University of Toronto

Faculty of Arts and Science

Quiz 1

MAT2371Y - Advanced Calculus

Duration - 1 hour

No Aids Permitted

Surname: ______________________________

First Name: ______________________________

Student Number: ______________________________

Tutorial:

T0101	**T5101**	**T5102**
T4/R4	**T5/R5**	**T5/R5**
Chris	**Anne**	**Ivan**
SS1074	**SS1070**	**BA1240**

This exam contains 5 pages (including this cover page) and 3 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	12	
2	13	
3	5	
Total:	30	

1. Let S be a set, and suppose that $A \subseteq S$ and $B \subseteq S$.
   (a) (8 points) Prove that $A \subseteq B^c$ if and only if $A \cap B = \emptyset$

($\Rightarrow$) By contrapositive, show $A^c \cup B^c = S \Rightarrow B \subseteq A^c$

+4

Let $x \in B$ then since $A^c \cup B^c = S$ and $x \notin B^c$

we must have $x \in A^c$

($\Leftarrow$) Let $x \in A$ then since $A \cap B = \emptyset$ there are

no elements which are in both A and B,

+4

so $x \in A^c$

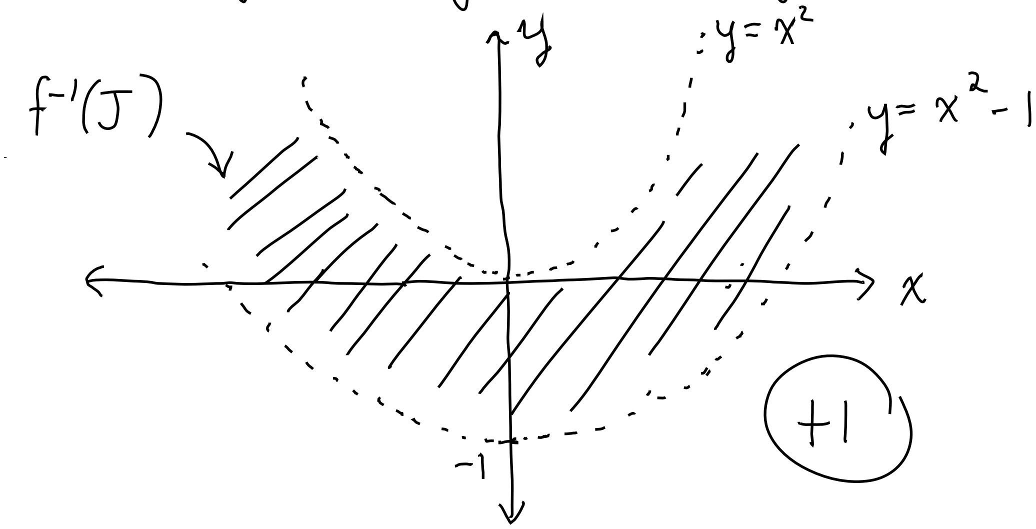
   (b) (4 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x^2 - y$, and let J be the interval $(0, 1) \subseteq \mathbb{R}$. Describe $f^{-1}(J)$ using set builder notation, and make a sketch of the region in $\mathbb{R}^2$ by shading the set of points in $f^{-1}(J)$. Label your sketch, and be as precise as possible.

+1 $f^{-1}(J) = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) \in (0,1)\}$

For $f(x,y) \in (0,1)$, $\exists\, c \in (0,1)$ s.th.

+2

$f(x,y) = x^2 - y = c \iff y = x^2 - c$



+1

2. (a) (4 points) Prove that if U and V are open subsets of $\mathbb{R}^n$, then $U \cap V$ is open.

+1 If $U \cap V = \emptyset$ then it is open, otherwise assume $U \cap V \neq \emptyset$.

Choose $x \in U \cap V$, then $x \in U$ and $x \in V$. As U, V are open, $\exists\ \varepsilon_1, \varepsilon_2 > 0$ s.th. $B(\varepsilon_1, x) \subseteq U$ and

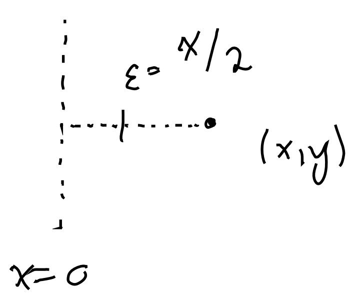
+3 $B(\varepsilon_2, x) \subseteq V$. Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then $B(\varepsilon, x) \subseteq B(\varepsilon_i, x)$ for $i = 1, 2$, so $B(\varepsilon, x)$ is contained in both U and V.

