Mathematics
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Generating functions of modules over preprojective algebras and cohomology of MV cycles

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Abstract

The data of fixed $\lambda, \mu \vdash n$ and the difference $\nu = \lambda - \mu$ define

- $\overline{\mathcal{G}^\lambda} \cap S_\mu$ (Coulomb branch)
- $\mathcal{M}_\mu^\lambda$ (Higgs branch)
- $M_\mu^\lambda = \overline{\mathcal{O}_\lambda} \cap \mathcal{T}_\mu$ (Mirković—Vybornov slice)
- $\Lambda(\nu)$ (Lusztig's nilpotent variety)

Definition. The MV cycles of coweight (μ, λ) are the irreducible components of $\overline{\mathcal{G}^\lambda} \cap S_\mu$.

Fact. Let $\mu = 0$. There is a bijection

$$\mathrm{Irr}(\Lambda(\nu)) \longleftrightarrow \{\text{MV polytopes of weight } \nu\} \longleftrightarrow \mathrm{Irr}(\overline{\mathcal{G}^\nu} \cap S_0).$$

The first correspondence is obtained by sending $M \in \Lambda(\nu)$ to its Harder-Narasimhan polytope $\mathrm{Pol}(M) = \mathrm{Conv}(\underline{\dim} N | N \subset M \text{ as } \Lambda\text{-submodule})$ where $\underline{\dim} N = \sum (\dim N_i)\alpha_i$.

Example 1. $M = 2 \to 1$ has submodules $0, 1, 1 \leftarrow 2$ hence $\mathrm{Pol}(M) = \mathrm{Conv}(0, \alpha_1, \alpha_1 + \alpha_2)$.

One would like to upgrade the (combinatorial) correspondence to a geometric one.

Fact. Let $X \subset \mathbb{P}^N$ be a projective variety with a torus action. Its moment map image can be expressed in terms of the induced torus action on sections $\Gamma(X, \mathcal{O}(n))$.

This description suggests that a geometric correspondence may be found by studying (representation theory of) cohomology of MV cycles and corresponding Λ-modules.

Theorem (Conjecture)**.** For $M \in \Lambda(\nu)$ there is a top-dimensional subscheme $X \subset \overline{\mathcal{G}^\nu \cap S_0}$ such that

$$H^\bullet(G^{(n)}(M)) \cong \Gamma(X, \mathcal{O}(n)).$$

Problem. The watered-down version of this conjecture and what I am working on is the claim that the "generating function" of an irreducible component of $\Lambda(\nu)$ coincides with the equivariant multiplicity of the corresponding MV cycle.

Definition. Let $[M] \in \mathrm{Irr}(\Lambda)$. Let $M \in [M]$ be generic. The generating function of $[M]$ is

$$\chi(M) = \sum_{\mathbf{i}=(i_1,\dots,i_N)} \frac{\dim H^\bullet(F^{(\mathbf{i})}(M))}{\alpha_{i_1}(\alpha_{i_1}+\alpha_{i_2})\cdots(\alpha_{i_1}+\cdots+\alpha_{i_N})}$$

where $F^{(\mathbf{i})}$ is a permissible flag of submodules of M.

Example 2. Suppose $M = \ 2 \longrightarrow 1\ $ is generic. It has exactly one two-step flag $F^{(1,2)} = 1 \subset 1 \leftarrow 2$ so $\chi = \frac{1}{\alpha_1(\alpha_1+\alpha_2)}$.

All sets of irreducible components we consider are indexed by (semi-)standard Young tableaux of shape λ and content μ. It is easier to study cohomology of MV cycles in Mirković—Vybornov coordinates and the (generalized) Spalsenstein algorithm tells us how to cook up a generic matrix $A = A_\alpha$ in M_μ^λ given a tableau α.

By means of another algorithm, one produces a generic module $M = M_\alpha$ in $\Lambda(\nu)$. In the case of two-row tableaux one can apply a special algorithm using Dyck paths (see examples).

