda_i M_i, q_i) \xrightarrow{G-H} (C_{q_\infty} Y, o) = (\mathbb{R}^k, 0)$ for some $k > 0$. Let $\tilde{N}_i$ denote the universal cover of $B_1(p_i)$ and $\tilde{q}_i$ a lift of q_i. Let Γ_i be the subgroup of the deck transformation group generated by those elements which displace q_i by at most δ_i. By construction, Γ_i contains a nilpotent group of rank n. Put $N_i = \tilde{N}_i/\Gamma_i$. Since $(\lambda_i N_i, q_i) \xrightarrow{G-H} (\mathbb{R}^k, 0)$ with $k > 0$, we get a contradiction to the Induction Theorem (6.1).

Thus, $\mathrm{diam}(M_i) \to 0$ and in particular, $M_i = B_1(p_i)$ is a closed manifold for all large i. After a slow rescaling we may assume in addition that $\mathrm{Ric}_{M_i} \geq -h_i \to 0$. Next we plan to show that the universal cover of M_i converges to $\mathbb{R}^n$. This will follow from

Claim 1. Suppose a sequence of complete n-manifolds (N_i, p_i) with lower Ricci curvature bound $> -h_i \to 0$ converges to $(\mathbb{R}^k \times K, p_\infty)$ with K compact. Also assume that $\pi_1(N_i)$ is generated by loops of length $\leq R$ and contains a nilpotent subgroup of rank $n-k$. Then the universal cover of $(\tilde{N}_i, \tilde{p}_i)$ converges to $(\mathbb{R}^n, 0)$.

We argue by reverse induction on k. The base of induction $k = n$ is obvious.

By the Induction Theorem, after passing to a bounded cover, we may assume that $\pi_1(N_i)$ itself is a torsion free nilpotent group of rank $n-k$.

Choose $\mathsf{N}_i \lhd \pi_1(N_i)$ with $\pi_1(N_i)/\mathsf{N}_i \cong \mathbb{Z}$. Then $(\tilde{N}_i/\mathsf{N}_i, \hat{p}_i)$ converges to $(\mathbb{R}^l \times K', (0, p_\infty))$ for some $l > k$ and K' compact and the claim follows from the induction assumption.

Therefore $(\tilde{M}_i, \tilde{p}_i)$ converges to $(\mathbb{R}^n, 0)$. From the Cheeger–Colding Stability Theorem (see Theorem 1.2) it follows that for any $R > 0$ the ball $B_R(p)$ is contractible in $B_{R+1}(p)$ for all $p \in B_R(\tilde{p}_i)$ and $i \geq i_0(R)$. Since we have a cocompact deck transformation group with nearly dense orbits, the result actually holds for all $p \in \tilde{M}_i$.

In order to show that M_i is aspherical we may replace M_i by a bounded cover and thus, by Theorem 1 without loss of generality, $\pi_1(M_i)$ has a nilpotent basis of length $\leq n$. Because $\mathrm{rank}(\pi_1(M_i)) \geq n$ it follows that $\pi_1(M_i)$ is torsion free. Therefore we can choose subgroups $\{e\} = \mathsf{N}_0^i \lhd \cdots \lhd \mathsf{N}_n^i = \pi_1(M_i)$ with $\mathsf{N}_j^i/\mathsf{N}_{j-1}^i \cong \mathbb{Z}$.

In the rest of the proof we will slightly abuse notations and sometimes drop basepoints when talking about pointed Gromov–Hausdorff convergence when the base points are clear.

Note that the above claim easily implies that $\tilde{M}_i/\mathsf{N}_j^i \to \mathbb{R}^{n-j}$ for all $j = 0, \ldots n$.

Claim 2. $\mathsf{N}_j^i \star B_r(\tilde{p}_i)$ is contractible in $\mathsf{N}_j^i \star B_{4^{4^j} r}(\tilde{p}_i)$ for $j = 0, \cdots, n$ and $r \in \left[1, 4^{4^{(2n-j)^2}}\right]$ and all large i.

We want to prove the statement by induction on j. For $j = 0$ it holds as was pointed out above. Suppose it holds for $j < n$ and we need to prove it for $j+1$. Choose $g \in \mathsf{N}_{j+1}^i$ representing a generator of $\mathsf{N}_{j+1}^i/\mathsf{N}_j^i$.

Notice that $\mathsf{N}^i_{j+1} \star B_R(\tilde p_i)/\mathsf{N}^i_j$ converges to $\mathbb{R} \times B_R(0) \subset \mathbb{R}^{n-j}$ as $i \to \infty$ where $B_R(0)$ is the ball in $\mathbb{R}^{n-j-1}$. Moreover, the action of $\mathsf{N}^i_{j+1}/\mathsf{N}^i_j$ on the set converges to the $\mathbb{R}$ action on $\mathbb{R} \times B_R(0)$ given by translations.

We can also find finite index subgroups $\bar{\mathsf{N}}^i_{j+1} \subset \mathsf{N}^i_{j+1}$ with $\mathsf{N}^i_j \subset \bar{\mathsf{N}}^i_{j+1}$ such that $\mathsf{N}^i_{j+1} \star B_R(\tilde p_i)/\bar{\mathsf{N}}^i_{j+1}$ converges to $\mathbb{S}^1 \times B_R(0) \subset \mathbb{S}^1 \times \mathbb{R}^{n-j}$ where $\mathbb{S}^1$ has diameter 10^{-n}. It is easy to construct a smooth map $\bar\sigma\colon \mathsf{N}^i_{j+1} \star B_R(\tilde p_i)/\bar{\mathsf{N}}^i_{j+1} \to \mathbb{S}^1$ that is arbitrary close to the projection map $\mathbb{S}^1 \times B_R(0) \to \mathbb{S}^1$ in the Gromov–Hausdorff sense.

We can lift $\bar\sigma$ to a map $\sigma\colon \mathsf{N}^i_{j+1} \star B_R(\tilde p_i) \to \mathbb{R}$. Notice that σ commutes with the action of $\bar{\mathsf{N}}^i_{j+1}$ where $\bar{\mathsf{N}}^i_{j+1}/\mathsf{N}^i_j$ can be thought of as the deck transformation group of the covering $\mathbb{R} \to \mathbb{S}^1$.

It suffices to show that the inclusion $\mathsf{N}^i_{j+1} \star B_r(\tilde p_i) \to \mathsf{N}^i_{j+1} \star B_{4^{4^{j+1}}r}(\tilde p_i)$ induces the trivial map on the level of homotopy groups. Thus, we have to show that for any $k > 0$ any map $\iota\colon \mathbb{S}^k \to \mathsf{N}^i_{j+1} \star B_r(\tilde p_i)$ is null homotopic in $\mathsf{N}^i_{j+1} \star B_{4^{4^{j+1}}r}(\tilde p_i)$.

We can assume that ι is smooth. The image of $\sigma \circ \iota$ is given by an interval $[a, b]$ in $\mathbb{R}$. We choose a 10^{-n}-fine finite subdivision $a < t_1 < \cdots < t_h < b$ of the interval such that the t_α are regular values. Thus $\iota^{-1}(\sigma^{-1}(t_\alpha)) = H_\alpha$ is a smooth hypersurface in $\mathbb{S}^k$ for every α.

Notice that by construction, the image $\iota(H_\alpha)$ is contained in $g\mathsf{N}_j \star B_{2r}(\tilde p_i)$ for some $g \in \bar{\mathsf{N}}^i_{j+1}$. By induction assumption, $\iota_{|H_\alpha}$ is homotopic to a point map in $g\mathsf{N}^i_j \star B_{4^{4^j}2r}(\tilde p_i)$. We homotope ι into $\tilde\iota$ such that $\tilde\iota(H_\alpha)$ is a point for all α and $\tilde\iota$ is $4^{4^j}4r$ close to ι.

Consider now all components of H_α for all α. They divide the sphere into connected regions such that each boundary component of a region is mapped by $\tilde\iota$ to a point and the whole region is mapped to a set $g\mathsf{N}^i_j \star B_{4^{4^j}8r}(\tilde p_i)$ for some g.

Thus the map $\tilde\iota$ restricted to a region with crushed boundary components is null homotopic in $g \star B_{4^{2\cdot 4^j}8r}(\tilde p_i)$ by the induction assumption. Clearly, this implies that ι is null homotopic in $\mathsf{N}^i_{j+1} \star B_{4^{4^{j+1}}r}(\tilde p_i)$.

