University of Toronto

Faculty of Arts and Science
Quiz 2
MAT2371Y - Advanced Calculus
Duration - 50 minutes
No Aids Permitted

Surname: ______________________________

First Name: ______________________________

Student Number: ______________________________

Tutorial:

T0101	**T5101**	**T5102**
T4/T5	**R4/R5**	**T5/R5**
Chris	**Anne**	**Ivan**
SS1074	**SS1070**	**BA1240**

This exam contains 8 pages (including this cover page) and 3 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	8	
2	13	
3	9	
Total:	30	

1. (a) (2 points) Give any equivalent definition of the statement, "$K \subseteq \mathbb{R}^n$ is compact".

K is compact $\Leftrightarrow$ $\forall \{x_k\}_{k=1}^{\infty} \subseteq K$, $\exists \{x_{k_n}\} \subseteq \{x_k\}$ and $x \in K$ s.th. $x_{k_n} \to x$.

$\Leftrightarrow$ K is closed and bounded

$\Leftrightarrow$ Every open cover has a finite subcover

(+2) for any of the above.

(b) (3 points) Give an example of a compact set. You do not need to prove the set is compact. Give an example of a non-compact set, and say why your set is not compact.

- $K = [0,1] \subseteq \mathbb{R}$ is a compact set.
- $S = [0,1)$ is not compact because it is not closed
- $S = [0,\infty)$ is not compact because it is not bounded

(+1) Correct example

(+1) Correct counterexample

(+1) Correct reason

(c) (3 points) Prove that if $K \subseteq \mathbb{R}^n$ is compact and $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous, then $f(K)$ is compact. (Make note of the point value of this problem and budget your time accordingly)

Let $\{y_k\}_{k=1}^{\infty} \subseteq f(K)$ be any sequence; for every k, $\exists\, x_k \in K$ s.th. $y_k = f(x_k)$. Since K is compact, $\exists \{x_{k_n}\} \subseteq \{x_k\}$ and $x \in K$ s.th. $x_{k_n} \to x$. Now,

$$f(x) = f\left(\lim_{n\to\infty} x_{k_n}\right) \quad \text{as } f \text{ is continuous,}$$
$$= \lim_{n\to\infty} f(x_{k_n})$$

So $f(x_{k_n}) \to f(x) \in f(K)$ is a subsequence of $\{y_k\}$ converging in $f(K)$. This shows $f(K)$ is compact.

(+1) Build sequence x_k

(+1) Apply compactness of K

(+1) Use continuity

2. (a) (4 points) Prove that $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

Fix $\varepsilon > 0$.

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x|\,|y|}{\sqrt{x^2+y^2}} \qquad \text{since } \frac{|x|}{\sqrt{x^2+y^2}} \leq 1$$

$$\leq |y|$$

$$\leq \sqrt{x^2+y^2}$$

$$< \delta \qquad \text{Choose } \delta = \varepsilon.$$

$$= \varepsilon$$

So $\lim_{(x,y)\to 0} \frac{xy}{\sqrt{x^2+y^2}} = 0$

(+1) Estimate correct quantity

(+1) Use $|x| \leq \sqrt{x^2+y^2}$ or some variant

(+1) Use $|y| \leq \sqrt{x^2+y^2}$

(+1) Choose δ.

(b) (3 points) State the ϵ-δ definition of continuity for a function $f : \mathbb{R}^n \to \mathbb{R}^k$. State the definition of what it means for $f : \mathbb{R}^n \to \mathbb{R}^k$ to be uniformly continuous.

$f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous $\Leftrightarrow \forall x \in \mathbb{R}^n\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{R}^n$

if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$f : \mathbb{R}^n \to \mathbb{R}^k$ is uniformly cont. $\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \mathbb{R}^n$

if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Each def. worth +1.5

(c) (2 points) Give an example of a continuous function which is not uniformly continuous. You do not need to prove that your example is not uniformly continuous.

