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EINSTEIN STRUCTURES: EXISTENCE VERSUS UNIQUENESS

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Abstract

Two well-known questions in differential geometry are "Does every compact manifold of dimension greater than four admit an Einstein metric?" and "Does an Einstein metric of a negative scalar curvature exist on a sphere?" We demonstrate that these questions are related: For every $n \geq 5$ the existence of metrics for which the deviation from being Einstein is arbitrarily small on every compact manifold of dimension n (or even on every smooth homology sphere of dimension n) implies the existence of metrics of negative Ricci curvature on the sphere S^n for which the deviation from being Einstein is arbitrarily small. Furthermore, assuming either a version of the Palais-Smale condition or the plausibly looking existence of an algorithm deciding when a given metric on a compact manifold is close to an Einstein metric, we show for any $n \geq 5$ that: 1) If every n-dimensional smooth homology sphere admits an Einstein metric then S^n admits infinitely many Einstein structures of volume one and of negative scalar curvature; 2) If every compact n-dimensional manifold admits an Einstein metric then every compact n-dimensional manifold admits infinitely many distinct Einstein structures of volume one and of negative scalar curvature.

Einstein metrics are metrics of constant Ricci curvature, i.e. solutions of the Einstein equation $Ric_g = \lambda g$, where λ is a constant. Isometry classes of Einstein metrics are called Einstein structures. Many facts about Einstein structures can be found in [Be] and [S]. A study of Einstein structures is motivated by possible applications to General Relativity and also, according to [Be], by the following question of R. Thom: "Are there any best (or nicest) Riemannian structures on a given compact manifold M?" (Einstein structures are natural candidates to be considered as nice structures.) One of our results (Theorem 3) is that there is no quite satisfactory positive answer to Thom's question. More precisely, assume that one defines nice structures in such a manner that: 1) (Existence) A nice structure exists on every compact manifold of a fixed dimension $n \geq 5$; 2) (Scale-invariance) If a metric is nice, then metrics obtained by the multiplication of this metric by a positive constant are nice; and 3) (Recognizability) It is possible to recognize when a given Riemannian structure is close to a nice structure. Then the set of nice structures of volume one for *every* compact manifold of the dimension n is infinite. This result follows from the S. Novikov theorem stating the non-existence of an algorithm recognizing S^n in the class of n-dimensional manifolds (or even n-dimensional homology spheres). (In the Appendix, we present the proof of a smooth version of this theorem which we use in this paper.) The main idea of this paper is that a sufficiently good positive answer for Thom's question would imply the existence of an algorithm recognizing S^n, which is impossible. The same idea will be used to prove other results of the paper. Informally, Theorem 1 below states that the existence of almost Einstein metrics on homology spheres with "large" fundamental groups implies the existence of almost Einstein metrics with negative scalar curvature on S^n, $(n \geq 5)$. Note that in view of results of [L] the existence of such metrics seems quite plausible. Theorem 2 states that the existence of Einstein metrics on homology spheres implies the existence of infinitely many distinct Einstein structures of volume one and of negative scalar curvature on S^n if a plausible conjecture about algorithmic recognizability of Einstein metrics (namely, statement (D) below) is true. Formulating these theorems the author was, in particular, motivated by the hope that more complicated topology of the space of Riemannian structures on homology spheres with "large" fundamental groups will make proving the existence of Einstein metrics of negative scalar curvature on such homology spheres easier than proving the existence of such metrics on S^n (see also the discussion before the proof of Theorem 1 below).

