CURVATURE BOUNDS VIA RICCI SMOOTHING

VITALI KAPOVITCH

Abstract. We give a proof of the fact that the upper and the lower sectional curvature bounds of a complete manifold vary at a bounded rate under the Ricci flow.

Let (M^n, g) be a complete Riemannian manifold with $|\sec(M)| \le 1$. Consider the Ricci flow of g given by

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}(g) \tag{0.1}$$

It is known (see [Ham82, BMOR84, Shi89]) that (0.1) has a solution on $[0, T]$ for some $T > 0$ which smoothes out the metric. Namely, g_t satisfies

$$e^{-c(n)t} g \le g_t \le e^{c(n)t} g \quad |\nabla - \nabla_t| \le c(n)t \quad |\nabla^m R_{ijkl}(t)| \le \frac{c(n,m,T)}{t^m} \tag{0.2}$$

In particular, the sectional curvature of $g(t)$ satisfies

$$|K_{g_t}| \le C(n, T) \tag{0.3}$$

This result proved to be a very useful technical tool in many situations and in particular in the theory of convergence with two-sided curvature bounds (see [CFG92, Ron96, PT99] etc). However, it turns out that in applications to convergence with two-sided curvature bounds in addition to the above properties, it is often convenient to know that $\sup K_{g_t}$ and $\inf K_{g_t}$ also vary at the bounded rate and in particular, the upper and the lower curvature bounds for g_t are almost the same as for g for sufficiently small t. For example, it is very useful to know that if g_0 has pinched positive [Ron96] or negative [Kan89, BK] curvature, then g_t has almost the same pinching.

This fact has apparently been known to some experts and it was used without a proof by various people (see e.g [Kan89, Fuk90, FJ98]). A careful proof was given in [Ron96] in case of a compact M. To the best of our knowledge, no proof exists in the literature in case of a noncompact M. The purpose of this note is to rectify this situation. To this end we prove

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Proposition 0.4. *In the above situation one has*

$$\inf K_g - C(n,T)t \le K_{g_t} \le \sup K_g + C(n,T)t$$

Proof. Throughout the proof we will denote by C various positive constants depending only on n, T. The proof in [Ron96] relies on the maximum principle applied to the evolution equation for the curvature tensor Rm which can be computed to have the form [Shi89]

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + P(Rm) \tag{0.5}$$

where $P(Rm)$ is a homogeneous quadratic polynomial in Rm. However, in noncompact case the maximum principle can not be applied directly. We will use a local version of the maximum principle often employed in [Shi89]. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

(1) $\chi \ge 0$ and is nonincreasing

(2) $\chi(x) = \begin{cases} 1 \text{ for } x \le 1 \\ \text{nonincreasing for } 1 \le x \le 2 \\ 0 \text{ for } x \ge 2 \end{cases}$

(3) $|\chi''(x)| \le 8$

(4) $\left|\frac{(\chi'(x))^2}{\chi(x)}\right| \le 16$

Fix $z \in M$ and let $d_z(x,t) = d_{g_t}(x,z)$ be the distance with respect to g_t. Put $\xi_z(x,t) = \chi(d_z(x,t))$. Using the properties of χ we obtain

(i) $0 \le \xi_z \le 1$

(ii) $\Delta \xi_z \ge C$ in the barrier sense

(iii) $\frac{|\nabla \xi_z|^2}{|\xi_z|} \le C$

(iv) $|\frac{\partial \xi_z(x,t)}{\partial t}| \le C$.

To see (ii) we compute $\Delta \xi_z = \chi''(d_z)|\nabla d_z|^2 + \chi'(d_z)\Delta d_z \ge C$ because $\chi' \le 0$ and $\Delta d_z \le C$ for $d_z \ge 1$ by Laplace comparison for spaces with $\sec \ge -C$. Finally, (iv) holds by the evolution equation of the metric (0.1) and the estimate (0.3).

Assume for now that $\sup K_{g_t} \ge 0$ for all $t \in [0,T]$. Let $\bar{A}(t) = \sup K_{g_t}$ and $\bar{A}_z(t) = \max\{0, \max_{(x,\sigma)} K_{g_t}(x,\sigma)\xi_z(x,t)\}$ where $x \in M$, σ is a 2-plane at x. Clearly $\bar{A}(t) = \sup_z \bar{A}_z(t)$.

We want to show that the upper right derivative of $\bar{A}_z(t)$ (which with a slight abuse of notations we will denote by $\bar{A}'_z(t)$) satisfies $\bar{A}'_z(t) \le C$ independent of z, t. Fix $t_0 \in [0,T]$ and let $\phi_z(x,\sigma,t) = K_{g_t}(x,\sigma)\xi_z(x,t)$. By a standard maximum principle argument, it is enough to check that $\frac{\partial \phi_z}{\partial t}(x_0,\sigma_0,t_0) \le C$ for any point of maximum of $\phi_z(\cdot, t_0)$.

Let U, V be a basis of σ_0 orthonormal with respect to g_{t_0}. Extend U, V to constant vector fields in normal coordinates at x_0 with respect to g_{t_0}.

Let $\Phi_z(x,t) = K_{g_t}(x,U,V)\xi_z(x) = \frac{Rm(t)(U,V,U,V)}{|U\wedge V|^2_{g_t}}\xi_z(x)$.

