# MAT237 - Tutorial 7 - 9 June 2015

## 1 Coverage

Everything up to the definition of a tangent space in the notes. This tutorial will focus on the definition of differentiability and some exercises surrounding it.

## 2 Problems

I suggest the following problems.

1. (BL 8.10) Let $S \subseteq \mathbb{R}^n$ be open and connected, and let $f : S \to \mathbb{R}$ be differentiable on $S$. Show that if $\nabla f(p) = 0$ for all $p \in S$, then $f$ is constant.
2. (BL 8.6) Show that $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \sqrt{|xy|}$ is not differentiable at the origin.
3. (A particular case of Theorem 9.12 in the notes) Let $U \subseteq \mathbb{R}^2$ be open, let $p \in U$, and let $f = (f_1, f_2) : U \to \mathbb{R}^2$ be a function. Suppose the partial derivatives of $f_1$ and $f_2$ exist and are continuous in an open neighbourhood of $p$. Show that $f$ is differentiable at $p$.

## 3 Solutions and Comments

The two proof questions here hope to showcase how important the mean value theorem is in multivariable calculus. We really drummed into their heads last term in MAT137 that the MVT would be critical to doing everything in this course. I'll comment more after each solution. I expect the students to be pretty lost with this material, so feel free to do only one of the two proofs if time doesn't allow you to do both. As always we would prefer they understand one of them well than both of them poorly.

**Retrospective note**: After having just written out the proof for the third question in some detail, you would be lucky to do just that in one tutorial. I think your best bet is to stick with the first two questions. I may ask James to post the proof of the third question as a supplement to his notes, which mostly gloss over this proof. I won't comment on that proof further in this note.

1. ***Solution***: The proof will be in two parts: a claim, then a proof that the result follows from that claim.

   **Claim.** *The result is true if $S$ is convex (in particular, it's true for open balls).*

   *Proof.* Let $S \subseteq \mathbb{R}^n$ be convex and $f : S \to \mathbb{R}$ be a differentiable function with zero gradient on $S$. Suppose for a contradiction that $f$ is not constant. This means there are two points

$x, y \in S$ such that $f(x) \neq f(y)$. Since $S$ is convex, the straight line connecting $x$ and $y$ lies in $S$. Let $\gamma : [0,1] \to S$ be a differentiable parametrization of this line. Then the function $f \circ \gamma : [0,1] \to \mathbb{R}$ is differentiable, and continuous on $(0,1)$, so we can apply the MVT to this situation.

The MVT says there is a $c \in (0,1)$ such that

$$(f \circ \gamma)'(c) = f(\gamma(1)) - f(\gamma(0)) = f(y) - f(x) \neq 0.$$

We know from the chain rule, however, that $(f \circ \gamma)'(c) = \nabla f(\gamma(c)) \cdot \gamma'(c) = 0 \cdot \gamma'(c) = 0$, contradicting the above. □

This claim shows that if $\nabla f = 0$ on $S$, then $f$ takes the same value on any points in $S$ that can be connected to one another along piecewise-straight paths. We would be done if we could show that any two points in an open, connected subset of $\mathbb{R}^n$ can be connected this way. We essentially do that, but not quite calling it that. We instead mirror the proof that an open, connected subset of $\mathbb{R}^n$ is path connected.

Fix a $p \in S$ arbitrarily. Define the following two sets:

$$S_1 = \{\, x \in S \,:\, f(x) = f(p) \,\}$$
$$S_2 = \{\, x \in S \,:\, f(x) \neq f(p) \,\}$$

Clearly $S = S_1 \cup S_2$, $S_1 \cap S_2 = \emptyset$, and $S_1 \neq \emptyset$ since $p \in S_1$. We show that both $S_1$ and $S_2$ are open. From this it follows that $S_2 = \emptyset$, since if they were both nonempty they would disconnect $S$.

$S_2$ is open more or less by definition of continuity: $S_2$ is the preimage of the open set $\mathbb{R} \setminus \{f(p)\}$ under $f$.

To see that $S_1$ is open, fix $x \in S_1$. Since $x \in S$ and $S$ is open, there is an $\epsilon > 0$ such that $B(\epsilon, x) \subseteq S$. Noting that $B(\epsilon, x)$ is convex, by the claim we have that $f$ is constant on this ball. Since $x$ is the centre of the ball and $f(x) = f(p)$ by assumption, the whole ball is in $S_1$. This completes the proof that $S_1$ is open, which completes the whole thing.