Baby steps towards verifying the claim

Since we are working in coordinates, with MVyb slices, rather than in $\mathcal{G}$, the first step is to check that this is OK, i.e. that the combinatorial data is intact.

In particular, we check that the Lusztig datum of an irreducible component of an MVyb slice agrees with the Lusztig datum of the corresponding MV cycle.

The Lusztig datum of an MV cycle in $\mathcal{G}$ is defined using certain functions D_γ [Kam10]. Under an alternate $\overline{\mathcal{G}^\lambda} \cap S_\mu \cong M_\mu^\lambda$ isomorphism [CK16]

$$D_\gamma(A) = \min_{|J|=|\gamma|} \mathrm{val}\ \det(tI - A)_{\gamma \times J}.$$

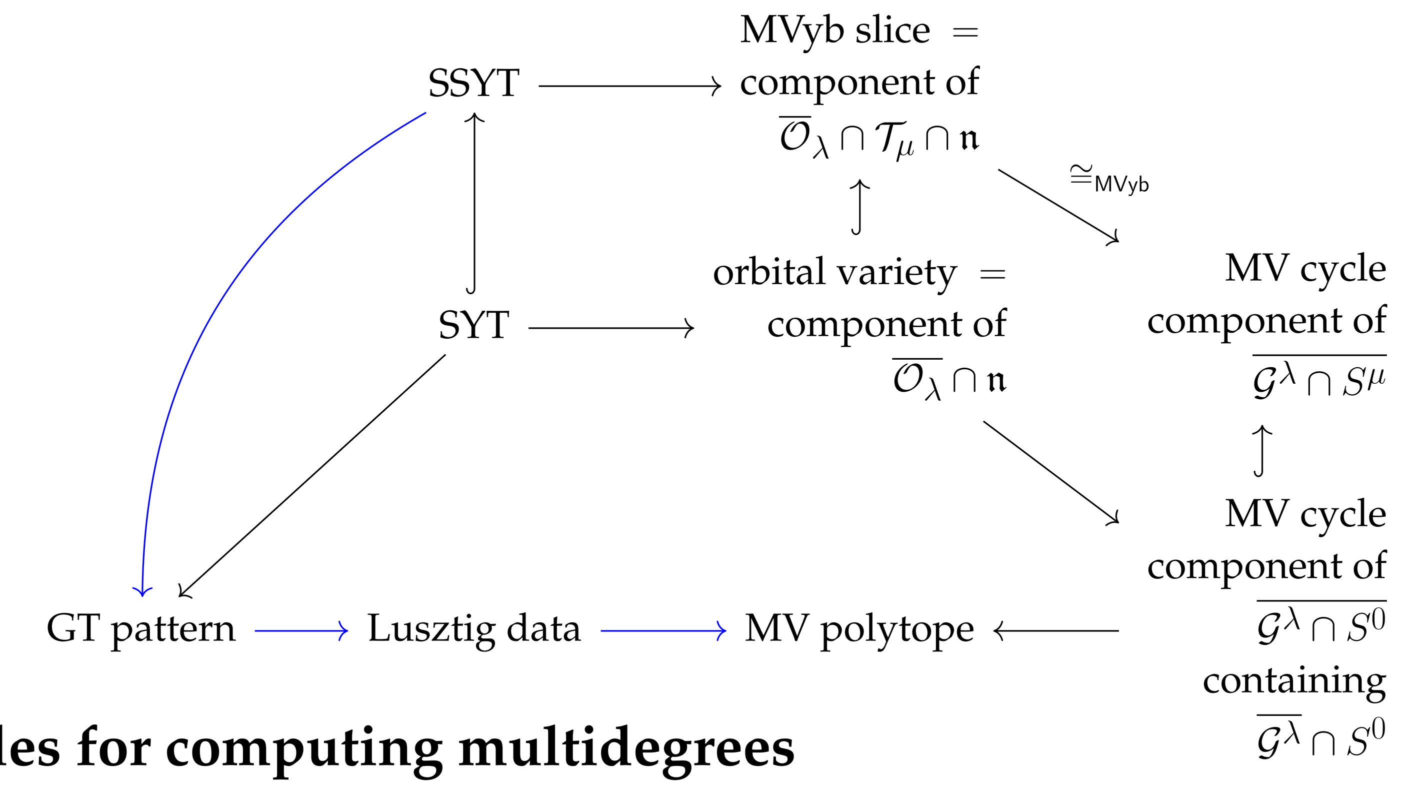
Lemma. $D_{[an]} = \sum_{i=a}^n \lambda_i$.