Let us say that a compact manifold M *almost admits* an Einstein metric of negative scalar curvature if there exists an infinite sequence of metrics $\{\mu_i\}$ on M such that the (C^0) norm of the tensor $Ric_{\mu_i} + \mu_i$ tends to zero. (It is clear that if M admits an Einstein metric of negative scalar curvature, then multiplying this metric by a positive constant one gets an Einstein metric g of scalar curvature $-n$ (hence, $Ric_g + g = 0$).) Note that this definition of the intuitive concept of "almost Einstein metrics" can be modified in various ways, if necessary. It can be replaced, for example, by the following definition (and all results below will remain true): Fix arbitrary positive integer k_1. We could demand the existence of a sequence of Riemannian metrics on M such that $\exp(\|R\|_{C^{k_1}} Diam^2 \max\{1,\ 1/vol\})\|Ric + g\| \to 0$ for this sequence. (R is the curvature tensor of a metric, vol is its volume, and $Diam$ is its diameter). Or, alternatively, we could demand the existence of a sequence of Riemannian metrics on M such that $Diam^2 \|Ric + g\| \to 0$ for this sequence. More generally, we could put before $\|Ric + g\|$ any computable function of volume, diameter and $\|R\|_{C^{k_1}}$.

THEOREM 1. *Assume that for some $n \geq 5$ every n-dimensional smooth homology sphere with fundamental group of exponential growth almost admits an Einstein metric of negative scalar curvature. Then S^n almost admits an Einstein metric of negative scalar curvature.*

Remark: Note, that Milnor's well-known result (cf. [Be, Theorem 6.61]) implies that if the fundamental group of a compact manifold has exponential growth then it does not admit metrics of non-negative Ricci curvature. Hence, it does not admit Einstein metrics of zero or positive scalar curvature.

A corollary of Theorem 1 is that three important problems in the theory of Einstein metrics are related. Namely, for a fixed $n \geq 5$ consider the following three statements:

(A) *Every smooth n-dimensional homology sphere with fundamental group of exponential growth admits an Einstein metric;*

(B) *The sphere S^n does not admit Einstein metrics of negative scalar curvature.*

(C) (*A weak form of the Palais-Smale condition*) *If S^n almost admits an Einstein metric of negative scalar curvature, then it admits an Einstein metric of negative scalar curvature.*

COROLLARY. *For any $n \geq 5$ at least one of the statements* (A),(B),(C) *is false.*

As we will see from the proof of Theorem 1 below, the statement (C) in the corollary can be replaced by the following more awkward but very plausible statement:

(D) (*Recognizability of Einstein metrics; an informal version*) *If a deviation of a metric on S^n from being Einstein of scalar curvature $-n$ does not exceed a certain decreasing positive computable function of the diameter of this metric, the inverse of its volume,the C^k norm of its curvature tensor (for some fixed k), then this metric is close to an Einstein metric of scalar curvature $-n$ on S^n.*

(D) (*One of the possible rigorous versions*) *For any positive δ there exists a positive $\epsilon(\delta, x, v, D)$ such that:*

(i) *The function ϵ is effectively minorizable, i.e. for every $\delta > 0$ there exists a recursive (i.e. Turing computable) increasing function $E_\delta : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\epsilon(\delta, x, v, D) \geq 1/E_\delta([x], [1/v], [D])$;*

(ii)*Let g be a metric on S^n such that $vol_g(S^n) \geq v$, $diam_g(S^n) \leq D$, the C^k-norm of the curvature tensor R_g does not exceed x (for some fixed k), and*

$$\|Ric_g + g\|_{C^0} \leq \epsilon(\delta, x, v, D) \ .$$

Then there exists an Einstein metric g_E of scalar curvature $-n$ on S^n such that $\|g - g_E\|_{C^2} \leq \delta$. (All norms mentioned in condition (D) *are taken in the considered metric g).*

Note that for $k \geq 2$ the existence of a function ϵ, satisfying (ii), follows from the Gromov-Cheeger compactness theorem (cf. [A, Theorem 2.2 and Corollary B, part (4)]). In fact, statement (D) is the statement that this function can be effectively minorized. This statement implies the possibility of recognizing when a metric is sufficiently close to an Einstein metric on S^n. If statement (D) is true, then statement (A) implies not only the existence of an Einstein metric of negative scalar curvature on S^n, but the existence of infinitely many Einstein structures of scalar curvature $-n$ on S^n.

THEOREM 2. *Let statement* (D) *above be true for some $n \geq 5$. Then either there exists a smooth n-dimensional homology sphere not admitting any Einstein metric, or the sphere S^n admits infinitely many distinct Einstein structures of scalar curvature $-n$.*

The proofs of Theorems 1 and 2 are based on the result of S. Novikov stating that for any $n \geq 5$ there is no algorithm deciding whether or not a given manifold is diffeomorphic to S^n. As was noted in [N1]–[N3], this theorem is true even in the case when given manifolds are compact non-singular algebraic hypersurfaces in $\mathbb{R}^{n+1}$, presented as zero sets of polynomials with rational coefficients. The degree and the vector of coefficients of such a polynomial are regarded as the input data for the algorithm. More precisely, one can effectively construct a sequence of smooth homology spheres $\{S_i\}$ realized as non-singular algebraic hypersurfaces in $\mathbb{R}^{n+1}$ in such a manner that the set of indices i for which the fundamental group of S_i is trivial (and, thus, S_i is diffeomorphic to S^n) is a non-recursive set. If necessary, one can construct the sequence S_i in such a manner that all non-trivial fundamental groups of hypersurfaces S_i have exponential growth (see the Remark at the end of Appendix). We give the proof of the above-formulated version of Novikov's theorem in the Appendix. Note that Novikov's theorem immediately implies the algorithmic unrecognizability of any compact manifold M of dimension ≥ 5. Indeed, given a compact manifold M of dimension ≥ 5 (represented, say, as a Nash (i.e. smooth semialgebraic) submanifold of an Euclidean space), one can consider the sequence of connected sums of the elements of the sequence S_i with M. (Any compact manifold can be represented as a Nash submanifold of an Euclidean space in such a manner that all coefficients of all defining polynomials are rational, cf. [BooHPo, pp. 57-63] for details. In fact, according to [BT], it can be represented even as an algebraic set over $\mathbb{Q}$.)

One possible application of Theorems 1 and 2 is the following. Often a more complicated topology of a manifold makes it easier to prove the existence of solutions of variational problems on a space of certain geometric objects associated with this manifold. For example, Gromov noticed in [G1] (see also [G2, p. 70]) that if the fundamental group of a compact Riemannian manifold has an unsolvable word problem then there are infinitely many contractible closed geodesics on this manifold. (But it is still unknown whether or not there are always infinitely many closed geodesics on S^n.) So, it may happen that it is easier to prove the existence of Einstein or almost Einstein metrics on smooth homology spheres with large fundamental groups than the existence of Einstein metrics or almost Einstein metrics of negative scalar curvature on S^n (if such metrics exist at all). In this case one of Theorems 1 or 2 will be a step in a proof of the existence of such metrics on S^n. In this connection note that in order to establish that S^n almost admits (or admits) Einstein metrics of negative scalar curvature one can consider only homology spheres with non-trivial fundamental groups in the sequence constructed in the proof of Novikov's theorem. Since there exists a significant arbitrariness in the construction of their fundamental groups, one can ensure, if necessary, that these groups will satisfy certain additional properties.

Proof of Theorem 1: We prove Theorem 1 by contradiction. Assume that every n-dimensional smooth homology sphere with infinite fundamental group almost admits an Einstein metric of negative scalar curvature. Assume that for a certain positive N and for any metric on S^n the norm of the tensor $Ric_g + g$ is not less than $\frac{1}{N}$. We are going to construct an algorithm recognizing which algebraic hypersurfaces in the above-mentioned sequence $\{S_i\}$ are diffeomorphic to S^n and, thus, to obtain a contradiction with the Novikov theorem. The idea of the algorithm is very simple. We organize a trial and error search over a dense in C^2-norm countable subset of the space of Riemannian metrics on the considered manifold for either a Riemannian metric of positive Ricci curvature or a metric g such that $\|Ric_g + g\| \leq \frac{1}{2N}$. Our assumptions imply that for every i either a metric of positive Ricci curvature or a metric g such that $\|Ric_g + g\| < \frac{1}{2N}$ exists on S_i. Moreover, in the first case we can be sure that the manifold S_i is diffeomorphic to S^n (if S_i is not diffeomorphic to S^n, then $\pi_1(S_i)$ is infinite and, thus, S_i does not carry a metric of positive Ricci curvature), and in the second case our assumptions imply that the manifold cannot be diffeomorphic to S^n.

Now we will describe how exactly this trial and error search is conducted. Initially, the manifold was presented as a non-singular algebraic hypersurface S_i in $\mathbb{R}^{n+1}$. The outer unit normal ν is a smooth semialgebraic function on this hypersurface. Below we will deal only with semialgebraic functions defined using only algebraic constants (but these functions will be regarded as semialgebraic over $\mathbb{R}$). So, every time when we will mention that a function is semialgebraic, we implicitly mean that it was defined using only algebraic constants. Consider the covering of S_i by semialgebraic open (in S_i) subsets $\bar{O}_j$, defined for every $j \in \{1, \dots, n+1\}$ as the set of points x of S_i such that the vector e_j of the standard basis of $\mathbb{R}^{n+1}$ is not "almost" tangent to S_i at x, that is, for example, $(e_j, \nu(x)) \geq 1/(n+1)$. Consider a covering of every $\bar{O}_j$ by semialgebraic open subsets O_{lj} with the following properties: (i) For every $x \in O_{lj}$ $(e_j, \nu(x)) \geq 1/(2(n+1))$; (ii) The orthogonal projection of O_{lj} onto hyperplane spanned by all vectors of the standard basis of $\mathbb{R}^{n+1}$ but e_j is a bijective map of O_{lj} onto an open ball of rational radius centered at a point with rational coordinates. Thus, O_{lj} are open semialgebraic subsets of S_i, homeomorphic to $\mathbb{R}^n$, and covering together all S_i. $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}$ are local coordinates in O_{lj}. We are going to express Riemannian metrics (and their curvature tensors and Ricci tensors) in these local coordinates. For the computational purposes we need a countable dense subset of the space of Riemannian metrics on S_i, such that every metric in this subset is expressed by a finite set of data. We will consider the set of all metrics g, such that the metric coefficients $g_{pq} = g(\frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q})$ in the local coordinates on all sets O_{lj} are smooth semialgebraic functions, as such a subset. It is not difficult to prove using the Weierstrass approximation theorem and a C^k-smooth semialgebraic analog of partitions of unity (cf. [BoCoR, pp. 175-178]) that for any k every metric on S_i can be arbitrarily close in C^k-norm approximated by such metrics. (Let us call them for brevity *semialgebraic metrics.*) It is clear that the curvature tensor of a semialgebraic metric will have semialgebraic components when expressed in the local coordinates in every domain O_{lj}. Hence, the analogous statement will be true for the Ricci curvature tensor. Thus, for every fixed semialgebraic metric g on S_i the conditions $Ric_g > 0$ and $\|Ric_g + g\| \leq 1/(2N)$ can be expressed by formulas of the first order language of the theory of real closed fields (i.e., by formulas obtained starting from polynomial sign conditions $P_i ?_i 0$, where $?_i$ means $<$, $>$, or $=$, and using a finite constant number of conjunctions, disjunctions, negations, and universal or existential quantifications of variables. All variables must be real. Cf. [Co], [BoCoR]. Moreover, in our case all variables will be quantified and, also, all coefficients of the involved polynomials will be algebraic.) Now the Tarski-Seidenberg theorem implies that there exists an algorithm verifying validity of every given formula of such type. That is, there exists an algorithm verifying conditions $Ric_g > 0$ and $\|Ric_g + g\| \leq 1/(2N)$ for every given semialgebraic metric on S_i, represented by a system of $n(n+1)/2$ semialgebraic metric coefficients on every domain O_{lj}. (Of course, the existence of an algorithm which precisely verifies the inequality $\|Ric_g+g\| \leq \frac{1}{2N}$ for semialgebraic metrics is more a luxury than necessity for us. For our goals it would be sufficient to be able to evaluate $\|Ric_g+g\|$ to the accuracy of $\frac{1}{100N}$ and then to compare this approximation with $\frac{1}{2N}$. The existence of an algorithm evaluating $\|Ric_g+g\|$ to a prescribed accuracy is entirely obvious.)

Now, it remains to organize a trial and error search over the described countable and dense set of metrics. The basic idea is to search over all possible collections of semialgebraic metrics on O_{lj}. (Of course, for every such collection it will be necessary first to check that the expressions in the local coordinates give the same metric on all overlaps of domains O_{lj}. This condition can be represented by a formula of the first order language of the theory of real closed fields and, thus, can be verified by an algorithm.) On every domain O_{lj} the metric is given by a collection of $n(n+1)/2$ metric coefficients. Every metric coefficient can be regarded as a smooth semialgebraic function of n variables, defined on the corresponding open ball. Every semialgebraic function can be described by a semialgebraic presentation of its graph of the form $\bigcup_{i=1}^{r}\bigcap_{j=1}^{h}\{x \in \mathbb{R}^n; y \in \mathbb{R}|P_{ij}(x,y)?_{ij}0\}$, where P_{ij} are polynomials (and in our case they are polynomials with algebraic coefficients) and $?_{ij}$ denotes one of the signs $<$, $>$, or $=$. Observe that every algebraic number α can be represented by a finite sequence of integer coefficients of a polynomial p such that $p(\alpha)=0$. For any positive integer D and K there exists only finitely many algebraic numbers α such that there exists a polynomial p of a degree not exceeding D with integer coefficients in the interval $[-K,K]$ such that $p(\alpha)=0$. Let us call such algebraic numbers α (D,K)-numbers. Now it is clear how one can organize an exhaustive search over the set of all semialgebraic metrics: We fix a positive integer L. Then we construct all semialgebraic metrics on S_i such that the number r of "pieces" and the number h of polynomials defining a "piece" in the semialgebraic presentation of every metric coefficient do not exceed L, the degrees of all polynomials used in the semialgebraic presentation of every metric coefficient do not exceed L and all coefficients of these polynomials are (L,L)-numbers. There will be only finitely many such semialgebraic metrics and all of them can be written down. As it was already mentioned, we are looking for a metric g such that either $Ric_g>0$ or $\|Ric_g+g\| \leq 1/(2N)$. If there is no such metric in the list corresponding to the chosen value of L we increase L by one and look for g in the finite set of metrics corresponding to this new value of L, etc. Our assumptions imply that eventually (for a sufficiently large L) such a metric will be found. Then we stop the search.

This completes the proof of Theorem 1. Note that the value of N used in the algorithm is unknown to us. This fact, however, is of no importance since it does not affect the *existence* of the algorithm described above which uses this value of N. □

Proof of the Corollary: Assume that (A) and (C) hold. (A) implies that every n-dimensional smooth homology sphere with the fundamental group with exponential growth will have an Einstein metric of negative scalar curvature (since by Milnor's result mentioned above such manifolds do not admit metrics of non-negative Ricci curvature). Hence, every such manifold almost admits an Einstein metric of negative scalar curvature. Now the successive application of Theorem 1 and then (C) implies that S^n admits at least one Einstein metric of negative scalar curvature. □

Proof of Theorem 2: Assume that statement (D) is true and that every smooth n-dimensional homology sphere admits an Einstein metric. We must prove that S^n admits infinitely many Einstein structures of scalar curvature $-n$. We are going to prove this by contradiction. Assume that there exists only a finite set of such structures. We are going to use the same idea as in the proof of Theorem 1 and build an algorithm, recognizing S^n in the sequence of smooth algebraic hypersurfaces S_i constructed in the proof of Novikov's theorem, thus getting a contradiction to Novikov's theorem. We can assume that if $\pi_1(S_i)$ is non-trivial, then $\pi_1(S_i)$ has exponential growth (see the Remark at the end of the Appendix). So, in this case S_i does not admit Einstein metrics of non-negative scalar curvature. Let $g_1, \dots, g_p$ be a set of representatives from all isometry classes of Einstein metrics on S^n of scalar curvature $-n$. Again we will do the trial and error search over the space of semialgebraic metrics on S_i (as in the proof of Theorem 1) aimed to find either a metric of positive Ricci curvature or an approximation to an Einstein metric of negative curvature. By "an approximation to an Einstein metric" we mean here and below in the proof of Theorem 2 a semialgebraic metric g (see the definition in the text of the proof of Theorem 1) such that

$$\|Ric_g + g\| \le 1/E_\delta\big([\|R_g\|_{C^k}], [1/vol_g], [Diam_g]\big) ,$$

where k is the same number as in the statement (D), R_g denotes the curvature tensor of g, vol_g and $Diam_g$ denote, correspondingly, the volume and the diameter of (S_i, g), and E_δ is the computable function introduced in the text of the statement (D). An appropriate value of δ will be chosen below. In order to ensure that this condition holds we first evaluate $\|R_g\|_{C^k}$, $1/vol_g$ and $Diam_g$ to an accuracy of 0.5 (this clearly can be done for any semialgebraic metric on S_i). Denote these approximations by R^*, $1/vol^*$, $Diam^*$. Then we will verify a stricter condition

$$\|Ric_g + g\| \le 1/E_\delta\big([R^*] + 1, [1/vol^*] + 1, [Diam^*] + 1\big) .$$

Since every metric on S_i can be arbitrarily closely approximated in the C^{k+2}-norm by a semialgebraic metric (see the proof of Theorem 1), we will eventually find a semialgebraic metric for which this (stricter) condition is satisfied if S_i has an infinite fundamental group of exponential growth. (Recall, that our assumption implies that such S_i admits an Einstein metric of negative scalar curvature.) So, the trial and error search will always stop. If we found a metric of positive Ricci curvature, then the manifold is diffeomorphic to S^n. Assume that we found an approximation $\bar{g}$ to an Einstein metric on S_i of negative scalar curvature. Below we will explain how we proceed in this case. But first let us explain how the value of δ in the trial and error search is chosen.

If the manifold is an n-dimensional sphere, then the assumption that statement (D) is valid implies that the approximation $\bar{g}$ to an Einstein metric must be δ-close in the C^2-norm to an Einstein structure of scalar curvature $-n$ on S^n. Hence, it must be close in the Gromov-Hausdorff metric to one of the chosen representatives $g_1, \ldots, g_p$ of all Einstein structures of scalar curvature $-n$ on S^n. On the other hand, let M_1^n and M_2^n be compact Riemannian manifolds of volume $\geq v$, diameter $\leq D$ and with sectional curvature the absolute value of which does not exceed Λ^2. The Cheeger-Gromov compactness theorem implies that if the Gromov-Hausdorff distance between M_1^n and M_2^n is less than a certain function of n, v, D, Λ, which can be written explicitly, then these manifolds are homotopy equivalent. Theorem 8 in [Pe] implies that for this function one can take any function of the form $\phi_n(v, D, \Lambda)/(2n^2+12n+16)$, where $\phi_n(v, D, \Lambda)$ is a positive lower bound for the injectivity radius of both manifolds. The existence of such a bound for the injectivity radius is a classical result of Cheeger ([C]). An explicit expression for ϕ_n can be found in [P]. Note that this function increases with the second and the third arguments and decreases with the first argument. Since we consider only homology spheres of dimension $n \geq 5$, embeddable (as manifolds) in $\mathbb{R}^{n+1}$, the homotopy equivalence of S_i and S^n implies that S_i is diffeomorphic to S^n (cf. [Mi1, ch. 9, Theorem A]). Let $v = \frac{1}{2}\min_{j\in\{1,\ldots,p\}} vol_{g_i}$, $D = 2\max_j diam_{g_j}$, $\Lambda^2 = 2\max_j |K_{g_j}|$. Choose any positive rational number τ not exceeding $\phi_n(v,\ D,\ \Lambda)/(2n^2+12n+16)$. Choose a rational $\delta > 0$ such that for any $j \in \{1, \ldots, p\}$ the condition $\|g - g_j\|_{C^2} \leq \delta$ implies that $(S^n,\ g_j)$ is $\tau/2$-close in the Gromov-Hausdorff metric to $(S^n,\ g)$ and that the diameter, maximal absolute sectional curvature and inverse value of the volume of (S^n, g) do not exceed D, Λ^2 and v^{-1}, correspondingly. This value of δ is the value which must be used in the trial and error part of the algorithm described above. Observe that, by the definition, τ and δ depend only on $g_1, \ldots, g_p$. Since the chosen values of τ and δ do not depend on i, we can regard τ and δ as constants, used in the algorithm. (The fact that we do not know their values is of no importance, since we are only demonstrating *the existence* of an algorithm recognizing S^n in the sequence $\{S_i\}$.) Also, we will need another set of constants in the remaining part of the algorithm. Namely, we will need a set of semialgebraic metrics $\bar{g}_j,\ j \in \{1, \dots, p\}$, such that $(S^n, \bar{g}_j)$ is $\tau/100$-close in the Gromov-Hausdorff metric to (S^n, g_j). Let us fix some such set of semialgebraic metrics.

