MAT237 - Tutorial 13 - 14 July 2015

1 Coverage

Riemann integration and Jordan measure in $\mathbb{R}$ and $\mathbb{R}^2$

2 Problems

I suggest the following problems.

This week is a little weird, as they've already been tested on the definition of Riemann integrability and Jordan measure, but haven't really moved on to other subjects except those same things in $\mathbb{R}^2$. There aren't many questions on the Big List in these sections that aren't either trivial given what was on the quiz, or almost exactly the same as proofs in the notes.

I had considered putting in BL 12.1.4, which asks them to show that all the different definitions of integrable are equivalent, but it turns out to be very technical in some cases, and I think beyond the reasonable scope of what they should be expected to know. If you have extra time, you might consider doing one or two of those implications. The (a) implies (c) implication in particular is interesting, I think.

1. (BL 12.2.1) Let $f : [a, b] \to \mathbb{R}$ be a bounded, integrable function.
   (a) Show that the graph of f, $\Gamma(f) = \{ (x, f(x)) \in \mathbb{R}^2 : x \in [a, b] \}$ has measure zero.
   (b) If f is non-negative, show that $S = \{ (x, y) \in \mathbb{R}^2 : x \in [a, b],\ y \in [0, f(x)] \}$ is measurable, and $m(S) = \int_a^b f(x)dx$.
2. (BL 12.1.6) Let $f : [a, b] \to \mathbb{R}$ be integrable, and let $g : [a, b] \to \mathbb{R}$ be such that $S = \{ x \in [a, b] : f(x) \neq g(x) \}$ consists of exactly n points. Show that g is also integrable.

3 Solutions and Comments

1. ***Solution***: (a) Fix $\epsilon > 0$, and we'll show that we can cover $\Gamma(f)$ by a finite union of rectangles with total area less than ϵ.

   Since f is integrable, we can find a partition $P = \{a = x_0, \ldots, x_n = b\}$ of $[a, b]$ such that $U_f(P) - L_f(P) < \epsilon$. For each $i = 1, \ldots, n$, we have the obvious fact that:

   $$f([x_{i-1}, x_i]) \subseteq [m_i, M_i],$$

   where as usual we define $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$, and similarly for M_i with the supremum. This means the graph of f restricted to $[x_{i-1}, x_i]$ is inside the rectangle $[x_{i-1}, x_i] \times [m_i, M_i]$, and therefore the union of these rectangles covers $\Gamma(f)$. The sum of their areas is:

$$\sum_{i=1}^{n}(x_i - x_{i-1})(M_i - m_i) = U_f(P) - L_f(P) < \epsilon,$$

as required.

(b) To see that S is measurable, note that its boundary consists of $\Gamma(f)$, along with three straight lines. These all have zero measure, and therefore so does their union, and so in turn S is measurable.

To see that $m(S) = \int_a^b f(x)dx$ we'll show for every partition P of $[a, b]$, that $L_f(P) \leq m(S) \leq U_f(P)$.

On the one hand, the upper sum $U_f(P)$ is by definition the sum of the areas of a finite number of rectangles which cover S, and so $m(S)$, being a lower bound of all such sums, is less than it. That is, $m(S) \leq U_f(P)$.

On the other hand, the lower sum $L_f(P)$ is by definition the sum of the areas of a finite number of rectangles whose union is a subset of S. Therefore, this sum is less than the sum of the areas of any finite collection of rectangles that cover S. This exactly means that $L_f(P)$ is a lower bound for the set of sums of areas of rectangles that cover S. Since $m(S)$ is the greatest lower bound of the same set, it follows that $L_f(P) \leq m(S)$.

Comments: The proof in (a) is a deceptively straightforward one. If you draw the usual picture of the upper and lower sums of a function, the area in between them is a bunch of rectangles, and those are exactly the rectangles we use to cover the graph. It's very intuitive once you've done that, so long as they have a handle on the definition of integrability. The students may be scared by the proof that images of C^1 parameterizations have measure zero in Tyler's notes, but this one is much easier.

The proof for the second one is easy to do, but I think the hard part for them is figuring out what the strategy of the proof will be. $m(S)$ is defined as an infimum, and the integral has a bunch of definitions all of which are either suprema/infima or $\epsilon - \delta$ things. This makes it hard to approach. Once you decide that showing $L_f(P) \leq m(S) \leq U_f(P)$ for all P is the way to go, it more or less falls into place when you consider the definition of a greatest lower bound.

2. ***Solution***: To show that g is integrable, we can use any of the equivalent definitions of integrability. We'll use the third equivalent condition from BL 12.1.4. So, fix $\epsilon > 0$. Note first that g is bounded (since f is bounded, and g differs from f at only finite many places), so let m and M denote lower and upper bounds of g on $[a, b]$, respectively.

   Enumerate $S = \{x_1, \ldots, x_n\}$. For ease of exposition, assume that none of the x_i are endpoints of $[a, b]$. (This assumption doesn't make the proof easier, it just makes for fewer cases with notation.)

   Let W_i, $i = 1, \ldots, n$ be mutually disjoint open intervals that are subsets of $[a, b]$, and such that $x_i \in W_i$, and such that

$$\sum_{i=1}^{n} \ell(W_i) < \frac{\epsilon}{2(M - m)}.$$

Let $W = \cup_{i=1}^{n} W_i$, and let $V = [a,b] \setminus W$. Then by assumption, f and g are equal on V, and so g is integrable on V. This means we can find a partition P such that $U_{g\restriction V}(P) - L_{g\restriction V}(P) < \frac{\epsilon}{2}$. If necessary, refine P by adding in the endpoints of the intervals W_i.

It remains only to check what happens on W. We can check:

$$U_{g\restriction W}(P) - L_{g\restriction W}(P) \leq \sum_{i=1}^{k} (M-m)\ell(W_i) < (M-m)\frac{\epsilon}{2(M-m)} = \frac{\epsilon}{2}.$$

Noting that $U_g(P) = U_{g\restriction W}(P) + U_{g\restriction V}(P)$ completes the proof.

Comments: Not too much to say about this. It very much mirrors the proof of theorem 2.17 from Tyler's notes, since they're basically the same thing. The result generalizes to the case where S is any measure zero set. The key of course is just that you can put intervals around the bad places with a small total length. This trick of bounding g feels like cheating, but it's fine because we're only doing it in finitely many places.