$f : (0, \infty) \to \mathbb{R}$ is continuous but not
$x \mapsto \frac{1}{x}$

uniformly continuous.

+2 correct example.

(d) (4 points) Show that if $\{x_k\} \subseteq \mathbb{R}^n$ is Cauchy and $f : \mathbb{R}^n \to \mathbb{R}^k$ is uniformly continuous, then $\{f(x_k)\} \subseteq \mathbb{R}^k$ is Cauchy.

Fix $\varepsilon > 0$. Since f is uniformly continuous, $\exists\, \delta > 0$ s.th. $\forall x, y \in \mathbb{R}^n$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. The sequence $\{x_k\}$ is Cauchy, so choose N so that if $i, j > N$ then $|x_i - x_j| < \delta$. But now $|x_i - x_j| < \delta$ implies $|f(x_i) - f(x_j)| < \varepsilon$ whenever $i, j > N$. This has shown that $\{f(x_k)\}$ is Cauchy.

$\square$

4 marks TA discretion.

3. (a) (2 points) State the definition of, "$S \subseteq \mathbb{R}^n$ is not connected"

S is not connected $\Leftrightarrow \exists\, S_1, S_2 \subseteq \mathbb{R}^n$ s.th.

① $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$

② $S = S_1 \cup S_2$

③ $\overline{S_1} \cap S_2 = \emptyset$ and $S_1 \cap \overline{S_2} = \emptyset$

+2 for correct defn

(b) (4 points) Show that a finite set of points $S = \{p_1, \ldots, p_k\} \subseteq \mathbb{R}^n$, where $p_1 \neq p_2 \neq \cdots \neq p_k$, is not connected.

Let $S_1 = \{p_1\}$, $S_2 = \{p_2, \ldots, p_n\}$

① $p_1 \in S_1 \Rightarrow S_1 \neq \emptyset$. $p_2 \in S_2 \Rightarrow S_2 \neq \emptyset$

② $S_1 \cup S_2 = \{p_1, p_2, \ldots, p_n\} = S$

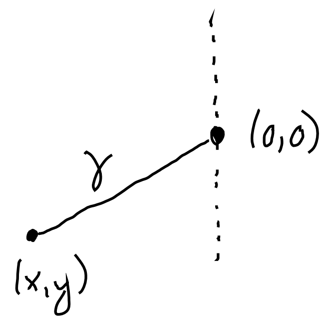
③ S_1 and S_2 are closed, so $\overline{S_1} = S_1$ and $\overline{S_2} = S_2$.
But since $S_1 \cap S_2 = \emptyset$ then $\overline{S_1} \cap S_2 = \emptyset$ and
$S_1 \cap \overline{S_2} = \emptyset$

(+1) for any disconnection

(+1) for each axiom.

(c) (3 points) Recall from class that $\{(x,y) \in \mathbb{R}^2 \mid x \neq 0\}$ is not connected.
Prove that $S = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\} \cup \{(0,0)\}$ is connected. (Hint: Showing this by directly negating the definition is hard. Do you know a better way?)

If S is path connected, then S is connected. It suffices to prove S is path connected. We show that any point $p=(x,y) \in S$ can be joined to $(0,0)$ through S using a straight line.



Define $\gamma : [0,1] \to \mathbb{R}^2$ by $\gamma(t) = (tx, ty)$

- If $t=0$ then $\gamma(0) = (0,0) \in S$
- If $t \neq 0$ then $\gamma(t) = (tx, ty)$ and $tx \neq 0$. b.c. $x \neq 0$ so $\gamma(t) \in S$.

This shows $\gamma \subseteq S$. Since $\gamma(0) = (0,0)$ and $\gamma(1) = (x,y)$, we have shown S is path conn. $\square$

(+1) path conn $\Rightarrow$ conn.

(+1) Constructing γ.

(+1) $\gamma \subseteq S$