Corollary. $D_{[ab]} = b - \sum_{i=1}^{a-1} \lambda_i^{(b)}$.

Corollary. Lusztig data agree, $n(A_\alpha) = n(\alpha)$.

Problem. In [ZJ15] the multidegrees of irreducible components of MVyb slices are shown to satisfy the qKZ equations (and more). One idea which I have not made any progress with is to check that generating functions satisfy qKZ too.

Below is a diagram of some of the maps involved in this story. What follows are several examples.



Rules for computing multidegrees

Let $T = (\mathbb{C}^\times)^M$ be a torus, and suppose $(X \subset W)$ is a pair of linear T-reps, with X a T-invariant closed subscheme.

The multidegree of such a pair is a polynomial $\mathrm{mdeg}_W X \in \mathrm{Sym}\, T^* \cong \mathbb{Z}[z_1, \dots, z_M]$ computed as follows.

1. $X = W = \{0\} \Rightarrow \mathrm{mdeg}_W X = 1$
2. If $X \subset W$ has top-dimensional components X_i, then $\mathrm{mdeg}_W X = \sum [X : X_i] \mathrm{mdeg}_W X_i$ where $[X : X_i]$ denotes multiplicity of X_i in X. Thus the case of schemes is reduced to the case of varieties (as reduced irreducible schemes).
3. If X is a variety and $H \subset W$ is a T-invariant hyperplane, then
   (a) $X \not\subset H \Rightarrow \mathrm{mdeg}_W X = \mathrm{mdeg}_H (X \cap H)$
   (b) $X \subset H \Rightarrow \mathrm{mdeg}_W X = \mathrm{mdeg}_H X \cdot (\text{weight of } T \text{ on } W/H)$

Example 3. Let $X = \begin{bmatrix} 0 & 0 & a_2 & a_3 \\ & & a_4 & a_5 \\ & & & 0 \\ & & & 0 \end{bmatrix} \in \mathfrak{n}$. Let $W = V(a_1, a_6) \subset \mathfrak{n}$ and $H = V(a_1, a_6, a_2, a_3, a_4, a_5) \subset W$. Since $X \in W$ and $X \cap H = 0$

$$\mathrm{mdeg}_{\mathfrak{n}}(X) = \{\text{weight of } T \text{ on } \mathfrak{n}/W\} \cdot \mathrm{mdeg}_W(X) = (z_1 - z_2)(z_3 - z_4)$$

Example 4. Let $X = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ & 0 & a_4 & a_5 \\ & & & a_6 \end{bmatrix} \in \mathfrak{n}$ such that $a_1 a_5 + a_2 a_6 = 0$. Let $W = V(a_4) \subset \mathfrak{n}$ and $H = V(a_4, a_2) \subset W$. Since $X \in W$ and $X \cap H = V(a_1 a_5) \subset H$

