Lusztig datum of an open MV cycle

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Motivation. Lusztig datum originally defined for canonical basis elements as exponents on PBW elements in PBW expansion of canonical basis elements that survive the $q = 0$ limit.

what is an open MV cycle?

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$$A = \left[\begin{array}{ccc|ccc|cc|c}
0 & 1 & & & & & & & \\
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* & * & 0 & * & * & * & * & * & * \\
\hline
 & & & 0 & 1 & 0 & & & \\
 & & & & 0 & 1 & & & \\
* & * & * & * & * & 0 & * & * & * \\
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The Mirkovic-Vybornov slice M_μ^λ. Fix two partitions $\lambda \geq \mu \vdash N$ having at most n parts. Form the subspace T_μ of gl_N whose elements look like $\mu_i \times \mu_j$ almost zero block matrices

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Denote by Z_μ^λ the uppertriangular elements of this slice, i.e. elements of the form

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 & & & & 0 & 1 & & & \\
 & & & & & 0 & * & * & * \\
\cline{4-9}
 & & & & & & 0 & 1 & 0 \\
 & & & & & & & 0 & * \\
\cline{7-9}
 & & & & & & & & 0
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Fact-Aside. Z_μ^λ is Lagrangian in M_μ^λ.

Definition. Irreducible components of Z_μ^λ will be called *open MV cycles.* E.g. when $\mu = (1 \ldots 1)$ these are the *orbital varieties*!

Theorem A

Denote by $SSYT_{\mu}^{\lambda}$ the set of semistandard young tableaux of shape λ and weight μ.

Here the weight of a tableau σ is defined by

$$\text{wt}(\sigma) = (\text{number of boxes in } \sigma \text{ labeled } i : 1 \leq i \leq n)$$

so weight $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ means μ_1 1s, μ_2 2s and so on.

For example

$$\mathrm{wt}\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 2 \\ \hline 1 & 3 \\ \hline \end{array} = (3, 2, 1)$$

For $A \in Z_\mu^\lambda$ define the tableau $\sigma(A) \in SSYT_\mu^\lambda$ by viewing the sequence of Jordan types of principal submatrices

$$\text{shape}(A\big|_{\mathbb{C}^{\mu_1}}), \text{shape}(A\big|_{\mathbb{C}^{\mu_1+\mu_2}}), \ldots, \text{shape}(A)$$

as a sequence of nested Young diagrams and filling boxes in excess regions

$$\text{shape}(A\big|_{\mathbb{C}^{\mu_1+\cdots+\mu_k}}) - \text{shape}(A\big|_{\mathbb{C}^{\mu_1+\cdots+\mu_{k-1}}})$$

with ks for $1 \le k \le n$.

Theorem A. *Fibres of the map $A \mapsto \sigma(A)$ are irreducible and their closures are the irreducible components of Z^{λ}_{μ}.*

Spalsenstein's Theorem. *Let F be the flag variety of n-step flags in an n-dimensional vector space over an algebraically closed field. Let u be a unipotent transformation of F and let F^u be its fixed points. Irreducible components of F^u are in bijection with standard Young tableaux of shape* $\mathrm{shape}(u)$ *and*

1. $\cup_{\tau \geq \sigma} F^u_\tau$ *is closed in* F^u *and* F^u_σ *is locally closed*
2. $\dim F^u_\sigma = \sum_{s \geq 1} d_s(d_s - 1)/2$
3. $F^u_\sigma = \cup Y_j$ *for some* Y_j *affine*

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$$\text{shape}(A\big|_{\mathbb{C}^2}) = \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \rightsquigarrow A\big|_{\mathbb{C}^2} = \begin{bmatrix} 0 & a \\ & 0 \end{bmatrix}$$

and generically $a \neq 0$

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and $ax = 0 \Rightarrow x = 0$

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Conversely, the set $Z_\mu^\lambda = \{A : \dim \operatorname{Ker} A = 2, \dim \operatorname{Ker} A^2 = 4\}$ decomposes into two irreducible components

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and generically elements of the first component map to $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$

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Denote by Z_σ the fibre over σ and consider the restriction

$$Z_\sigma \to Z_\tau : A \mapsto A\big|_{\mathbb{C}^{|\tau|}}$$

for τ a "subtableau" of σ like $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \subset \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$

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Lemma. When τ is the tableau obtained from σ by deleting the last occurrence of the highest weight the fibres of the restriction map are equidimensional affine of dimension

$$\text{highest weight} - \text{length of row containing last occurrence}$$

Note, the lemma is proved by changing basis to the Jordan basis where the claim is trivial.

Another example. Let $A \in Z_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}} \subset Z_{(2,2,1)}^{(3,3)}$ so

$$\dim \operatorname{Ker} A = 2 \quad \dim \operatorname{Ker} A^2 / \operatorname{Ker} A = 4 \quad \dim \operatorname{Ker} A^3 / \operatorname{Ker} A^2 = 6$$

and further constraints imposed by $\sigma(A) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ force it to take the form

$$\left[\begin{array}{cc|cc|c|c} 0 & 1 & & & & \\ & 0 & & x & y & a \\ \hline & & 0 & 1 & & \\ & & & 0 & & b \\ \hline & & & & 0 & c \\ \hline & & & & & 0 \end{array}\right]$$

with $bx + y = 0$

$$Z_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}} \longrightarrow Z_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array}} \longrightarrow Z_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}} \longrightarrow Z_{\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \end{array}} \longrightarrow Z_{\begin{array}{|c|}\hline 1 \\ \hline \end{array}}$$

$$\{\, bx+y=0 \,\} \longmapsto \begin{bmatrix} 0 & 1 & & & \\ & 0 & 0 & x & y \\ & & 0 & 1 & \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 & & \\ & 0 & & x \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \longmapsto [0]$$

$$\mathbb{C}^2_{a,c} \qquad \mathbb{C}_y \qquad \mathbb{C}_x \qquad pt \qquad pt$$

Theorem B

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what are MV cycles? what are open MV cycles?

In type A $\mathrm{Gr} = G((t))/G[[t]]$ has a lattice description in which an MV cycle is an irreducible component of the set of lattices $L \subset L_0 := \mathbb{C}[[t]]^n$ such that multiplication by t on L_0/L has fixed Jordan type λ and $\lim_{s\to 0} s \cdot L = L_\mu$ for fixed μ.

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Question. Does the isomorphism identify irreducible components of Z_μ^λ and MV cycles?

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Theorem. Not quite, so call an irreducible component of Z_μ^λ an *open* MV cycle. Call the image of an irreducible component under the isomorphism by the same name. The Lusztig datum of an open MV cycle however is equal to that of the corresponding MV cycle.

Thank you for listening