MAT237 - Tutorial 7 - 9 June 2015

1 Coverage

They now know a bit more about tangent spaces, and have seen that gradients are perpendicular to them, and point in the direction of greatest change.

2 Problems

This is slightly more vague than usual. I think the students really just want to talk about parametrizations and derivatives and tangent planes a lot, so I'd like this tutorial to be less structured than usual to allow for this.

1. Compute the tangent spaces of the set S in $\mathbb{R}^3$ which is the graph of the function $f(x,y) = x^2 + y^2$. That is, compute the spaces TS_p, where $S = \left\{ (x,y,z) \in \mathbb{R}^3 \,:\, z = x^2 + y^2 \right\}$.

2. (Stuff from BL section 7) Talk about surfaces and parametrizations. I know this is vague, but over the course of the last few days, from reading Piazza and talking to students it's become very clear that most students aren't comfortable with picturing surfaces, nor with parametrizing them. You can do any parts of BL 7.2 and 7.3, for example.

3. If you/the students want, do the third problem from last tutorial that there wasn't time for on Tuesday.

3 Solutions and Comments

1. ***Solution***: By now they've seen the proposition (Prop 9.15 in their notes) that tells them how to compute tangent spaces. I'll go over essentially the proof of that proposition here, which I think is important for them to get, then actually do the computation.

   The definition of TS_p is

   $$TS_p = \left\{ v \in \mathbb{R}^3 \,:\, \exists \gamma : (-\epsilon, \epsilon) \to S \text{ differentiable with } \gamma(0) = p, \text{ such that } v = p + \gamma'(0) \right\}.$$

   Our goal is to phrase this in terms of derivatives somehow. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $F(x,y,z) = z - (x^2 + y^2)$. Then notice that our set S is precisely $F^{-1}(0)$.

   Fix a point $v \in TS_p$. By definition, there is a γ of the form in the definition of TS_p such that $v = p + \gamma'(0)$. Since γ is a curve in S, we know that $F(\gamma(t))$ is identically 0, and also notice that this composition $F \circ \gamma$ is a function from an interval in $\mathbb{R}$ to $\mathbb{R}$. Using the chain rule on this expression, we get

   $$0 = (F \circ \gamma)'(0) = \nabla F(\gamma(0))\gamma'(0) = \nabla F(p)\gamma'(0),$$

where just to be clear, $\nabla F(p)$ is a row vector with three components and $\gamma'(0)$ is a column vector with three components, so this matrix multiplication is just like a dot product. Noting that $\gamma'(0) = v - p$, we have that $\nabla F(p) \cdot (v - p) = 0$. Since v was chosen arbitrarily, this is true for all $v \in TS_p$.

We notice that this equation defines a plane in $\mathbb{R}^3$, since a plane is exactly the set of all vectors orthogonal to a given, fixed vector. Expanding our equation out, the equation of our plane is $\nabla F(p) \cdot v = \nabla F(p) \cdot p$.

Returning to our particular case, we can easily compute that $\nabla F(x, y, z) = (-2x, -2y, 1)$. Therefore, at a point $p = (p_1, p_2, p_3)$, the equation of the tangent plane to S at p is:

$$-2p_1x - 2p_2y + z = -2p_1^2 - 2p_2^2 + p_3^2.$$

To be concrete, let's check that this does what we expect it to do. At the origin, we expect this plane to be horizontal. Plugging in the point $p = (0, 0, 0)$, we get the plane $z = 0$, as expected.

Comments: I think the students in the class at the moment don't have any idea what tangent spaces are, and the definition of the total derivative has totally confused everyone. The main problem, I think, is that they're too fixated on thinking of surfaces as graphs of functions, rather than level sets of functions. As a warm up exercise to this, I suggest going through all this machinery for a function from $\mathbb{R}$ to $\mathbb{R}$, for which they already know a way of finding the tangent line at a point. Express the graph of $y = f(x)$ as the level set at 0 of $y - f(x)$, etc. I think that should be somewhat convincing that this new stuff isn't complete wizardry. You can mention that not all surfaces they will want to consider are graphs of functions. Some of them are implicitly defined functions. A great benefit of this machinery is that it works exactly the same for explicit or implicitly defined surfaces.

By the way, feel free to use a more interesting function that $f(x, y) = x^2 + y^2$ for this question. I wrote that one down because I had discussed it with my students last time, and because it's relatively easy to picture, but if you have a favourite surface then feel free to use that. Problem BL 8.4 has three examples.

General Comments about the second thing:

Like I said, I think students are not comfortable with thinking about what surfaces look like given their equations, nor with parametrizations. For example, I'm sure that most students in the class haven't internalized that graphs of functions are "automatically" parametrized by the functions.

A nice one of these to talk about is $S = \{ (x, y, z) : x^2 + y^2 - z^2 = 4 \}$. This is a one-sheeted hyperboloid, about the z-axis. Its most natural parametrization is

$$(2\cosh(v)\cos(\theta), 2\cosh(v)\sin(\theta), 2\sinh(v)),$$

where $\theta \in [0, 2\pi)$ and $v \in (-\infty, \infty)$. This is more complicated than we'd ever ask them to come up with on a test or something, but it's nice to talk about. It's entirely possible that they've never realised that hyperbolic trig functions parametrize hyperbolas the same way regular trig functions parametrize circles...