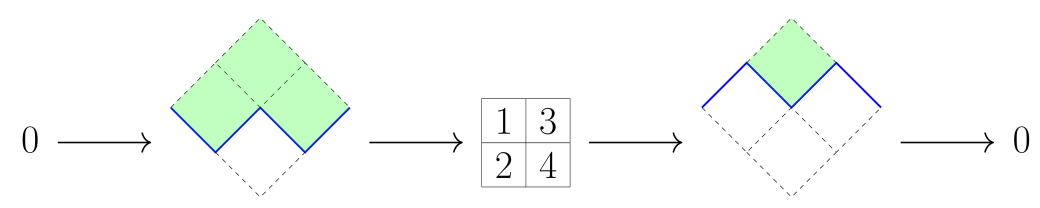
$$\begin{aligned}\mathrm{mdeg}_{\mathfrak{n}}(X) &= \{\text{weight of } T \text{ on } \mathfrak{n}/W\} \cdot \mathrm{mdeg}_W(X) = (z_2 - z_3) \cdot \mathrm{mdeg}_H(X \cap H) \\ &= (z_2 - z_3) \cdot (\mathrm{mdeg}_H(V(a_1)) + \mathrm{mdeg}_H(V(a_5))) \\ &= (z_2 - z_3) \cdot (\{\text{weight of } T \text{ on } H/V(a_1)\} + \{\text{weight of } T \text{ on } H/V(a_5)\}) \\ &= (z_2 - z_3)(z_1 - z_2 + z_2 - z_4) = (z_2 - z_3)(z_1 - z_4)\end{aligned}$$

Bricks

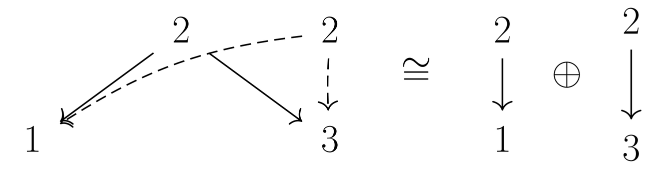
Bricks are submodules of $\Lambda(\nu)$ attached to elements $\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_k$ of the root lattice, computed wrt $\mathbf{i} = (12 \dots n12 \dots n-1 \dots 121)$. Warning! I abuse notation and denote the brick $M(\beta_k)$ by β_k and similarly the module $M(\alpha)$ by α.

Example 5. Let $\mathbf{i} = (123121)$. Then $\mathbf{B} = \left\{ \beta_1 = 1 \not\subset \beta_2 = \begin{smallmatrix} 1 \\ \downarrow \\ 2 \end{smallmatrix} \not\subset \beta_3 = \begin{smallmatrix} 1 \\ \downarrow \\ 2 \\ \downarrow \\ 3 \end{smallmatrix} \not\subset \beta_4 = 2 \not\subset \beta_5 = \begin{smallmatrix} 2 \\ \downarrow \\ 3 \end{smallmatrix} \not\subset \beta_6 = 3 \right\}$.

Example 6. $\alpha = \begin{smallmatrix} 1 & 3 \\ 2 & 4 \end{smallmatrix}$ has Lusztig datum (100110) and determines M_α as an iterated central extension by bricks $\beta_1, \beta_4, \beta_5$, $\ 0 \longrightarrow \beta_1 \longrightarrow \alpha \longrightarrow \beta_4 \oplus \beta_5 \longrightarrow 0\ $. Alternatively, the mnemonic short exact sequence

