MAT237 - Tutorial 1

1 Coverage

Here's what they've covered in class so far, which roughly comprises the material covered in this tutorial.

Set Theory: Set containment, indexed families of sets, intersection, union, complement, functions, image, preimage

Topology of $\mathbb{R}^n$: $\mathbb{R}^n$ as a vector space, norms and distances, Cauchy-Schwarz inequality, cross products, boundaries, interiors, exteriors, open, closed, closures

2 Problems

I suggest going over the following four problems. Some are from the Big List, some not. I'll give you a quick idea of the solution if it seems necessary, and discuss what's worth stressing or notable connections to other things.

Give them a bit of time to think about each one, and ask for ideas about how to proceed. Really try to drag the proofs out of them if it seems like they're being reluctant. Note that this tutorial will be more proof-heavy than most, since the only non-proof questions they can do now are about finding interiors and closures of things, checking that sets are open, etc.

1. (Big List 1.5) Let A, B be sets and $f : A \to B$ a function. Prove the following statements:

   (a) $\forall X \subseteq A$, $X \subseteq f^{-1}(f(X))$.

   (b) $\forall Y \subseteq B$, $f(f^{-1}(X)) \subseteq Y$.

   (c) If f is injective, $\forall X \subseteq A$, $X = f^{-1}(f(X))$.

   (d) If f is surjective, $\forall Y \subseteq B$, $f(f^{-1}(X)) = Y$.

   Comments: I'm sure I don't need to write out solutions for this. This is of course very basic. Some students on Piazza seemed to have difficulties getting their heads around it though. We, as mathematicians, look at this as just "unwinding definitions", and they see it as a mess of notation they don't understand. When explaining solutions to the first two parts especially, emphasize how little you have to actually think to solve this problem, so long as you know what all the symbols mean. I also suggest using this chance to emphasize the difference between preimages of sets and inverses of functions. That's another thing I saw people confused about.

   I suspect many people will have already done and understood this question, so feel free to go through it quickly if it seems like everyone is on top of it. This is an easy warm up.

2. Let $U_1, \ldots, U_n$ be open subsets of $\mathbb{R}^n$.

(a) Show that $\bigcap_{k=1}^{n} U_k$ is open.

(b) (Big List 2.5, partly) Is an intersection of infinitely distinct many open subsets of $\mathbb{R}^n$ necessarily open? Support your answer completely.

Solution: (a) For ease of notation, let $U = \bigcap_{k=1}^{n} U_k$. First, note that if the intersection is empty, then there's nothing to check and we're done. Otherwise, let $x \in U$. Then $x \in U_k$ for all $k = 1, \dots, n$. Since each U_k is open, by definition there is an $\epsilon_k > 0$ such that $B(\epsilon_k, x) \subseteq U_k$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_k\}$. Then since $\epsilon \le \epsilon_k$ for all k, we have that $B(\epsilon, x) \subseteq B(\epsilon_k, x) \subseteq U_k$ for all k. This implies that $x \in B(\epsilon, x) \subseteq U$, and so U is open.

(b) This is not necessarily true: the intersection of infinitely many open sets can be open, closed, both or neither. Let's give examples just in $\mathbb{R}$.

(i) Let $U_k = (0, k)$, for $k = 1, 2, \dots$. Then their intersection is $(0, 1)$, which is open.

(ii) Let $U_k = (-\frac{1}{k}, 1 + \frac{1}{k})$. Then their intersection is $[0, 1]$, which is closed.

(iii) Let $U_k = (0, 1 + \frac{1}{k})$. Then their intersection is $(0, 1]$, which is neither.

(iv) Let $U_k = (0, \frac{1}{k})$. Their their intersection is empty, which is both open and closed.

Comments: The proof in part (a) is probably hard for them. Even those that could write it correctly probably would miss the case where the intersection is empty. Focus on the logical structure of the proof: Given a point x in U, we're looking for the ϵ that will work for that point, and order to get it we use the only other information available to us which is that the other sets are open. That stuff about taking the minimum of the epsilons looks tricky for them, but once they draw a picture it's all straightforward, so do a lot of pointing at pictures.

For (b), feel free to choose examples in higher dimensions for some of them. I would also note that this is what you need to do to fully answer the question of what can arise as the intersection of infinitely many open sets: give an example of every thing that can happen actually happening.