***Comments***: The main moral of this story for them should be the argument in the claim. It's exactly the same as the proof of the same fact for functions of one variable that they've already seen in MAT137. The key of course is that in convex sets, points can be joined by differentiable paths. It seems like the same proof should work without assuming convexity, the problem being that in that case you only have continuous paths, not differentiable paths, and you need differentiable ones to apply the Chain Rule and in turn the MVT. The ability to parametrize things and compose with the parametrization lets you reduce to a single variable situation. This is going to be a recurring theme throughout the course.

The part after that can be done in many ways. One could show that open, connected sets in $\mathbb{R}^n$ are "polygonally connected" or something, but that proof is basically the same as this one, so I prefer the more direct one for this problem. The fact that dividing $S$ into two open, disjoint, nonempty sets shows it's disconnected isn't something they explicitly know, but it's pretty easy to see that it implies the more technical definition of a disconnection they have (the one with $S_1 \cap \overline{S_2} = \overline{S_1} \cap S_2 = \emptyset$). They should have already reviewed the proof that open connected sets are path connected which uses this idea, so this idea isn't new to them.

It's also worth mentioning to them here that this fact generalizes to functions into $\mathbb{R}^m$ instead of just $\mathbb{R}$, but to prove it you need a more general version of the Mean Value Theorem, which they'll surely see soon enough.

2. ***Solution***: If $f$ were differentiable at the origin, then in particular its directional derivatives in all directions would exist. However, along the line $y = x$, the directional derivative doesn't exist. Indeed, letting $u = (\sqrt{2}, \sqrt{2})$ be the unit vector along this line, we have:

$$\begin{aligned}\partial_u f(0) &= \lim_{h \to 0} \frac{f(0 + hu) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\sqrt{|h\sqrt{2}|^2}}{h} \\ &= \sqrt{2} \lim_{h \to 0} \frac{|h|}{h}\end{aligned}$$

which doesn't exist.

***Comments***: This is pretty straightforward once they figure out what it is they want to show, which is that one of the directional derivatives doesn't exist. If they start trying to show this by assuming there is a gradient then trying to derive a contradiction from the definition of the derivative in general, they're not going to get anywhere.

I suppose the moral of this story is then that directional derivatives are somehow the bridge for their understanding between what they know about single variable derivatives, and derivatives of functions from $\mathbb{R}^n$ to $\mathbb{R}$. Left to their own devices they would probably realise that the function isn't differentiable along this line, so all that needs to be stressed is the fact that the existence of the general derivative gives you the existence of directional derivatives in all directions, at which point their intuition can take over.

3. ***Solution***: Let $[Df_p]_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_p$. If $f$ is indeed differentiable at $p$, then we know these must be the components of its derivative. We show that that's the case. That is, we show:

$$\lim_{(h_1,h_2) \to (0,0)} \frac{\|f(p + h) - f(p) - Df_p(h)\|}{\|h\|} = 0,$$

where $h = (h_1, h_2)$.

First, let's simplify notation. Let $F(h) = f(p + h) - f(p) - Df_p(h)$. Phrased this way, we need to show that

$$\lim_{h \to 0} \frac{\|F(h)\|}{\|h\|} = 0.$$

So, fix $\epsilon > 0$. Note that by the triangle inequality,

$$\|F(h)\| \leq |F_1(h)| + |F_2(h)|$$

(where $F_1$ and $F_2$ are the component functions of $F$), and so it suffices to show that we can make $\frac{|F_i(h)|}{\|h\|} < \frac{\epsilon}{2}$, for $i = 1, 2$ by making $h$ sufficiently close to the origin. This allows us to restrict to the somewhat easier world of functions from $\mathbb{R}^2$ to $\mathbb{R}$. We show this for $F_1$, since the case for $F_2$ is analogous.

To be explicit, we always call $h = (h_1, h_2)$ as before and call $p = (p_1, p_2)$. Then:

$$\begin{aligned} |F_1(h)| &= |f_1(p+h) - f_1(p) - \nabla f(p) \cdot h| \\ &= |f_1(p_1 + h_1, p_2 + h_2) - f_1(p_1, p_2) - \partial_x f_1(p) h_1 - \partial_y f_1(p) h_2|, \end{aligned}$$

where by $\partial_x f_1(p)$ we mean $\frac{\partial f_1}{\partial x}\Big|_p$, and similarly for $\partial_y f_1(p)$.