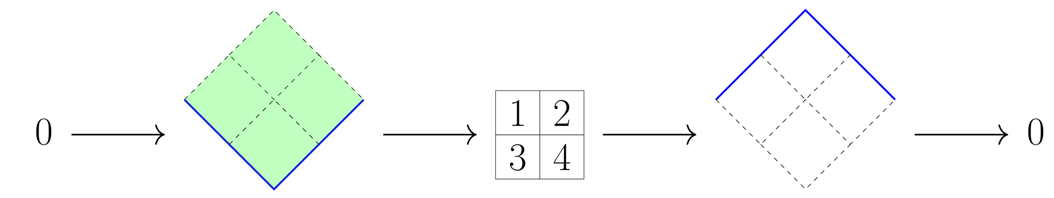


determines M_α =

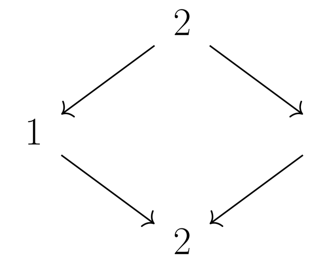


whose permissible flags are $(1,3,2,2), (3,1,2,2), (1,2,3,2), (3,2,1,2)$. Note $2 = \chi((1,3,2,2)) = \chi((3,1,2,2)) = \chi(\mathbb{P}^1)$. Find $\chi(M_\alpha) = \frac{1}{\alpha_1 \cdot \alpha_3 \cdot (\alpha_1+\alpha_2) \cdot (\alpha_2+\alpha_3)}$ agreeing with the multidegree computation in example 4.

Example 7. $\alpha = \begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}$ has Lusztig datum (010010) and determines M_α as an iterated central extension of bricks β_2, β_5, $\ 0 \longrightarrow \beta_2 \longrightarrow \alpha \longrightarrow \beta_5 \longrightarrow 0\ $. Alternatively

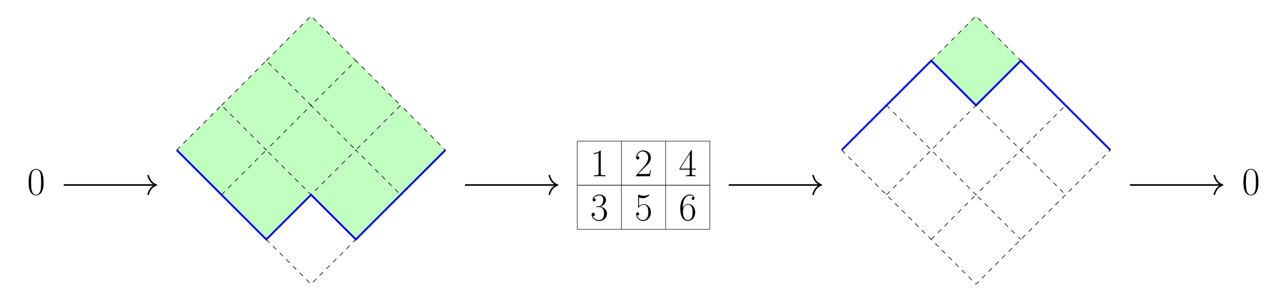


In either case, we get the diamond-shaped module M_α =

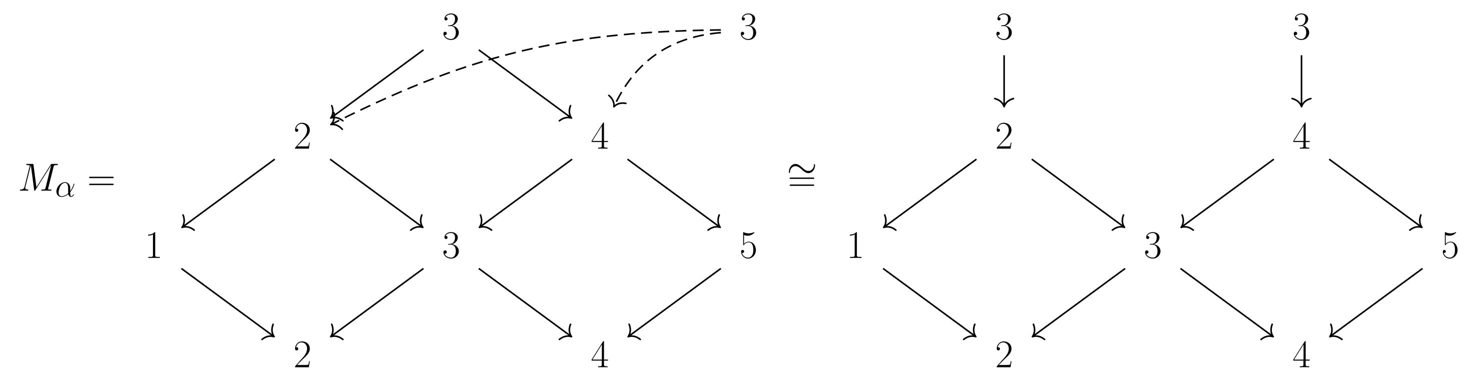


whose permissible flags are $(2,1,3,2), (2,3,1,2)$. Find $\chi(M_\alpha) = \frac{1}{\alpha_2 \cdot (\alpha_1+\alpha_2) \cdot (\alpha_2+\alpha_3) \cdot (\alpha_1+\alpha_2+\alpha_3)}$ agreeing with multidegree computation in example 3.