(b) (3 points) For the following parts, let $S = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 1\}$. Prove that S is open. (Hint: Apply part (a)!)

We show that $U = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ is open; the proof for $V = \{(x,y) \in \mathbb{R}^2 \mid y > 1\}$ is identical.

Let $p = (x, y) \in \mathbb{R}^2$ be in U, so $x > 0$.

+2



Set $\varepsilon = x/2$. Claim that $B(\varepsilon, p) \subseteq U$. If $q = (x', y') \in B(\varepsilon, p)$ then $(x - x')^2 \leq (x - x')^2 + (y - y')^2$

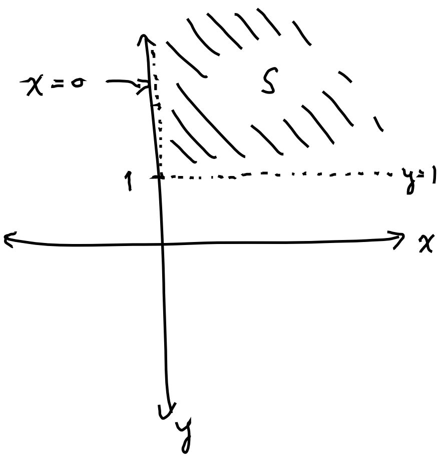
so $|x - x'| \leq \|p - q\| < \varepsilon \Rightarrow x' - x > -\varepsilon$

$x' = x + x' - x > x - \varepsilon = \frac{x}{2} > 0$ so $x' \in U$. (cont on last page)

(c) (3 points) Give the definition of ∂S. Write down (without proof) what is ∂S for the set S from part (b).

+1

The boundary of S is $\partial S = \{ x \in S \mid \forall \varepsilon > 0 \ B(\varepsilon, x) \cap S \neq \emptyset \text{ and } B(\varepsilon, x) \cap S^c \neq \emptyset \}$



For the set from pt (b)

$$\partial S = \{ (x,y) \in \mathbb{R}^2 \mid x = 0, y \geq 1 \} \cup \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y = 1 \}$$

+2

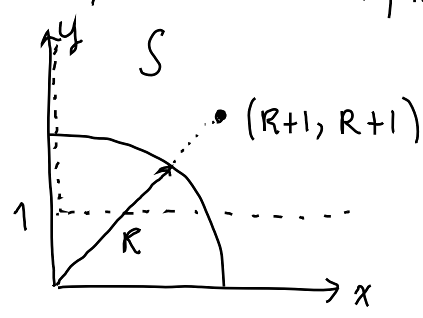
(d) (3 points) Give the definition of what it means for a set S to be bounded. Prove that the set S from part (b) is not bounded.

+1 A set $S \subseteq \mathbb{R}^n$ is bounded iff $\exists R > 0$ s.th. $S \subseteq B(R, 0)$

Claim: The set from pt (b) is not bounded +1

Proof: Need to show $\forall R > 0$, $\exists p \in S$ and $p \notin B(R, 0)$

- If $R \leq 1$, pick any $p \in S$
- If $R > 1$ then let $p = (R+1, R+1)$
  - $p \in S$ because $R+1 > 0$ and $R+1 > 1$
  - $p \notin B(R,0)$ because $\|p\| = \sqrt{2}(R+1) > R$



3. (a) (5 points) Prove that the sequence in $\mathbb{R}^2$ given by $(\cos(\pi k)/k, \sin(\pi k)/k)$ converges to $(0, 0)$.

Fix $\varepsilon > 0$, then estimate:

$$\|x_k\| = \sqrt{\left(\frac{\cos(\pi k)}{k}\right)^2 + \left(\frac{\sin(\pi k)}{k}\right)^2}$$

+5

$$= \sqrt{\frac{1}{k^2}\left(\cos^2 \pi k + \sin^2 \pi k\right)}$$

$$= \frac{1}{k}$$

Choosing any integer $N > \frac{1}{\varepsilon}$ ensures that if $k > N$ then $\|x_k\| < \frac{1}{k} < \varepsilon$, so $x_k \to (0,0)$.

Alternatively, you could show that both $\frac{\cos \pi k}{k} \to 0$ and $\frac{\sin \pi k}{k} \to 0$.

Question 2(b) cont;

+1

Now since both U and V are open and

$$S = \{(x,y) \mid x > 0 \text{ and } y > 1\} = U \cap V$$

by part (b) S is open.