University of Toronto

Faculty of Arts and Science
Quiz 3
MAT2371Y - Advanced Calculus
Duration - 50 minutes
No Aids Permitted

Surname: ______________________________

First Name: ______________________________

Student Number: ______________________________

Tutorial:

T0101	**T5101**	**T5102**
T4/T5	**R4/R5**	**T5/R5**
Chris	**Anne**	**Ivan**
SS1074	**SS1070**	**BA1240**

This exam contains 6 pages (including this cover page) and 3 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	8	
2	10	
3	12	
Total:	30	

1. (a) (8 points) Let C be the curve formed by intersecting the sphere, $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1\}$, with the plane $P = \{(x,y,z) \in \mathbb{R}^3 \mid x+z=1\}$. Find a map $\gamma : [0,1] \to \mathbb{R}^3$ which parameterizes C.

Eliminate the z-variable to find a simpler eqn. x and y must satisfy.

$$x^2+y^2+z^2=1 \qquad \rightsquigarrow \qquad x^2+y^2+(1-x)^2=1$$

$$z = 1-x \qquad\qquad 2x^2-2x+y^2=0 \qquad \text{Complete the square}$$

$$2\left(x-\tfrac{1}{2}\right)^2+y^2=\frac{1}{2}$$

$$(2x-1)^2+2y^2=1$$

Now we can solve for $x(t)$, $y(t)$. Notice the relation above is "almost" $x^2+y^2=1$

Set $\quad x(t) = \frac{1}{2}(\cos 2\pi t+1) = \cos^2 \pi t$

$\quad y(t) = \dfrac{\sin 2\pi t}{\sqrt{2}}$

Solve: $\quad z(t) = 1-x(t) = \sin^2 \pi t$

So $\gamma : [0,1] \longrightarrow C$

$$\gamma(t) = \begin{pmatrix} \cos^2 \pi t \\ \dfrac{\sin 2\pi t}{\sqrt{2}} \\ \sin^2 \pi t \end{pmatrix}$$

parameterizes C.

Marking guide:

- 3 marks for some attempt at substitution (i.e. $z=1-x$ into $x^2+y^2+z^2=1$)
- 2 marks for trying to find $x(t), y(t), z(t)$ in so that eqns are satisfied.
- 1 mark correct answer

2. (a) (2 points) Define what it means for $f : \mathbb{R}^n \to \mathbb{R}^k$ to be differentiable at a point $a \in \mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at $a \in \mathbb{R}^n$

$\Leftrightarrow \exists \; Df_a : \mathbb{R}^n \to \mathbb{R}^k$ a linear map s.th.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - Df_a(h) \|}{\|h\|} = 0$$

−1 mark for remembering Df is linear

−1 mark for the limit

(b) (4 points) Using any method you know, show that $f(x, y) = \sin(xy)$ is differentiable at any point (a, b)

Thm: $f \in C^1 \Longrightarrow f$ is differentiable

Suffices to show the partial derivatives are continuous.

Indeed, $$\frac{\partial f}{\partial x} = y \cos(xy)$$
$$\frac{\partial f}{\partial y} = x \cos(xy)$$

Which are continuous, as products and compositions of continuous functions.

−2 marks for theorem

−1 mark computation of $\partial_i f$

−1 mark justifying continuity

(c) (4 points) Let $\gamma : [0,1] \to \mathbb{R}^3$ be the curve $\gamma(t) = (\cos t, \sin t, t)$, and let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x,y,z) = x^2 + y^2 + z^2$. Find $(f \circ \gamma)'(0)$ using the chain rule.

The chain rule tells us that $(f \circ \gamma)'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t)$

$$\gamma(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \nabla f(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} \text{ so } \nabla f(\gamma(0)) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} \text{ so } \gamma'(0) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$(f \circ \gamma)'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0)$$
$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0$$

- 2 marks for knowing chain rule
- 2 marks for surrounding computations

3. (a) (2 points) Give a statement of the multivariable Taylor's theorem for a $C^{k+1}(U)$ function, where $U \subseteq \mathbb{R}^n$ is open. Present any form of the remainder. You do not need to define any of the multi-index notation.

Let $f \in C^{k+1}(U)$, where $U \subseteq \mathbb{R}^n$ is a convex open set containg points $a, a+h \in \mathbb{R}^n$. Then,

$$f(a+h) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha + R_{a,k}(h)$$

where $\exists\, c \in (0,1)$ s.th.

$$R_{a,k}(h) = \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(a+ch)}{\alpha!} h^\alpha$$

- 1 mark for formula of Taylor polynomial
- 1 mark for remainder

(b) (3 points) Use Lagrange's form of the remainder to estimate the error on the interval $h \in [0, 1]$ of the 3rd order Taylor expansion of $f(x) = e^x$ about the point $x = 0$.

Lagrange's remainder: $\exists\, c \in (0,1)$ s.th. $R_{0,3}(h) = \dfrac{f^{(4)}(c)\, h^4}{4!}$

Since $f(x) = e^x$, $f^{(4)}(x) = e^x$ and $|f^{(4)}(x)| < e < 3$ on $[0,1]$

Now $$|R_{0,3}(h)| = \frac{|f^{(4)}(c)|\,|h|^4}{4!} < \frac{3}{4!}(1) = 0.125$$

- 1 mark for Lagrange's remainder
- 2 mark for any correct estimates

(c) (7 points) Find the 3rd order Taylor polynomial of $f(x,y) = \log(1+x-y)$ based at the point $(0,0)$

Let $g(x,y) = x-y$. At the basepoint $(0,0)$ we have $g(0,0) = 0$.

We may use the 1-variable expansion of $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ expanded about $z=0$, then substitute $z = x-y$.

$$\log(1+x-y) = (x-y) - \frac{(x-y)^2}{2} + \frac{(x-y)^3}{3} - \ldots$$

$$= x - y - \frac{x^2}{2} + xy - \frac{y^2}{2} + \frac{x^3}{3} - x^2y + xy^2 - \frac{y^3}{3} + \cdots$$

By uniqueness of Taylor polynomials, this is the 3rd order Taylor polynomial.

- TA disgression for marks.