This finishes the proof of Claim 2. Notice that $\mathsf{N}^i_n \star B_1(\tilde p_i) = \tilde M_i$ for large i since $\mathrm{diam}(M_i) \to 0$. Thus $\tilde M_i$ is contractible by Claim 2. $\square$

8. Almost nonnegatively curved manifolds with maximal first Betti number

This is the only section where we assume lower sectional curvature bounds. The main purpose is to prove

Corollary 8.1. *In each dimension there are positive constants $\varepsilon(n)$ and $p_0(n)$ such that for all primes $p > p_0(n)$ the following holds.*

Any manifold with $\mathrm{diam}(M, g)^2 K_{sec} \geq -\varepsilon(n)$ and $b_1(M, \mathbb{Z}_p) \geq n$ is diffeomorphic to a nilmanifold. Conversely, for every p, every compact n-dimensional nilmanifold covers another (almost flat) nilmanifold M with $b_1(M, \mathbb{Z}_p) = n$.

The second part of the Corollary is fairly elementary but it provides counterexamples to a theorem of Fukaya and Yamaguchi [FY92, Corollary 0.9] which asserted that only tori should show up. The second part follows from the following Lemma and the fact that every nilmanifold is almost flat, i.e. for any $i > 0$ it admits a metric with $\mathrm{diam}(M, g)^2 |K_{sec}| \leq \frac{1}{i}$.

Lemma 8.2. *Let Γ be a torsion free nilpotent group of rank n, and let p be a prime number.*

a) There is a subgroup of finite index $\Gamma' \subset \Gamma$ for which we can find a surjective homomorphism $\Gamma' \to (\mathbb{Z}/p\mathbb{Z})^n$.

b) We can find a torsion free nilpotent group $\hat{\Gamma}$ containing Γ as finite index subgroup such that there a surjective homomorphism $\hat{\Gamma} \to (\mathbb{Z}/p\mathbb{Z})^n$.

Proof. a) We argue by induction on n. The statement is trivial for $n = 1$. Assume it holds for $n - 1$. Choose a subgroup $\Lambda \lhd \Gamma$ with $\Gamma/\Lambda \cong \mathbb{Z}$. By the induction assumption, we can find a subgroup $\Lambda' \subset \Lambda$ of finite index and $\Lambda'' \lhd \Lambda'$ with $\Lambda'/\Lambda'' \cong (\mathbb{Z}/p\mathbb{Z})^{n-1}$. Let $g \in \Gamma$ represent a generator of Γ/Λ. Since Λ contains only finitely many subgroups with of index $\leq [\Lambda : \Lambda'']$, it follows that g^l is in the normalizer of Λ' and Λ'' for a suitable $l > 0$. After increasing l further we may assume that the automorphism of $\Lambda'/\Lambda'' \cong (\mathbb{Z}/p\mathbb{Z})^{n-1}$ induced by the conjugation by g^l is the identity.

Define Γ' as the group generated by Λ' and g^l. It is now easy to see that Γ'/Λ'' is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{n-1} \times \mathbb{Z}$ and thus Γ' has a surjective homomorphism to $(\mathbb{Z}/p\mathbb{Z})^n$.

b). An analysis of the proof of a) shows that in a) the index of Γ' in Γ can be bounded by a constant C only depending on n and p. Let L be the Malcev completion of Γ, i.e. L is the unique n-dimensional nilpotent Lie group containing Γ as a lattice. Let $\exp\colon \mathfrak{l} \to \mathsf{L}$ denote the exponential map of the group. It is well known that the group $\bar{\Gamma}$ generated by $\exp(\frac{1}{C!}\exp^{-1}(\Gamma))$ contains Γ as a finite index subgroup. By part a) we can assume now that $\bar{\Gamma}$ contains a subgroup $\hat{\Gamma}$ of index $\leq C$ such that there is a surjective homomorphism to $\hat{\Gamma} \to (\mathbb{Z}/p\mathbb{Z})^n$. By construction, any subgroup of $\bar{\Gamma}$ of index $\leq C$ must contain Γ and hence we are done. □

Lemma 8.3. *For any n there exists $p(n)$ such that if Γ is a torsion free virtually nilpotent of rank n which admits an epimorphism onto $(\mathbb{Z}/p\mathbb{Z})^n$ for some $p \geq p(n)$ then Γ is nilpotent.*

Proof. By [LR85], Γ is an almost crystallographic group, that is, there is an n-dimensional nilpotent Lie group L, a compact subgroup $\mathsf{K} \subset \mathrm{Aut}(\mathsf{L})$ such that Γ is isomorphic to a lattice in $\mathsf{K} \ltimes \mathsf{L}$. By a result of Auslander (see [LR85] for a proof), the projection of Γ to K is finite and we may assume that the projection is surjective. Thus K is a finite group and it is easy to see that the action of K on $\mathsf{L}/[\mathsf{L},\mathsf{L}]$ is effective. It is known that $\mathsf{N}' := \Gamma \cap [\mathsf{L},\mathsf{L}]$ is a lattice in $[\mathsf{L},\mathsf{L}]$. Thus, the image $\bar{\Gamma} := \Gamma/\mathsf{N}'$ of Γ in $\mathsf{K} \ltimes \mathsf{L}/[\mathsf{L},\mathsf{L}]$ is discrete and cocompact. Since K acts effectively on $\mathsf{L}/[\mathsf{L},\mathsf{L}]$, we can view $\mathsf{K} \ltimes \mathsf{L}/[\mathsf{L},\mathsf{L}]$ as a cocompact subgroup of $\mathrm{Iso}(\mathbb{R}^k)$ where $k = \dim(\mathsf{L}/[\mathsf{L},\mathsf{L}])$.

In particular, $\bar{\Gamma}$ is a crystallographic group. As the projection of Γ to K is surjective, $\bar{\Gamma}$ is abelian if and only if K is trivial. If it is not abelian, then

$$\mathrm{rank}(\bar{\Gamma}/[\bar{\Gamma},\bar{\Gamma}]) < \mathrm{rank}(\bar{\Gamma}) = k.$$

By the third Bieberbach theorem there are only finitely many crystallographic groups of any given rank, and consequently there are only finitely many possibilities for the isomorphism type of $\bar{\Gamma}/[\bar{\Gamma},\bar{\Gamma}]$. Obviously, any homomorphism $\bar{\Gamma} \to (\mathbb{Z}/p\mathbb{Z})^k$ factors through $\bar{\Gamma} \to \bar{\Gamma}/[\bar{\Gamma},\bar{\Gamma}] \to (\mathbb{Z}/p\mathbb{Z})^k$; this easily implies that there is p_0 such that that there is no surjective homomorphism $\bar{\Gamma} \to (\mathbb{Z}/p\mathbb{Z})^k$ for $p \geq p_0$.

Therefore, there is no surjective homomorphism $\Gamma \to (\mathbb{Z}/p\mathbb{Z})^n$ for $p > p_0$. If we put $p(n) = p_0$, then $\bar{\Gamma}$ is abelian, $\mathsf{K} = \{e\}$ and hence Γ is nilpotent. □

Corollary 8.1 follows from Lemma 8.2, Lemma 8.3 and the following result

Corollary 8.4. *In each dimension n there are positive constants $C(n)$ and $\varepsilon(n)$ such that for any complete manifold with $K_{sec} > -1$ on $B_1(p)$ for some $p \in M$ one of the following holds*

a) The image $\pi_1(B_{\varepsilon(n)}(p)) \to B_1(p)$ contains a subgroup N of index less than $C(n)$ such that N has a nilpotent basis of length $\leq n-1$ or

b) M is compact and homeomorphic to an infranilmanifold. Moreover, any finite cover with a nilpotent fundamental group is diffeomorphic to a nilmanifold.

Proof. Consider a sequence of manifolds (M_i, p_i) with $K_{sec} > -1$ such that M_i does not satisfy a) with $C = i$ and $\varepsilon(n) = \frac{1}{i}$. As in proof of Corollary 7.1, it is clear that $\mathrm{diam}(M_i) \to 0$.

By the Margulis Lemma or by work of [KPT10], we may assume that $\pi_1(M_i)$ is nilpotent. We plan to show that $\pi_1(M_i)$ has rank n for large i.

The fact that $\pi_1(M_i)$ does not contain a subgroup of bounded index with a nilpotent basis of length $\leq n-1$, guarantees a certain extremal behavior if we run through the proof of the Induction Theorem. It will be clear that the blow up limits $\mathbb{R}^k \times K$, that occur if we go through the procedure provided in the proof of the Induction Theorem, are flat – more precisely each K has to be a torus. This follows from the fact that we can never run into a situation where Step 3 of the proof of the Induction Theorem applies.

In particular, if we rescale the manifolds to have diameter 1 they must converge to a torus $K = T^h$. We can now use Yamaguchi's fibration theorem [Yam91] to get a fibration $F_i \to M_i \to T^h$.

Let Γ_i denote the kernel of $\pi_1(M_i) \to \pi_1(T^h)$. In the next h steps of the procedure in the proof of the Induction Theorem we replace M_i by $\tilde{M}_i/\Gamma_i$ endowed with h deck transformations $f_i^1, \ldots, f_i^h$ projecting to generators of $\pi_1(T^h)$.

$(\hat{M}_i := \tilde{M}_i/\Gamma_i, \hat{p}_i)$ converges to $(\mathbb{R}^h, 0)$. The next step in the Induction Theorem would be to rescale $\hat{M}_i$ so that $(\lambda_i \hat{M}_i, \hat{p}_i) \xrightarrow{G-H} (\mathbb{R}^h \times K', \hat{p}_\infty)$ with K' not a point and it will be clear again that $K' \cong T^{h'}$ is a torus.

It is easy to see that $\frac{1}{\lambda_i}$ is necessarily comparable to the size of the fiber F_i of the Yamaguchi fibration.

From the exact homotopy sequence we can deduce that $\Gamma_i = \pi_1(\hat{M}_i) \cong \pi_1(F_i)$. The fibration theorem ensures that F_i fibers over a new torus $T^{h'}$. Let Γ_i' denote the kernel of $\pi_1(F_i) \to \pi_1(T^{h'})$. Clearly $\pi_1(M_i)$ has rank n if and only if Γ_i' has rank $n - h - h'$. After finitely many similar steps this shows that $\pi_1(M_i)$ has rank n. By Corollary 7.1 the first part of b) follows. To get the second part of part b) observe that if $\pi_1(M_i)$ is nilpotent and torsion free then by the proof of [KPT10, Theorem 5.1.1], for all large i we have that M_i smoothly fibers over a nilmanifold with simply connected factors. The fibers obviously have to be trivial as M_i is aspherical which means that M_i is diffeomorphic to a nilmanifold. □

Problem. *a) We suspect that in the case of almost nonnegatively curved manifolds Corollary 8.4 remains valid if one replaces the $n-1$ in a) by $n-2$.*

b) *The result of this section remain valid if one just has a uniform lower bound on sectional curvature and lower Ricci curvature bounds close enough to* 0*. However, it would be nice to know whether or not they remain valid without sectional curvature assumptions.*

9. Finiteness results

In the proof of the following theorem the work of Colding and Naber [CN10] is key.

Theorem 9.1 (Normal Subgroup Theorem)**.** *Given n, D, $\varepsilon_1 > 0$ there are positive constants $\varepsilon_0 < \varepsilon_1$ and C such that the following holds:*

If (M, g) is a compact n-manifold M with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \leq D$, then there is $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ and a normal subgroup $\mathsf{N} \lhd \pi_1(M)$ such that for all $p \in M$ we have:

- *the image of $\pi_1(B_{\varepsilon/1000}(p), p) \to \pi_1(M, p)$ contains N and*
- *the index of N in the image of $\pi_1(B_\varepsilon(p), p) \to \pi_1(M, p)$ is $\leq C$.*

To avoid confusion, we should recall that a normal subgroup $\mathsf{N} \lhd \pi_1(M, p)$ naturally induces a normal subgroup $\mathsf{N} \lhd \pi_1(M, q)$ for all $p, q \in M$.

If we put $\varepsilon_1 = \varepsilon_{Marg}(n)$, the constant in the Margulis Lemma, then by Theorem 1, the group N in the above theorem contains a nilpotent subgroup of index $\leq C_{Marg}$ which has a nilpotent basis of length $\leq n$. After replacing N by a characteristic subgroup of controlled index Theorem 6 follows.

Proof of Theorem 9.1. We will argue by contradiction. It suffices to rule out the existence of a sequence (M_i, g_i) with the following properties.

(1) $\mathrm{Ric}_{M_i} > -(n-1)$, $\mathrm{diam}(M_i) \leq D$;
(2) for all $\varepsilon \in (2^{-i}, \varepsilon_1)$ and any normal subgroup $\mathsf{N} \lhd \pi_1(M_i)$ one of the following holds
  (i) N is not contained in the image of $\pi_1(B_{\varepsilon/1000}(p))$ for some $p \in M_i$ or
  (ii) the index of N in the image of $\pi_1(B_\varepsilon(q_i), q_i) \to \pi_1(M_i, q_i)$ is $\geq 2^i$ for some $q_i \in M_i$.

Since any subsequence of (M_i, g_i) also satisfies this condition, we can assume that the universal covers $(\tilde{M}_i, \pi_1(M_i), \tilde{p}_i)$ converge to $(Y, \Gamma_\infty, \tilde{p}_\infty)$. Put $\Gamma_i := \pi_1(M_i)$. For $\delta > 0$ define

$$S_i(\delta) := \left\{ g \in \Gamma_i \mid d(q, gq) \leq \delta \text{ for all } q \in B_{1/\delta}(\tilde{p}_i) \right\} \text{ and } \Gamma_{i\delta} := \langle S_i(\delta) \rangle \subset \Gamma_i$$

for $i \in \mathbb{N} \cup \{\infty\}$.

By Colding and Naber [CN10], Γ_∞ is a Lie group. In particular, the component group $\Gamma_\infty / \Gamma_{\infty 0}$ is discrete and hence there is some positive $\delta_0 \leq \min\{\varepsilon_1, \frac{1}{2D}\}$ such that $\Gamma_{\infty\delta}$ is given by the identity component of Γ_∞ for all $\delta \in (0, 2\delta_0)$.

It is easy to see that this property carries over to the sequence in the following sense. We fix a large constant $R \geq 10$, to be determined later, and put $\varepsilon_0 := \delta_0 / R$. Then $\Gamma_{i\delta} = \Gamma_{i\varepsilon_0}$ for all $\delta \in [\varepsilon_0/1000, R\varepsilon_0]$ and all $i \geq i_0(R)$.

We claim that $\Gamma_{i\varepsilon_0}$ is normal in Γ_i. In fact, Γ_i can be generated by elements which displace $\tilde{p}_i$ by at most $2D \leq \frac{1}{\varepsilon_0}$. It is straightforward to check that for any $g_i \in \Gamma_i$ satisfying $d(g_i \tilde{p}_i, \tilde{p}_i) \leq \frac{1}{\varepsilon_0}$ we have $g_i S(\varepsilon_0/2) g_i^{-1} \subset S(\varepsilon_0)$. Thus,

$$g_i \Gamma_{i\varepsilon_0/2} g_i^{-1} \subset \Gamma_{i\varepsilon_0} = \Gamma_{i\varepsilon_0/2}$$

and clearly the claim follows.

In summary, $\Gamma_{i\varepsilon_0/1000} = \Gamma_{iR\varepsilon_0} \lhd \Gamma_i$ with $\varepsilon_0 \le \varepsilon_1$ for all $i \ge i_0(R)$. By the choice of our sequence this implies that for some $q_i \in M$ the index of $\Gamma_{i\varepsilon_0}$ in the image of $\pi_1(B_{\varepsilon_0}(q_i), q_i)) \to \pi_1(M_i)$ is larger than 2^i.

Similarly to Step 2 in the proof of the Margulis Lemma in section 7, this gives a contradiction provided we choose R sufficiently large. In fact, the present situation is quite a bit easier as $\Gamma_{iR\varepsilon_0}$ is normal in $\pi_1(M_i)$. □

Lemma 9.2. *a) Let $\varepsilon_0, D, H, n > 0$. Then there is a finite collection of groups such that the following holds:*

Let (M, g) be a compact n-dimensional manifold with $\mathrm{diam}(M) < D$, $\mathrm{Ric}(M) > -(n-1)$, $\mathsf{N} \lhd \pi_1(M)$ and $\varepsilon \ge \varepsilon_0$. Suppose that

- *The image I of $\pi_1(B_\varepsilon(q)) \to \pi_1(M, q)$ satisfies that the index $[I : I \cap \mathsf{N}]$ is bounded by H, for all $q \in M$.*
- *The group N is in the image of $\pi_1(B_{\varepsilon/1000}(q_0)) \to \pi_1(M, q_0)$ for some $q_0 \in M$.*

Then $\pi_1(M)/\mathsf{N}$ is isomorphic to a group in the a priori chosen finite collection.

b) (Bounded presentation) Given a manifold as in a) one can find finitely many loops in M based at $p \in M$ whose length and number is bounded by an a priori constant such that the following holds: the loops project to a generator system of $\pi_1(M, p)/\mathsf{N}$ and there are finitely many relations (with an a priori bound on the length of the words involved) such that these words provide a finite presentation of the group $\pi_1(M, p)/\mathsf{N}$.