It is easy to see (cf. [Ron96]) that

(0.6) $|U\wedge V(x_0)|_{g_{t_0}} \le C, |\nabla|U\wedge V(x_0)|_{g_{t_0}}| \le C$ and $|\nabla^2|U\wedge V(x_0)|_{g_{t_0}}| \le C$

Therefore

$$|\frac{\partial|U\wedge V(x_0,t_0)|}{\partial t}| \le C(n,T) \text{ by (0.1) and (0.3).} \tag{0.7}$$

By construction, $\Phi_z(x,t_0)$ has a local maximum at x_0 and $\frac{\partial\phi_z(x_0,\sigma_0,t_0)}{\partial t} = \frac{\partial\Phi_z(x_0,t_0)}{\partial t}$. Therefore $\nabla\Phi_z(x_0,t_0) = 0$ and $\Delta\Phi_z(x_0,t_0) \le 0$. Using (0.5) we compute

(0.8)
$$\frac{\partial\Phi_z(x_0,t_0)}{\partial t} = \Delta\Phi_z(x_0,t_0) - Rm(x_0,t_0)(U,V,U,V)\xi_z(x_0,t_0)\frac{\partial}{\partial t}\left(\frac{1}{|U\wedge V|^2}\right)$$
$$-2\nabla Rm(x_0,t_0)(U,V,U,V)\nabla\left(\frac{\xi_z(x_0,t_0)}{|U\wedge V|^2}\right) - Rm(x_0,t_0)(U,V,U,V)\Delta\left(\frac{\xi_z(x_0,t_0)}{|U\wedge V|^2}\right) -$$
$$\frac{P(Rm(x_0,t_0))\xi_z(x_0,t_0)}{|U\wedge V|^2} - K_{g_t}(x,U,V)\frac{\partial\xi_z(x_0,t_0)}{\partial t}$$

We claim that the RHS is bounded above by C. The only terms that need explaining are the third and the forth summands. Let $f(x) = \frac{\xi_z(x,t_0)}{|U\wedge V|^2}$.

To see that the third term is bounded we observe that $\nabla\Phi_z(x_0,t_0) = 0$ yields $\nabla Rm(x_0,t_0)(U,V,U,V)f(x_0) + Rm(x_0,t_0)(U,V,U,V)\nabla f(x_0) = 0$,
$\nabla Rm(x_0,t_0)(U,V,U,V) = -\frac{\nabla f(x_0)}{f(x_0)}Rm(x_0,t_0)(U,V,U,V)$ and hence
$|\nabla Rm(x_0,t_0)(U,V,U,V)\nabla f(x_0)| \le C$ by the property (iii) of ξ_z above. The fourth term is bounded above because $Rm(x_0,t_0)(U,V,U,V) \ge 0$ and $\Delta f = \Delta\xi_z(x_0)\frac{1}{|U\wedge V|^2} + 2\nabla\xi_z(x_0)\nabla\left(\frac{1}{|U\wedge V|^2}\right) + \xi_z(x_0)\Delta\left(\frac{1}{|U\wedge V|^2}\right) \ge C$ by (0.6) and the property (ii) of ξ_z. Thus by (0.8) we have $\frac{\partial\phi_z}{\partial t}(x_0,\sigma_0,t_0) = \frac{\partial\Phi_z(x_0,t_0)}{\partial t} \le C$. Thus $\bar{A}'_z(t) \le C$ for all $z \in M, t \in [0,T]$ and hence $\bar{A}'(t) \le C$ for all $t \in [0,T]$ This concludes the proof in the case $\sup K_{g_t} \ge 0$. The general case can be easily reduced to this one by replacing the function $K_{g_t}(x,\sigma)$ by $K_{g_t}(x,\sigma)+C$. The argument for $\inf K_{g_t}$ is the same except we have to change $K_{g_{t_0}}(x,\sigma)$ to $K_{g_{t_0}}(x,\sigma) - C$ to ensure that $\inf(K_{g_{t_0}}(x,\sigma) - C) \le 0$.

□

Remark 0.9. In the proof of Proposition 0.4 we can actually always assume that $\inf K_{g_t} \le 0$ since otherwise the manifold M is compact and our statement is known by [Ron96].

Remark 0.10. By changing the cutoff function $\xi_z(\cdot)$ to $\chi(d(\cdot,z)/R)$ in the proof of Proposition 0.4 we see that the same proof actually shows that the *local* maximum and minimum of the curvature vary linearly. Namely, under condition of the Proposition, for any $R > 0$ there exists $C = C(T,R)$ such that for any $z \in M$ we have

$$\inf_{B(z,R)} K_g - C(n,R,T)t \leq K_{g_t}|_{B(z,R)} \leq \sup_{B(z,R)} K_g + C(n,R,T)t$$

However, as constructed, $C(n,R,T) \to \infty$ as $R \to 0$.

Remark 0.11. A slightly more careful examination of the proof of Proposition 0.4 shows that the local rate of change of the curvature bounds is proportional to the local absolute curvature bounds, i.e $\bar{A}'_z(t) \leq C(n,T)\cdot sup_{x\in B(z,2)}|Rm(x,t)|$. In particular, if (M^n,g) is asymptotically flat then so is (M^n,g_t) and it has the same curvature decay rate as (M^n,g). The only difference is that one has to notice that when we change $K_{g_t}(x,\sigma)$ by $K_{g_t}(x,\sigma)+C$ to ensure that $\sup_{x\in B(z,2)}(K_{g_t}(x,\sigma)+C) \geq 0$, the size of C is comparable to $sup_{x\in B(z,2)}|Rm(x,t)|$.

Alternatively one can argue as follows. Equation (0.5) yields

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 + P(Rm) \tag{0.12}$$

And the rest of the proof is the same as before if we apply the maximum principle to $|Rm|^2\xi_z(x,t)$.

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Vitali Kapovitch, Department of Mathematics, University of California, Santa Barbara, CA 93110

E-mail address: vitali@math.ucsb.edu