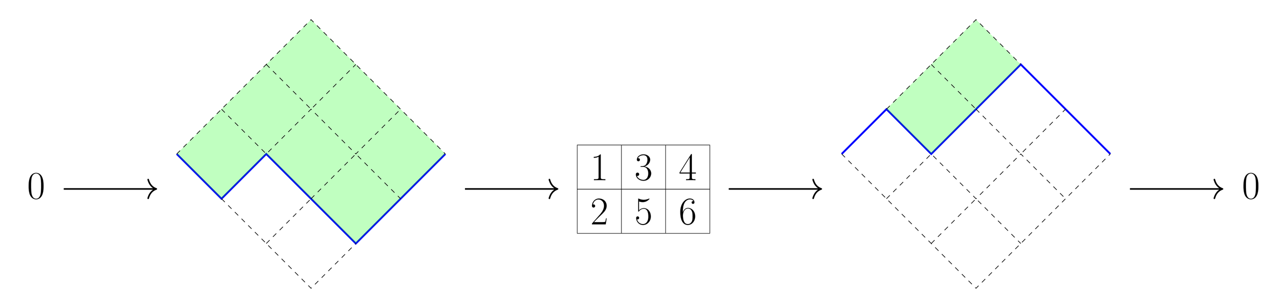
Example 8. Let $\mathbf{i} = (123451234123121)$. Lusztig datum (010000010101000) for $\alpha = \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix}$ corresponds to



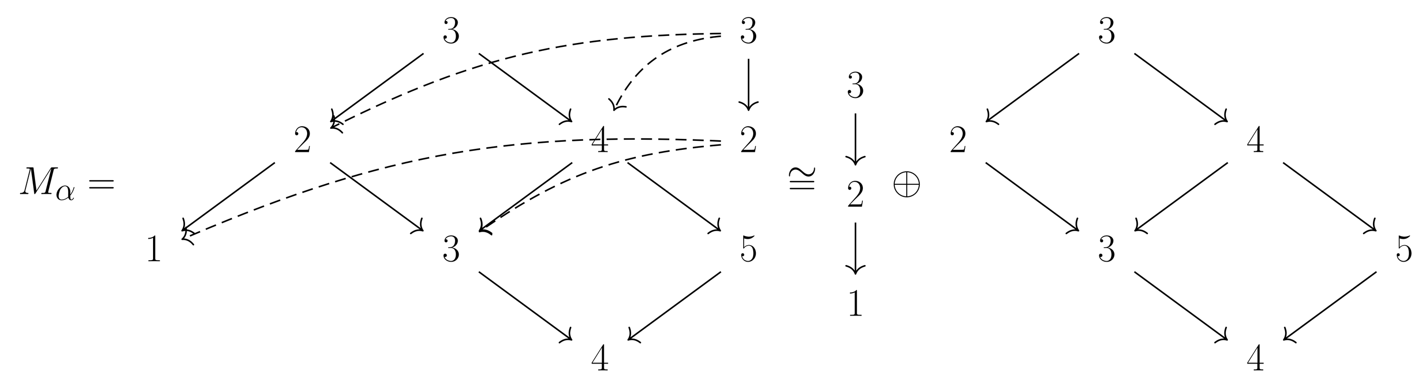
and determines



Example 9. Lusztig datum (100001010101000) for $\alpha = \begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{smallmatrix}$ correspondends to



determines



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