Instead of compact manifolds one could also consider manifolds satisfying the condition that the image of $\pi_1(B_{r_0}(p_1)) \to \pi_1(B_{R_0}(p_1))$ is via the natural homomorphism isomorphic to $\pi_1(M_i)$, where r_0 and R_0 are given constants. In that case one would only need the Ricci curvature bound in the ball $B_{R_0+1}(p_1)$. In fact, the only purpose of the Ricci curvature bound is to guarantee the existence of an $\varepsilon/100$–dense finite set in M with an a priori bound on the order of the set.

Of course, b) implies a). However, we prove a) first and then see that b) follows from the proof. In the context of equivariant Gromov–Hausdorff convergence a bounded presentation can often be carried over to the limit. That is why part b) has some advantages over a).

Proof. a). The idea is to define for each such manifold a 2-dimensional finite CW-complex C whose combinatorics is bounded by some a priori constants and which has the fundamental group $\pi_1(M)/\mathsf{N}$.

Choose a maximal collection of points $p_1, \ldots, p_k$ in M with pairwise distances $\ge \varepsilon/100$. Clearly, there is an a priori bound on k depending only on n, D and ε_0. We also may assume that $p_1 = q_0$.

The points $p_1, \ldots, p_k$ will represent the 0-skeleton C_0 of our CW-complex.

For each point p_i define F_i as the finite group given by $\mathrm{im}_{\varepsilon/5}(p_i)/(\mathsf{N} \cap \mathrm{im}_{\varepsilon/5}(p_i))$ where $\mathrm{im}_\delta(p_i)$ denotes the image of $\pi_1(B_\delta(p_i), p_i) \to \pi_1(M, p_i)$. For each of the at most H elements in F_i we choose a loop in $B_{\varepsilon/5}(p_i)$ representing the class modulo N. We attach to p_i an oriented 1-cell for each loop and call the loop the model path of the cell.

For any two points p_i, p_j with $d(p_i, p_j) < \varepsilon/20$ we choose in the manifold a shortest path from p_i to p_j. In the CW-complex we attach an edge (1-cell) between these two points. We call the chosen path the model path of the edge.

This finishes the definition of the 1-skeleton C_1 of our CW-complex.

For each point $p_i \in C_0$ we consider the part A_i of the 1-skeleton C_1 containing all points p_j with $d(p_i, p_j) < \varepsilon/2$ and also all 1-cells connecting them including the cells attached initially. If A_i has more than one connected component we replace A_i by the connected component containing p_i.

We choose loops in A_i based at p_i representing free generators of the free group $\pi_1(A_i, p_i)$. We now consider the homomorphism $\pi_1(A_i, p_i) \to \pi_1(M, p_i)$ induced by mapping each loop to its model path in M. The induced homomorphism $\alpha_i \colon \pi_1(A_i, p_i) \to \pi_1(M, p_i)/\mathsf{N}$ has finite image of order $\leq H$ since the model paths are contained in $B_\varepsilon(p_i)$.

The kernel of α_i is a normal subgroup of finite index $\leq H$ in $\pi_1(A_i, p_i)$. Thus, it is finitely generated and there is an a priori bound on the number of possibilities for the kernel. We choose a free finite generator system of the kernel and attach the corresponding 2-cells to the CW-complex for each generator of the kernel.

This finishes the definition of the CW-complex C.

Note that by construction, the total number of cells in our complex is bounded by a constant depending only on n, D, H and ε_0 and that the attaching maps of the two cells are under control as well. Hence there are only finitely many possibilities for the homotopy type of C. Thus, we can finish the proof by establishing that $\pi_1(M)/\mathsf{N}$ is isomorphic to $\pi_1(C)$.

First notice that there is a natural surjective map $\pi_1(C) \to \pi_1(M)/\mathsf{N}$: Consider the 1-skeleton C_1 of C. Its fundamental group is free and there is a homomorphism $\pi_1(C_1) \to \pi_1(M) \to \pi_1(M)/\mathsf{N}$ induced by mapping a path onto its model path. The two cells that where attached to the 1-skeleton only added relations to the fundamental group contained in the kernel of this homomorphism. Thus, the homomorphism induces a homomorphism $\pi_1(C, p_1) \to \pi_1(M, p_1)/\mathsf{N}$. Moreover, it is easy to see that the homomorphism is surjective.

In order to show that the homomorphism is injective we need to show that homotopies in M can be lifted in some sense to C.

Suppose we have a closed curve γ in the 1-skeleton C_1 which is based at p_1 and whose model curve c_0 in M is homotopic to a curve in $\mathsf{N} \subset \pi_1(M)$. We parametrize γ on $[0, j_0]$ for some large integer j_0. We will assume that $\gamma(t) = p_j$ for some j implies $t \in [0, j_0] \cap \mathbb{Z}$ – we do not assume that the converse holds.

Using that $p_1 = q_0$ we can find a homotopy $H \colon [0,1] \times [0, j_0] \to M$, $(s,t) \mapsto H(s,t) = c_s(t)$ such that each c_s is a closed loop at p_1, c_0 is the original curve and c_1 is a curve in the ball $B_{\varepsilon/1000}(p_1)$. If we choose j_0 large enough, we can arrange for $L(c_{s[i,i+1]}) < \varepsilon/1000$ for all $i = 0, \ldots, j_0 - 1$, $s \in [0,1]$.

After an arbitrary small change of the homotopy we can assume that $c_s(i)$ intersects the cut locus of the 0-dimensional submanifold $\{p_1, \ldots, p_k\}$ only finitely many times for every $i = 1, \ldots, j_0 - 1$. We can furthermore assume that at any given parameter value s there is at most one i such that $c_s(i)$ is in the cut locus of $\{p_1, \ldots, p_k\}$.

For each $c_s(i)$ we choose a point $\tilde{c}_s(i) := p_{l_i}(s)$ with minimal distance to $c_s(i)$. By construction $\tilde{c}_s(i)$ is piecewise constant in s and at each parameter value s_0

at most one of the chosen points is changed. Moreover, $\tilde{c}_s(0) = \tilde{c}_s(1) = p_1$ and $\tilde{c}_1(i) = p_1$.

We now define $\tilde{c}_{s|[i,i+1]}$ in three steps. In the middle third of the interval we run through $c_{s|[i,i+1]}$ with triple speed. In the last third of the interval we run through a shortest path from $c_s(i+1)$ to $p_{l_{i+1}}(s)$. In the first third of the interval we run through a shortest path from $p_{l_i}(s)$ to $c_s(i)$ and if there is a choice we choose the same path that was chosen in the last third of the previous interval.

Each curve $\tilde{c}_s$ is homotopic to c_s. Moreover, $\tilde{c}_s(i)$ is piecewise constant equal to a vertex. Clearly, $s \mapsto \tilde{c}_s$ is only piecewise continuous.

The 'jumps' for $\tilde{c}$ occur exactly at those parameter values at which $c_s(i)$ is the cut locus of the finite set $\{p_1, \dots, p_k\}$ for some i.

We introduce the following notation: for two continuous curves c_1, c_2 in M from p to $q \in M$ we say that c_1 and c_2 are homotopic modulo N if the homotopy class of the loop obtained by prolonging c_1 with the inverse parametrized curve c_2 is in N.

We define a 'lift' γ_s in C of the curve $\tilde{c}_s$ in M as follows: $\gamma_{s|[i,i+1]}$ should run in the one skeleton of our CW complex from $\tilde{c}_s(i)$ to $\tilde{c}_s(i+1) \in C_0$ and the model path of $\gamma_{s|[i,i+1]}$ should be modulo N homotopic to $\tilde{c}_{s|[i,i+1]}$. We choose $\gamma_{s[i,i+1]}$ in such a way that on the first half of the interval it runs through the edge from $\tilde{c}_s(i)$ to $\tilde{c}_s(i+1)$ and in order to get the right homotopy type for the model path, in the second half of the interval it runs through one of the 1-cells that were attached initially to the vertex $\tilde{c}_s(i+1) \in C_0$.

Thus, the model curve of $\gamma_{s|[i,i+1]}$ is homotopic to $\tilde{c}_{s|[i,i+1]}$ modulo N. In particular, the model curve of γ_s and the loop c_s are homotopic modulo N.

The map $s \mapsto \gamma_s$ is piecewise constant in s. We have to check that at the parameter where γ_s jumps the homotopy type of γ_s does not change.

However, first we want to check that the original curve γ is homotopic to γ_0 in C. By our initial assumption, there are integers $0 = k_0 < \ldots < k_h = j_0$ such that $\gamma(t)$ is in the 0-skeleton if and only if $t \in \{k_0, \dots, k_h\}$. Going through the definitions above it is easy to deduce that $\gamma(k_i) = c_0(k_i) = \tilde{c}_0(k_i) = \gamma_0(k_i)$. Thus, it suffices to check that $\gamma_{|[k_i,k_{i+1}]}$ is homotopic to $\gamma_{0|[k_i,k_{i+1}]}$. We have just seen that the model curve of $\gamma_{0|[k_i,k_{i+1}]}$ is homotopic to $\tilde{c}_{0|[k_i,k_{i+1}]}$ modulo N. This curve is homotopic to $c_{0|[k_i,k_{i+1}]}$, which is the model curve of $\gamma_{|[k_i,k_{i+1}]}$.

Moreover, the model curve of $\gamma_{|[k_i,k_{i+1}]}$ is in an $\varepsilon/5$-neighborhood of $\gamma(k_i)$. By the definition of $\tilde{c}_0$ this implies that $\tilde{c}_{0|[k_i,k_{i+1}]}$ is in an $\varepsilon/4$-neighborhood of $\gamma(k_i)$. Therefore, $\gamma_{|[k_i,k_{i+1}]}$ as well as $\gamma_{0|[k_i,k_{i+1}]}$ only meet points of the zero skeleton with distance $< \varepsilon/4$ to $\gamma(k_i)$. Thus, they are contained in A_j, where j is defined by $p_j = \gamma(k_i)$. Since their model curves are homotopic modulo N, our rules for attaching 2-cells imply that $\gamma_{0|[k_i,k_i+1]}$ is homotopic to $\gamma_{[k_i,k_{i+1}]}$ in C.

It remains to check that the homotopy type of γ_s does not change at a parameter s_0 at which $s \mapsto \gamma_s$ is not continuous. By construction there is an i such that $\gamma_{s|[0,i-1]}$ and $\gamma_{s|[i+1,j_0]}$ are independent of $s \in [s_0 - \delta, s_0 + \delta]$ for some $\delta > 0$.

The model curve of $\gamma_{s_0-\delta|[i-1,i+1]}$ is modulo N homotopic to $\tilde{c}_{s_0-\delta|[i-1,i+1]}$ which in turn is homotopic to $\tilde{c}_{s_0+\delta|[i-1,i+1]}$ and, finally, this curve is modulo N homotopic to the model curve of $\gamma_{s_0+\delta|[i-1,i+1]}$.

Furthermore, the model curves of $\gamma_{s_0\pm\delta|[i-1,i+1]}$ are in an $\varepsilon/2$-neighborhood of $\tilde{c}_{s_0}(i-1)$. Since they are homotopic modulo N, this implies by definition of our complex that $\gamma_{s_0-\delta|[i-1,i+1]}$ is homotopic to $\gamma_{s_0+\delta|[i-1,i+1]}$ in C .

Thus, the curve γ is homotopic to γ_1, which is the point curve by construction.

b) Since the number of CW-complexes constructed in a) is finite, we can think of C as fixed. We choose loops in C_1 representing a free generator system of the free group $\pi_1(C_1, p_1)$. The model curves of these loops represent generators of $\pi_1(M, p_1)/\mathsf{N}$ and the lengths of the loops are bounded.

For each of the attached 2-cells we consider the loop based at some vertex p_i in the one skeleton given by the attaching map. We choose a path in C_1 from p_1 to p_i and conjugate the loop back to $\pi_1(C_1, p_1)$.

We can express this loop as word in our free generators and if collect all these words for all 2-cells, we get a finite presentation of $\pi_1(C, p_1) \cong \pi_1(M)/\mathsf{N}$. □

It will be convenient for the proof of Theorem 7 to restate it slightly differently.

Theorem 9.3. *a) Given D and n there are finitely many groups $\mathsf{F}_1, \dots, \mathsf{F}_k$ such that the following holds: For any compact n-manifold M with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \le D$ we can find a nilpotent normal subgroup $\mathsf{N} \lhd \pi_1(M)$ which has a nilpotent basis of length $\le n-1$ such that $\pi_1(M)/\mathsf{N}$ is isomorphic to one of the groups in our collection.*

b) In addition to a) one can choose a finite collection of irreducible rational representations $\rho_i^j \colon \mathsf{F}_i \to \mathrm{GL}(n_i^j, \mathbb{Q})$ $(j = 1, \dots, \mu_i, i = 1, \dots, k)$ such that for a suitable choice of the isomorphism $\pi_1(M)/\mathsf{N} \cong \mathsf{F}_i$ the following holds: There is a chain of subgroups $\mathrm{Tor}(\mathsf{N}) = \mathsf{N}_0 \lhd \cdots \lhd \mathsf{N}_{h_0} = \mathsf{N}$ which are all normal in $\pi_1(M)$ such that $[\mathsf{N}, \mathsf{N}_h] \subset \mathsf{N}_{h-1}$ and $\mathsf{N}_h/\mathsf{N}_{h-1}$ is free abelian. Moroever, the action of $\pi_1(M)$ on N by conjugation induces an action of F_i on $\mathsf{N}_h/\mathsf{N}_{h-1}$ and the induced representation $\rho \colon \mathsf{F}_i \to \mathrm{GL}\big((\mathsf{N}_h/\mathsf{N}_{h-1}) \otimes_{\mathbb{Z}} \mathbb{Q}\big)$ is isomorphic to ρ_i^j for a suitable $j = j(h)$, $h = 1, \dots, h_0$.

Addendum: In addition one can assume in a) that $\mathrm{rank}(\mathsf{N}) \le n-2$.

Proof of Theorem 9.3. We consider a contradicting sequence (M_i, g_i), that is, we have $\mathrm{diam}(M_i, g_i) \le D$ and $\mathrm{Ric}_{M_i} \ge -(n-1)$ but the theorem is not true for any subsequence of (M_i, g_i).

We may assume that $\mathrm{diam}(M_i, g_i) = D$ and (M_i, g_i) converges to some space X. Choose $p_i \in M_i$ converging to a regular point $p \in X$. By the Gap Lemma (2.4) there is some $\delta > 0$ and $\varepsilon_i \to 0$ with $\pi_1(M_i, p_i, \varepsilon_i) = \pi_1(M_i, p_i, \delta)$.

Claim. There is C and i_0 such that $\pi_1(M_i, p_i, \delta)$ contains a subgroup of index $\le C$ which has a nilpotent basis of length $\le n - \dim(X)$ for all large $i \ge i_0$.

We choose $\lambda_i < \frac{1}{\varepsilon_i}$ with $\lambda_i \to \infty$ such that $(\lambda_i M_i, p_i)$ converges to $(\mathbb{R}^u, 0)$ with $u = \dim(X)$. It is easy to see that $(\lambda_i \tilde{M}_i/\pi_1(M_i, p_i, \varepsilon_i), \tilde{p}_i)$ converges to $(\mathbb{R}^u, 0)$ as well. The claim now follows from the Induction Theorem (6.1) with $f_i^j = \mathrm{id}$.

We now apply the Normal Subgroup Theorem (9.1) with $\varepsilon_1 = \delta$. There is a normal subgroup $\mathsf{N}_i \lhd \pi_1(M_i, p_i)$, some positive constants ε and C such that N_i is contained in $\pi_1(B_{\varepsilon_i}(p_i)) \to \pi_1(M_i)$ and N_i is contained in the image of $\pi_1(B_\varepsilon(q_i), q_i) \to \pi_1(M_i, q_i)$ with index $\le C$ for all $q_i \in M_i$.

By the Claim above, N_i has a subgroup of bounded index with a nilpotent basis of length $\le n - \dim(X) \le n-1$. We are free to replace N_i by a characteristic subgroup of bounded index and thus, we may assume that N_i itself has a nilpotent basis of length $\le n - \dim(X)$. By Lemma 9.2 a), the number of possibilities for $\pi_1(M_i)/\mathsf{N}_i$ is finite. In particular, part *a)* of Theorem 9.3 holds for the sequence and thus either *b)* or the addendum is the problem.

By Lemma 9.2 b), we can assume that $\pi_1(M_i,p_i)/\mathsf{N}_i$ is boundedly presented. By passing to a subsequence we can assume that we have the same presentation for all i. Let $g_{i1},\dots,g_{i\beta}\in\pi_1(M_i)$ denote elements with bounded displacement projecting to our chosen generator system of $\pi_1(M_i)/\mathsf{N}_i$. Moreover, there are finitely many words in $g_{i1},\dots,g_{i\beta}$ (independent of i) such that these words give a finite presentation of the group $\pi_1(M_i)/\mathsf{N}_i$.

We can assume that g_{ij} converges to $g_{\infty j}\in\mathsf{G}\subset\mathrm{Iso}(Y)$ where $(Y,\tilde p_\infty)$ is the limit of $(\tilde M_i,\tilde p_i)$. Let $\mathsf{N}_\infty\lhd\mathsf{G}$ be the limit of N_i. By construction, $\mathsf{G}/\mathsf{N}_\infty$ is discrete. Since $\pi_1(M_i)/\mathsf{N}_i$ is boundedly presented, it follows that there is an epimorphism $\pi_1(M_i)/\mathsf{N}_i\to\mathsf{G}/\mathsf{N}_\infty$ induced by mapping g_{ij} to $g_{\infty j}$, for all large i.

b). We plan to show that a subsequence satisfies b). We may assume that $\mathsf{G}\star\tilde p_\infty$ is not connected and by the Gap Lemma (2.4)

$$2\rho_0 \quad:=\quad \min\{d(\tilde p_\infty,g\tilde p_\infty)\mid g\tilde p_\infty\notin\mathsf{G}_0\star\tilde p_\infty\}>0. \tag{25}$$

Let r be the maximal nonnegative integer such that the following holds. After passing to a subsequence there is a subgroup $\mathsf{H}_i\subseteq\mathsf{N}_i$ of rank r satisfying

- $\mathsf{H}_i\lhd\pi_1(M_i)$, $\mathsf{N}_i/\mathsf{H}_i$ is torsion free and
- there is a chain $\mathrm{Tor}(\mathsf{N}_i)=\mathsf{N}_{i0}\lhd\cdots\lhd\mathsf{N}_{ih_0}=\mathsf{H}_i$ such that each group N_{ih} is normal in $\pi_1(M_i)$, $[\mathsf{N}_i,\mathsf{N}_{ih}]\subset\mathsf{N}_{ih-1}$, $\mathsf{N}_{ih}/\mathsf{N}_{ih-1}$ is free abelian and for each $h=1,\dots,h_0$ the induced representations of $\mathsf{F}=\pi_1(M_i)/\mathsf{N}_i$ in $(\mathsf{N}_{ih}/\mathsf{N}_{ih-1})\otimes_{\mathbb Z}\mathbb{Q}$ and in $(\mathsf{N}_{jh}/\mathsf{N}_{jh-1})\otimes_{\mathbb Z}\mathbb{Q}$ are equivalent for all i,j.

Here we used implicitly that we have a natural isomorphism between $\pi_1(M_i)/\mathsf{N}_i$ and $\pi_1(M_j)/\mathsf{N}_j$ in order to talk about equivalent representations.

Notice that these statements hold for $\mathsf{H}_i=\mathrm{Tor}(\mathsf{N}_i)$. We need to prove that $\mathsf{H}_i=\mathsf{N}_i$. Suppose on the contrary that $\mathrm{rank}(\mathsf{H}_i)<\mathrm{rank}(\mathsf{N}_i)$. Consider $\hat M_i:=\tilde M_i/\mathsf{H}_i$ endowed with the action of $\Gamma_i:=\pi_1(M_i)/\mathsf{H}_i$, and let $\hat p_i$ be a lift of p_i to $\hat M_i$. By construction $\hat{\mathsf{N}}_i:=\mathsf{N}_i/\mathsf{H}_i$ is a torsion free normal subgroup of Γ_i.

We claim that there is a central subgroup $\mathsf{A}_i\subset\hat{\mathsf{N}}_i$ of positive rank which is normal in Γ_i and is generated by $\{a\in\mathsf{A}_i\mid d(\hat p_i,a\hat p_i)\le d_i\}$ for a sequence $d_i\to0$: Recall that $\hat{\mathsf{N}}_i$ is generated by $\{a\in\hat{\mathsf{N}}_i\mid d(\hat p_i,a\hat p_i)\le\varepsilon_i\}$. If $\hat{\mathsf{N}}_i$ is not abelian this implies that $[\hat{\mathsf{N}}_i,\hat{\mathsf{N}}_i]$ is generated by $\{a\in[\hat{\mathsf{N}}_i,\hat{\mathsf{N}}_i]\mid d(\hat p_i,a\hat p_i)\le2n\varepsilon_i\}$. In fact an arbitrary commutator in $\hat{\mathsf{N}}_i$ can be expressed as a product of iterated commutators of a generator system and since $\hat{\mathsf{N}}_i$ has a nilpotent basis of length $\le n-1$ one only needs to iterate at most $n-1$ times. If $[\hat{\mathsf{N}}_i,\hat{\mathsf{N}}_i]$ is not central in $\hat{\mathsf{N}}_i$ one replaces it by $[\hat{\mathsf{N}}_i,[\hat{\mathsf{N}}_i,\hat{\mathsf{N}}_i]]$. After finitely many similar steps this proves the claim.

For each positive integer l put $l\cdot\mathsf{A}_i=\{g^l\mid g\in\mathsf{A}_i\}\lhd\Gamma_i$. We define $l_i=2^{u_i}$ as the maximal power of 2 such that there is an element in $l_i\cdot\mathsf{A}_i$ which displaces $\hat p_i$ by at most ρ_0. Thus, any element in $\mathsf{L}_i:=l_i\cdot\mathsf{A}_i$ displaces $\hat p_i$ by at least $\rho_0/2$.

After passing to a subsequence we may assume that $(\hat M_i,\Gamma_i,\hat p_i)$ converges to $(\hat Y,\hat{\mathsf{G}},\hat p_\infty)$ and the action of L_i converges to an action of some discrete abelian subgroup $\mathsf{L}_\infty\lhd\hat{\mathsf{G}}$. Finally we let $\hat{\mathsf{N}}_\infty\lhd\hat{\mathsf{G}}$ denote the limit group of $\hat{\mathsf{N}}_i$. Let $g_i\in\mathsf{L}_i$ be an element which displaces $\hat p_i$ by at most ρ_0. Combining $d(g_i^k\hat p_i,g_i^{k+1}\hat p_i)\le\rho_0$ with our choice of ρ_0, see (25), gives that the sets $\{g_i^k\hat p_i\mid k\in\mathbb Z\}$ converge to a discrete subset in the identity component of the limit orbit $\hat{\mathsf{G}}_0\star\hat p_\infty$. Therefore,

$$\{g\in\mathsf{L}_\infty\mid g\star\hat p_\infty\in\hat{\mathsf{G}}_0\star\hat p_\infty\}$$

is discrete and infinite. Let $\mathsf{K} \subset \hat{\mathsf{G}}$ denote the isotropy group of $\hat{p}_\infty$. By Colding and Naber K is a Lie group and thus it only has finitely many connected components. Hence $\mathsf{L}'_\infty := \mathsf{L}_\infty \cap \hat{\mathsf{G}}_0 \lhd \hat{\mathsf{G}}$ is infinite as well. Since the abelian group L'_∞ is a discrete subgroup of a connected Lie group, it is finitely generated.

We choose a free abelian subgroup $\hat{\mathsf{L}} \subset \mathsf{L}'_\infty$ of positive rank which is normalized by $\hat{\mathsf{G}}$ such that the induced representation of $\hat{\mathsf{G}}$ in $\hat{\mathsf{L}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible. Notice that $\hat{\mathsf{L}} \subset \mathsf{L}_\infty$ commutes with $\hat{\mathsf{N}}_\infty$. Hence we can view this as a representation of $\hat{\mathsf{G}}/\hat{\mathsf{N}}_\infty$. Recall that $\pi_1(M_i)/\mathsf{N}_i \cong \Gamma_i/\hat{\mathsf{N}}_i$ is boundedly represented. Let $\hat{g}_{i1}, \dots, \hat{g}_{i\beta} \in \Gamma_i/\hat{\mathsf{N}}_i$ denote the images of $g_{i1}, \dots, g_{i\beta}$. There is an epimorphism $\Gamma_i/\hat{\mathsf{N}}_i \to \hat{\mathsf{G}}/\hat{\mathsf{N}}_\infty$ induced by sending $\hat{g}_{im}$ to its limit element $\hat{g}_{\infty m} \in \hat{\mathsf{G}}$ for all large i. Thus, $\hat{\mathsf{L}}$ is also naturally endowed with a representation of $\Gamma_i/\hat{\mathsf{N}}_i$.

Let $b_1, \dots, b_l \in \hat{\mathsf{L}} \cong \mathbb{Z}^l$ be a basis. For large i there are unique elements $h_i(b_j) \in \mathsf{L}_i$ which are close to b_j, $j = 1, \dots, k$. We extend h_i to a $\mathbb{Z}$-linear map $h_i \colon \hat{\mathsf{L}} \to \mathsf{L}_i$.

We plan to prove that $h_i \colon \hat{\mathsf{L}} \to \mathsf{L}_i$ is equivariant for large i. For any given linear combination $\sum_{\alpha=1}^{l} z_\alpha b_\alpha$ ($z_j \in \mathbb{Z}$) we know that $\sum_{\alpha=1}^{l} z_\alpha h_i(b_\alpha)$ is the unique element in L_i which is close to $\sum_{\alpha=1}^{l} z_\alpha b_\alpha$ for all large i. For each $\hat{g}_{\infty m}$ and each b_j we have $\hat{g}_{\infty m} b_j \hat{g}_{\infty m}^{-1} = \sum_{\alpha=1}^{l} z_\alpha b_\alpha$ for $z_\alpha \in \mathbb{Z}$ (we suppress the dependence on m and j). We have just seen that $h_i\big(\sum_{\alpha=1}^{l} z_\alpha b_\alpha\big)$ is close to $\hat{g}_{\infty m} b_j \hat{g}_{\infty m}^{-1}$ for all large i. On the other hand, $\hat{g}_{im} h_i(b_j) \hat{g}_{im}^{-1} \in \mathsf{L}_i$ is the unique element in L_i close to $\hat{g}_{\infty m} b_j \hat{g}_{\infty m}^{-1}$ for all large i.

In summary, $h_i(\hat{g}_{\infty m} b_j \hat{g}_{\infty m}^{-1}) = \hat{g}_{im} h_i(b_j) \hat{g}_{im}^{-1}$ for $m = 1, \dots, \beta$, $j = 1, \dots, k$ and all large i. This shows that h_i is equivariant with respect to the representation.

Thus, $h_i(\hat{\mathsf{L}})$ is a normal subgroup of Γ_i and the induced representation of $\mathsf{F} = \pi_1(M_i)/\mathsf{N}_i = \Gamma/\hat{\mathsf{N}}_i$ in $h_i(\hat{\mathsf{L}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and is isomorphic to the one in $h_j(\hat{\mathsf{L}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all large i, j.

There is a unique subgroup A'_i of $\hat{\mathsf{N}}_i$ such that $h_i(\hat{\mathsf{L}})$ has finite index in A'_i and $\hat{\mathsf{N}}_i/\mathsf{A}'_i$ is torsion free. Let $\mathsf{N}_{ih_0+1} \lhd \mathsf{N}_i$ denote the inverse image of $\mathsf{A}'_i \subset \hat{\mathsf{N}}_i = \mathsf{N}_i/\mathsf{H}_i$.

Clearly, the representation of $\pi_1(M_i)/\mathsf{N}_i$ in $(\mathsf{N}_{ih_0+1}/\mathsf{H}_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the one in $(\mathsf{N}_{jh_0+1}/\mathsf{H}_j) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all large i, j – a contradiction to our choice of H_i.

Proof of the addendum. Thus, the sequence satisfies a) and b) but not the addendum. Recall that N_i has a nilpotent basis of length $\le n - \dim(X)$. Therefore $\dim(X) = 1$ and N_i is a torsion free group of rank $n-1$. Since X is one-dimensional we deduce that $\mathsf{G}/\mathsf{N}_\infty$ is virtually cyclic, that is, it contains a cyclic subgroup of finite index. This in turn implies that $\pi_1(M_i)/\mathsf{N}_i$ is virtually cyclic and by a) we can assume that it is a fixed group F.

The result (contradiction) will now follow algebraically from b): Let $\rho_j \colon \mathsf{F} \to \mathrm{GL}(n_j, \mathbb{Q})$ be a finite collection of irreducible rational representations $j = 1, \dots, j_0$. Consider, for all nilpotent torsion free groups N of rank $n-1$, all short exact sequences

$$\mathsf{N} \to \Gamma \to \mathsf{F}$$

with the property that there is a chain $\{e\} = \mathsf{N}_0 \lhd \cdots \lhd \mathsf{N}_{h_0} = \mathsf{N}$ such that $[\mathsf{N}, \mathsf{N}_h] \subset \mathsf{N}_{h-1}$, each N_h is normal in Γ, $\mathsf{N}_h/\mathsf{N}_{h-1}$ is free abelian group of positive rank and the induced representation of F in $(\mathsf{N}_h/\mathsf{N}_{h-1}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is in the finite collection.

Using that F is virtually cyclic, it is now easy to see that this leaves only finitely many possibilities for the isomorphism type of $\Gamma/\mathsf{N}_{h_0-1}$. This shows that if we replace N by N_{h_0-1} we are still left with finitely many possibilities in a). – a contradiction. □

Corollary 9.4. *Given n, D there is a constant C such that any* finite *fundamental group of an n-manifold (M,g) with* $\mathrm{Ric} > -(n-1)$, $\mathrm{diam}(M) \le D$ *contains a nilpotent subgroup of index $\le C$ which has a nilpotent basis of length $\le (n-1)$.*

Example 2. Our theorems rule out some rather innocuous families of groups as fundamental groups of manifolds with lower Ricci curvature bounds.

a) Consider a homomorphism $h\colon \mathbb{Z} \to \mathrm{GL}(2,\mathbb{Z})$ whose image does not contain a unipotent subgroup of finite index. Put $h_d(x) = h(dx)$ and consider the group $\mathbb{Z} \ltimes_{h_d} \mathbb{Z}^2$, $d \in \mathbb{N}$. By part b) of Theorem 9.3 the following holds:

   In a given dimension n and for a fixed diameter bound D only finitely many of these groups can be realized as fundamental groups of manifolds with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \le D$.

b) Consider the action of $\mathbb{Z}_{p^{k-1}}$ on $\mathbb{Z}_{p^k}$ induced by $\big(1+p^k\mathbb{Z}\big) \mapsto \big((1+p)+p^k\mathbb{Z}\big)$. In a given dimension n we have that for $k \ge n$ and $p > C_{Marg}$ (constant in the Margulis Lemma) the group $\mathbb{Z}_{p^{k-1}} \ltimes \mathbb{Z}_{p^k}$ can not be a subgroup of a compact manifold with almost nonnegative Ricci curvature, since it does not have a nilpotent basis of length $\le n$.

   Similarly, by Corollary 9.4, for a given D and n there are only finitely many primes p such that $\mathbb{Z}_{p^{n-1}} \ltimes \mathbb{Z}_{p^n}$ is isomorphic to the fundamental group of an n-manifold M with $\mathrm{Ric} > -(n-1)$ and $\mathrm{diam}(M) \le D$.

Problem. *Let $D > 0$. Can one find a finite collection of* 3*-manifolds such that for any* 3*-manifold M with* $\mathrm{Ric} > -1$ *and* $\mathrm{diam}(M) \le D$, *there is a manifold T in the finite collection and a finite normal covering $T \to M$ for which the covering group contains a cyclic subgroup of index ≤ 2?*

10. The Diameter Ratio Theorem

The aim of this section is to prove Theorem 8.

Lemma 10.1. *Let X_i be a sequence of compact inner metric spaces Gromov–Hausdorff converging to a torus T. Then $\pi_1(X_i)$ is infinite for large i.*

The proof of the lemma is an easy exercise.

Proof of Theorem 8. Suppose, on the contrary, that (M_i, g_i) is a sequence of compact manifolds with $\mathrm{diam}(M_i) = D$, $\mathrm{Ric}_{M_i} > -(n-1)$, $\#\pi_1(M_i) < \infty$ and the diameter of the universal cover $\tilde{M}_i$ tends to infinity.

By Corollary 9.4, we know that $\pi_1(M_i)$ contains a subgroup of index $\le C(n,D)$ which has a nilpotent basis of length $\le (n-1)$.

Thus, we may assume that $\pi_1(M_i)$ itself has a nilpotent basis of length $u < n$. We also may assume that u is minimal with the property that a contradicting sequence exist. Put

$$\hat{M}_i := \tilde{M}_i/[\pi_1(M_i), \pi_1(M_i)].$$

Clearly $[\pi_1(M_i), \pi_1(M_i)]$ has a nilpotent basis of length $\le u-1$. By construction $\hat{M}_i$ can not be a contradicting sequence and therefore $\mathrm{diam}(\hat{M}_i) \to \infty$.

Let $\mathsf{A}_i := \pi_1(M_i)/[\pi_1(M_i), \pi_1(M_i)]$ denote the deck transformation group of the normal covering $\hat{M}_i \to M_i$.

Claim. The rescaled sequence $\frac{1}{\operatorname{diam}(\hat{M}_i)}\hat{M}_i$ is precompact in the Gromov–Hausdorff topology and any limit space has finite Hausdorff dimension.

The problem is, of course, that no lower curvature bound is available after rescaling. The claim will follow from a similar precompactness result for certain Cayley graphs of A_i.

Recall that $\operatorname{diam}(M_i) = D$. Choose a base point $\hat{p}_i \in \hat{M}_i$. Let $f_1, \ldots, f_{k_i} \in \mathsf{A}_i$ be an enumeration of all the elements $a \in \mathsf{A}_i$ with $d(\hat{p}_i, a\hat{p}_i) < 10D$.

There is no bound for k_i, but clearly $f_1, \ldots, f_{k_i}$ is a generator system of A_i. We define a weighted metric on the abelian group A_i as follows:

$$d(e,a) := \min\left\{ \sum_{j=1}^{k_i} |\nu_j| \cdot d(\hat{p}_i, f_j\hat{p}_i) \;\Big|\; \prod_{j=1}^{k_i} f_j^{\nu_j} = a \right\} \quad \text{for all } a \in \mathsf{A}_i$$

and $d(a,b) = d(ab^{-1}, e)$ for all $a, b \in \mathsf{A}_i$. Note that this metric coincides with the restriction to A_i of the inner metric on its Cayley graph where each edge corresponding to f_j is given the length $d(\hat{p}_i, f_j\hat{p}_i)$. It is easy to see that the map

$$\iota_i \colon \mathsf{A}_i \to \hat{M}_i,\ a_i \mapsto a_i\hat{p}_i$$

is a quasi isometry with uniform control on the constants involved. In fact, there is some L independent of i such that

$$\tfrac{1}{L} d(a,b) \le d(\iota_i(a), \iota_i(b)) \le L d(a,b) \quad \text{for all } a, b \in \mathsf{A}_i.$$

and the image is of ι_i is D-dense.

Therefore, it suffices to show that $\frac{1}{\operatorname{diam}(\mathsf{A}_i)}\mathsf{A}_i$ is precompact in the Gromov–Hausdorff topology and all limit spaces of convergent subsequences are finite dimensional. For the proof we need

Subclaim. There is an $R_0 > D$ (independent of i) such that the homomorphism $h \colon \mathsf{A}_i \to \mathsf{A}_i,\ x \mapsto x^2$ satisfies that the R_0 neighborhood of $h(B_R(e))$ contains $B_{\frac{3R}{2}}(e)$, for all R and all i.

Using that $B_{10D}(e) \subset \mathsf{A}_i$ is isometric to a subset of $B_{10D}(\hat{p}_i)$, we can employ the Bishop–Gromov inequality in order to find a universal constant k such that $B_{10D}(e)$ does not contain k points with pairwise distance $\ge D$. Put $R_0 := 10D \cdot k$.

There is nothing to prove if $R \le 2R_0/3$. Suppose the statement holds for $R' \le R - D$. We claim it holds for R.

Let $a \in B_{3R/2}(e) \setminus B_{R_0}(e)$. By the definition of the metric on A_i, there are $g_1, \ldots, g_l \in \mathsf{A}_i$ with $d(e, g_j) \le 10D$, $a = \prod_{j=1}^{l} g_j$ and $d(e,a) = \sum_{j=1}^{l} d(\hat{p}_i, g_j\hat{p}_i)$. If there is any choice we assume in addition that l is minimal with these properties. By assumption $l \ge \frac{R_0}{10D} = k$. By the choice of k, after a renumbering, we may assume that $d(g_1, g_2) \le D$. Our assumption on l being minimal implies that $d(e, g_1g_2) > 10D$ and we may assume $4D \le d(e, g_1) \le d(e, g_2)$. Thus,

$$\begin{aligned} d\big(e, g_1^{-2}a\big) &\le d\big(e, (g_1g_2)^{-1}a\big) + D = d(e,a) - d(e,g_1) - d(e,g_2) + D \\ &\le d(e,a) - 2d(e,g_1) + D \le d(e,a) - 7D. \end{aligned}$$

By assumption, this implies that $g_1^{-2}a$ has distance $\le R_0$ to some $b^2 \in \mathsf{A}_i$ with $d(e,b) \le \frac{2}{3}\big(d(e,a) - 2d(e,g_1) + D\big)$. Consequently, a has distance $\le R_0$ to $g_1^2b^2 \in \mathsf{A}_i$ with $d(e, g_1b) < \frac{2}{3}d(e,a)$. This finishes the proof of the subclaim.

Since the Ricci curvature of $\hat{M}_i$ is bounded below and the L-bilipschitz embedding ι_i maps the ball $B_{50R_0}(e) \subset \mathsf{A}_i$ to a subset of $B_{50R_0}(\hat{p}_i)$, we can use Bishop–Gromov once more to see that there is a number $Q > 0$ (independent of i) such that the ball $B_{50R_0}(e) \subset \mathsf{A}_i$ can be covered by Q balls of radius R_0 for all i.

We now claim that the ball $B_{2R}(e) \subset \mathsf{A}_i$ can be covered by Q balls of radius R for all $R \geq 20R_0$ and all i. This will clearly imply that $\frac{1}{\mathrm{diam}(\mathsf{A}_i)}\mathsf{A}_i$ is precompact in the Gromov–Hausdorff topology and that the limits have finite Hausdorff dimension.

Consider the homomorphism

$$h^8 \colon \mathsf{A}_i \to \mathsf{A}_i, \;\; x \mapsto x^{16}.$$

It is obviously 16-Lipschitz since A_i is abelian. Choose a maximal collection of points $p_1, \dots, p_{l_i} \in B_{\frac{2R}{5}}(e) \subset \mathsf{A}_i$ with pairwise distances $\geq 2R_0$.

From the subclaim it easily follows that $B_{3R}(e) \subset \bigcup_{j=1}^{l_i} B_{50R_0}(h^8(p_j))$. Hence we can cover $B_{3R}(e)$ by $l_i \cdot Q$ balls of radius R_0. Consider now a maximal collection of points $q_1, \dots, q_h \in B_{2R}(e)$ with pairwise distances $\geq R$. In each of the balls $B_{\frac{2R}{5}}(q_j)$ we can choose l_i points with pairwise distances $\geq 2R_0$. Thus, $B_{3R}(e)$ contains $h \cdot l_i$ points with pairwise distances $\geq 2R_0$. Since we have seen before that $B_{3R}(e)$ can be covered by $l_i \cdot Q$ balls of radius R_0, this implies $h \leq Q$ as claimed.

Thus, $\frac{1}{\mathrm{diam}(\hat{M}_i)}\hat{M}_i$ is precompact in the Gromov–Hausdorff topology. After passing to a subsequence we may assume that $\frac{1}{\mathrm{diam}(\hat{M}_i)}\hat{M}_i \to \mathsf{T}$. Notice that T comes with a transitive action of an abelian group. Therefore T itself has a natural group structure. Moreover, T is an inner metric space and the Hausdorff dimension of T is finite. Like Gromov in [Gro81] we can now deduce from a theorem of Montgomery Zippin [MZ55, Section 6.3] that T is a Lie group and thus, a torus.

By Lemma 10.1, this shows $\pi_1(\hat{M}_i)$ is infinite for large i – a contradiction.

□

Final Remarks

We would like to mention that in [Wil11] the following partial converse of the Margulis Lemma (Theorem 1) is proved.

Theorem. *Given C and n there exists m such that the following holds: Let $\varepsilon > 0$, and let Γ be a group containing a nilpotent subgroup N of index $\leq C$ which has a nilpotent basis of length $\leq n$. Then there is a compact m-dimensional manifold M with sectional curvature $K > -1$ and a point $p \in M$ such that Γ is isomorphic to the image of the homomorphism*

$$\pi_1\big(B_\varepsilon(p), p\big) \to \pi_1(M, p).$$

Apart from the issue of finding the optimal dimension another difference to Theorem 1 is that this theorem uses the homomorphism to $\pi_1(M)$ rather than to $\pi_1(B_1(p), p)$. This actually allows for more flexibility (by adding relations to the fundamental group at large distances to p). This is the reason why the following problem remains open.

Problem. *The most important problem in the context of the Margulis Lemma for manifolds with lower Ricci curvature bound that remains open is whether or whether not one can arrange in Theorem 1 for the torsion of N to be abelian. We* *refer the reader to* [KPT10] *for some related conjectures for manifolds with almost nonnegative sectional curvature.*

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