We add and subtract $f_1(p_1, p_2 + h_2)$ inside there then use the triangle inequality to get:

$$\begin{aligned} |F_1(h)| &= |f_1(p_1 + h_1, p_2 + h_2) - f_1(p_1, p_2 + h_2) + f_1(p_1, p_2 + h_2) - f_1(p_1, p_2) - \partial_x f_1(p) h_1 - \partial_y f_1(p) h_2| \\ &\leq |f_1(p_1 + h_1, p_2 + h_2) - f_1(p_1, p_2 + h_2) - \partial_x f_1(p) h_1| + |f_1(p_1, p_2 + h_2) - f_1(p_1, p_2) - \partial_y f_1(p) h_2|. \end{aligned}$$

Let's examine the first term in the second line above.

The arguments of first two terms in there, $f_1(p_1 + h_1, p_2 + h_2)$ and $f_1(p_1, p_2 + h_2)$, differ only in their first coordinate. Since the partial derivative of $f_1$ with respect to $x$ exists near $p$, if $\|h\|$ is small enough (say smaller than some $\delta_1 > 0$) we have that

$$g(x) := f_1(p_1 + x, p_2 + h_2) : [0, h_1] \to \mathbb{R}$$

is continuous on $[0, h_1]$ and differentiable on $(0, h_1)$. Applying the mean value theorem to this function, we get a $c_x \in (0, h_1)$ such that

$$g'(c_x) = \frac{g(h_1) - g(0)}{h_1},$$

or in other words

$$\partial_x f_1(p_1 + c_x, p_2 + h_2) h_1 = f_1(p_1 + h_1, p_2 + h_2) - f_1(p_1, p_2 + h_2).$$

Similarly, the arguments of the terms $f_1(p_1, p_2 + h_2)$ and $f_1(p_1, p_2)$ differ only in their second coordinate, and so if $\|h\|$ is smaller than some $\delta_1$ we can find a $c_y \in (0, h_2)$ such that

$$\partial_y f_1(p_1, p_2 + c_y) h_2 = f_1(p_1, p_2 + h_2) - f_1(p_1, p_2).$$

Putting these two results into our estimate for $|F_1(h)|$, we have that if $\|h\| < \delta_1$, then:

$$\begin{aligned} |F_1(h)| &\leq |\partial_x f_1(p_1 + c_x, p_2 + h_2) h_1 - \partial_x f_1(p) h_1| + |\partial_y f_1(p_1, p_2 + c_y) h_2 - \partial_y f_1(p) h_2| \\ &= |\partial_x f_1(p_1 + c_x, p_2 + h_2) - \partial_x f_1(p)|\,|h_1| + |\partial_y f_1(p_1, p_2 + c_y) - \partial_y f_1(p)|\,|h_2| \end{aligned}$$

At this point, we make use of the continuity of the partial derivatives of $f_1$. Since $\frac{\partial f_1}{\partial x}$ is continuous at $p$, there is a $\delta_x > 0$ such that if $\|q\| < \delta_x$, then

$$|\partial_x f_1(p + q) - \partial_x f_1(p)| < \frac{\epsilon}{4}.$$

Similarly, since $\frac{\partial f_1}{\partial y}$ is continuous at $p$ there is a $\delta_y > 0$ such that if $\|q\| < \delta_y$, then

$$|\partial_y f_1(p+q) - \partial_y f_1(p)| < \frac{\epsilon}{4}.$$

Combining all of this, set $\delta < \min\{\delta_x, \delta_y, \delta_1\}$. Then if $\|h\| < \delta$, we have:

$$\begin{aligned} |F_1(h)| &\leq |\partial_x f_1(p_1 + c_x, p_2 + h_2) - \partial_x f_1(p)|\,|h_1| + |\partial_y f_1(p_1, p_2 + c_y) - \partial_y f_1(p)|\,|h_2| \\ &< \frac{\epsilon}{4}|h_1| + \frac{\epsilon}{4}|h_2|. \end{aligned}$$

Then finally:

$$\frac{|F_1(h)|}{\|h\|} < \frac{\epsilon}{4}\frac{|h_1|}{\|h\|} + \frac{\epsilon}{4}\frac{|h_2|}{\|h\|} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},$$

(where the new information we've used here is that since $0 < c_x < h_1$, then $\|(c_x, h_2)\| < \|h\| < \delta_x$, and since $0 < c_y < h_2$, then $\|(0, c_y)\| < \|h\| < \delta_y$).

This, as we said at the beginning, suffices to show the result.

***Comments***: