MAT237 - Tutorial 11 - 7 July 2015

1 Coverage

Implicit Function Theorem, smooth curves and surfaces. Notably we haven't covered the Inverse Function Theorem in class yet.

2 Problems

I suggest the following problems. I don't think these will take up the whole tutorial (though I might be wrong), but that's fine because the students are getting their tests back today and that will take up some time.

1. (BL 11.1.4) Show that the following system of equations has a solution for sufficiently small values of a:

$$\begin{aligned} x + y + \sin(xy) &= a \\ \sin(x^2 + y) &= a. \end{aligned}$$

2. (BL 11.1.5) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant, C^1 function such that $f'(0) \neq 0$ and $f(x + y) = f(x) + f(y)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x)f(y)$. Determine what conditions (if any) must be imposed on y to ensure that y can be solved for as a function of x on the set $\{ (x, y) \in \mathbb{R}^2 : F(x, y) = 1 \}$. **Bonus: Write down an explicit formula for y as a function of x,**

3. (BL 11.2.4) For what values of c, c_1 and c_2 do the following sets of equations define smooth surfaces?

   (a) $xyz = c$.

   (b) $x^2 + y^2 + z^2 = c_1$ and $x^2 + y^2 - z^2 = c_2$.

3 Solutions and Comments

1. ***Solution***: First of all, note that when $a = 0$, the system obviously has a solution: $(x, y) = (0, 0)$. Our goal is then to show that for some $\epsilon > 0$, the system has a solution for any $a \in (-\epsilon, \epsilon)$. To do this, we define a function $F : \mathbb{R}^3 \to \mathbb{R}^2$ by:

$$F(a, x, y) = \begin{pmatrix} x + y + \sin(xy) - a \\ \sin(x^2 + y) - 2a \end{pmatrix}$$

Then a solution to our set of equations is precisely a point in $F^{-1}(\mathbf{0})$, and so we'd like to show that in a neighbourhood of 0, the points in this level set can be solved for x and y as functions of a. That is, we'd like to apply the Implicit Function Theorem to the point $(0,0,0)$.

We've already noted that $(0,0,0) \in F^{-1}(\mathbf{0})$, so all we need to check is that $\det(d_{(x,y)}F(0,0,0)) \neq 0$. We compute:

$$d_{(x,y)}F = \begin{pmatrix} 1 + y\cos(xy) & 1 + x\cos(xy) \\ 2x\cos(x^2+y) & \cos(x^2+y) \end{pmatrix},$$

and so

$$d_{(x,y)}F(0,0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The determinant of this matrix is 1, and so the Implicit Function Theorem guarantees the existence of an $r > 0$ and a unique function $f : B_r(0) \to \mathbb{R}^2$ such that for all $a \in B_r(0)$, $F(a, f(a)) = \mathbf{0}$, as required.

Comments: I like this problem because it feels like an odd usage of the Implicit Function Theorem. Usually you imagine solving for a few variables in terms of many other variables, but in this case we're solving for two variables as a function of one of them.

More generally though, the trickiest part of this problem for the students will be seeing how to phrase this as something they can apply the Implicit Function Theorem to. Defining that function F and wrapping your head around how the Implicit Function Theorem will solve the problem for you if you apply it to F is the part to stress.

2. ***Solution***: This is stated in explicitly "Implicit Function Theorem terms". So let's just proceed with checking the hypotheses of the theorem and see where we end up.

   Define $G : \mathbb{R}^2 \to \mathbb{R}$ by $G(x,y) = f(x)f(y) - 1 = F(x,y) - 1$. Then G is C^1, and the problem asks us to find when we can solve for y as a function of x on the level set $G^{-1}(0)$. To see where we can do this, we need to check when $\frac{\partial G}{\partial y} \neq 0$.

   We can check that $\frac{\partial G}{\partial y} = f(x)f'(y)$. We'll separately look at each of these terms, and see when they can equal zero.

   First, can $f(x) = 0$? Turns out that it can't. If $f(x) = 0$ and y is an arbitrary real number, then $f(y) = f(x + (y - x)) = f(x)f(y - x) = 0$. Since by assumption f is not constant, this is impossible. So, f has no roots. Along similar lines, the condition on f we're given implies that $f(0) = 1$. Indeed, if x is an arbitrary real number, then $f(x) = f(0 + x) = f(0)f(x)$, whence we have $f(0) = 1$ since $f(x) \neq 0$ as we just showed.

What about $f'(y)$? To examine this, we'll use the definition of the derivative.

$$\begin{aligned} f'(y) &= \lim_{h\to 0}\frac{f(y+h)-f(y)}{h} \\ &= \lim_{h\to 0}\frac{f(y)f(h)-f(y)}{h} \\ &= f(y)\lim_{h\to 0}\frac{f(h)-1}{h} \\ &= f(y)\lim_{h\to 0}\frac{f(h)-f(0)}{h} \\ &= f(y)f'(0) \end{aligned}$$

By assumption $f'(0)\neq 0$, and by the previous argument $f(y)\neq 0$, so we conclude that $\frac{\partial G}{\partial y}$ is never 0, meaning that there is no restriction on the values of y for which y can be solved for as a function of x.

As for the bonus part, simply note that since $f(0)=1$, the formula that will work to express y as a function of x is $y=-x$. In this case, we have: $F(x,y)=f(x)f(y)=f(x+y)=f(x-x)=1$. The astute observer will recognise that f must be an exponential function of the form $f(x)=Ca^x$.

Comments: This problem is *a lot* trickier than it looks, but I like the "follow your nose" feeling of it. At the beginning it's not at all clear it will be this complicated, then it all develops in front of you. I would present this problem by really nudging the students through every part of it. I think the only part that they might not come up with themselves is realising that $f(0)=1$, and then inserting that into the definition of the derivative when calculating $f'(y)$.

3. ***Solution***: (a) In the case where $c=0$, this equation defines the union of the coordinate planes in $\mathbb{R}^3$, which is obviously not smooth. If $c\neq 0$, then define the function $F:\mathbb{R}^3\to\mathbb{R}$ by $F(x,y,z)=xyz-c$. This function is C^1, and our surface is is $F^{-1}(0)$, so to check that our surface is smooth it suffices to show that $\nabla F\neq 0$ on it. Indeed, we compute $\nabla F=(yz,xz,xy)$, which is never zero on our surface by assumption.

   (b) In this case, we're looking at the zero locus of the C^1 function

$$F(x,y,z)=\begin{pmatrix} x^2+y^2+z^2-c_1 \\ x^2+y^2-z^2-c_2 \end{pmatrix}.$$

   As before, we compute the derivative:

$$DF=2\begin{pmatrix} x & y & z \\ x & y & -z \end{pmatrix},$$

   In order for this matrix to have rank 2, we must have that $z\neq 0$ and at least one of $x\neq 0$ or $y\neq 0$. Looking back at our original equations, we can add and subtract them to obtain:

$$2z^2=c_1-c_2 \quad \text{and} \quad 2(x^2+y^2)=c_1+c_2.$$

From the first equation and what we learned above, we need $c_1 - c_2 > 0$. From the second, we require $c_1 + c_2 > 0$. Combining these, we have that our equations define a smooth surface when $c_1 \pm c_2 > 0$.

Comments: Part (a) is entirely straightforward to solve. The resulting surface when $c \neq 0$ has four connected components.

Part (b) is trickier. The solution above is completely algebraic, but you can get good intuition for it by imagining what the intersection of a sphere and a hyperboloid looks like.

It's worth noting that the conditions determined by the solution above are sufficient but not necessary. For example, when $c_1 = c_2 = 1$ our condition fails but the intersection of the two shapes is the unit circle on the xy-plane, which is certainly smooth.