3. Let's learn about closures! Fix $S \subseteq \mathbb{R}^n$ for the duration.

(a) Prove that S is closed if and only if $S = \bar{S}$.

(b) (Big List 2.12) Let $\mathcal{K} = \{\, A \subseteq \mathbb{R}^n : S \subseteq A, \text{ and } A \text{ is closed} \,\}$. Prove that $\bar{S} = \bigcap \mathcal{K}$. That is, prove that the closure of S equals the intersection of all closed sets that contain it.

Solution: First, let's note what definitions of things they use. For them, a set is closed if its complement is open. They also have a proposition which says that a set is closed if and only if it contains its own boundary. Finally, they define the closure of a set as the union of the set and its boundary.

(a) The ($\Leftarrow$) direction is easy given that proposition: If $S = \bar{S} = S \cup \partial S$, then obviously S contains its boundary.

On the other hand, if S is closed it contains its boundary, and so $S \cup \partial S = S$.

(b) We show containment both ways. First note that if $A \in \mathcal{K}$, then $S \subseteq A$ by definition of $\mathcal{K}$, and therefore $\bar{S} \subseteq \bar{A} = A$. Now if we have $x \in \bar{S}$, then $x \in \bar{S} \subseteq A$ for every $A \in \mathcal{K}$, and so $x \in \bigcap \mathcal{K}$.

On the other hand, let $x \in \bigcap \mathcal{K}$. Since $\bar{S}$ is a closed set containing S we know that $\bar{S} \in \mathcal{K}$, and so in particular we have $x \in \bar{S}$.

Comments: First, I suspect the notation $\bigcap \mathcal{K}$ may be confusing for them. In the Big List, that intersection is just written as $\bigcap_{A \supseteq S,\ A \text{ closed}} A$, which I find clunky, but feel free to use either one. I would emphasize that even though these proofs seem annoying, the end result of them is that closed sets and closures work exactly like you hope they would.

(a) This is pretty straightforward. Some of them may be uncomfortable with the ($\Rightarrow$) direction, since it looks like we didn't do anything. Set unions are probably still weird, artificial seeming things to them, so it may not be clear to them that the union of a set with a subset of itself is equal to the set again.

(b) The second direction should be easy, but the first direction will be hard for them. In particular, the fact that $S \subseteq A \Rightarrow \bar{S} \subseteq \bar{A}$ may not be obvious to them at all even though it feels true. Feel free to prove this fact (which you can do by taking $x \in \bar{S}$ and considering the cases where x is in S or $\partial S \setminus S$ separately), or just sidestep stating that as a fact and prove it for this case.

4. Let's use Cauchy-Schwarz! Let $a_1, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 1$. Prove that

$$\sqrt{a_1^2 + \cdots + a_n^2} \geq \frac{1}{\sqrt{n}}.$$

Solution: Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and let $b = (1, \ldots, 1) \in \mathbb{R}^n$. Then:

$$|a||b| = \sqrt{a_1^2 + \cdots + a_n^2}\sqrt{1 + \cdots + 1} = \sqrt{n}\sqrt{a_1^2 + \cdots + a_n^2} \geq |a \cdot b| = |a_1 + \cdots + a_n| = 1$$

from which the result follows, where we've used the Cauchy-Schwarz inequality in the middle there.

Comments: This seems really easy, but it's tricky to come up with if you're not looking for Cauchy-Schwarz everywhere like we are.

There's a lot you can say about this problem. They were told the definition of a norm in general, so you're welcome to introduce the Taxicab Norm (which they haven't seen), prove it's a norm, and note that the set of points $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ satisfying the condition in the question is two lines which form the part of the "unit circle" in that norm in the first and third quadrants.

If you draw a picture of the situation in $\mathbb{R}^2$ it's quite nice, since the set of points (a_1, a_2) such that $|a_1| + |a_2| = 1$ is a diamond centred at the origin, and the inscribed circle is precisely the circle of radius $\frac{1}{\sqrt{n}}$, which you can read off from the picture and easy geometry. The situation is the same in higher dimensions, though harder to picture. If you can draw an octahedron with its inscribed sphere, go for it. It'll be cool. The picture also tells you that the problem is true for the whole unit circle in the Taxicab norm.