Thus, the situation is as follows: If S_i is not diffeomorphic to S^n, then the Gromov-Hausdorff distance from $(S_i, \bar{g})$ to (S^n, g) cannot be less than τ, where g is an arbitrary metric on S^n. On the other hand, if S_i is diffeomorphic to S^n then for some $j \in \{1, \dots, p\}$ the distance from $(S_i, \bar{g})$ to (S^n, g_j) does not exceed $\tau/2$. Denote this value of j by j_0. The Gromov-Hausdorff distance from $(S_i, \bar{g})$ to $(S^n, \bar{g}_{j_0})$ does not exceed $\frac{51}{100}\tau$. Thus, to find out whether or not S_i is diffeomorphic to S^n we can proceed as follows. We will approximately find the Gromov-Hausdorff distance between $(S_i, \bar{g})$ and every Riemannian manifold $(S^n, \bar{g}_j)$, $j \in \{1, \dots, p\}$ to an accuracy of $\tau/10$. The manifold S_i is diffeomorphic to S^n if and only if at least one of the found values of these distances is less than $\frac{3}{4}\tau$. The Gromov-Hausdorff distance from $(S_i, \bar{g})$ to $(S^n, \bar{g}_j)$ can be approximately found as follows. Construct a finite $\tau/100$-net N_1 on $(S_i, \bar{g})$ and a finite $\tau/100$-net N_2 on $(S^n, \bar{g}_j)$. The Gromov-Hausdorff distance between N_1 and N_2 is a $\tau/50$-approximation to the distance between $(S_i, \bar{g})$ and $(S^n, \bar{g}_j)$. Note that the Gromov-Hausdorff distance between two finite metric spaces is the minimal value of a semialgebraic function defined on a finite dimensional space (namely, on the space of all possible distances between a point of one metric space and a point of another metric space). Thus, there exists an algorithm which finds Gromov-Hausdorff distances between given finite metric spaces within to any prescribed accuracy. So, we can find $\tau/100$-approximations of all distances between points of N_1, $\tau/100$-approximations of all distances between points of N_2, and then find a $\tau/20$-approximation to the Gromov-Hausdorff distance between the resulting metric spaces. This last value will be the desired $\tau/10$-approximation to the distance between $(S_i, \bar{g})$ and $(S^n, \bar{g}_j)$.

Thus, we have constructed an algorithm deciding when S_i is diffeomorphic to S^n based on the assumption that statements (A),(B) and (D) are true. The resulting contradiction with S. Novikov's theorem implies Theorem 2. □

Using the algorithmic unrecognizability of any compact n-dimensional manifold instead of Novikov's theorem one can prove, quite similarly to the proof of Theorems 1 and 2, that there is no quite good positive solution to Thom's problem. Let us say that a set S of Riemannian structures on compact n-dimensional manifolds is *recognizable* if for any prescribed neighborhood U of S (in the C^2 topology), there exists a computable function ψ on the set of all Nash submanifolds of $\mathbb{R}^{100n^2}$, regarded as Riemannian manifolds with the Riemannian metric induced by the inclusion map, with the following property: The function ψ is identically equal to zero outside U and to one in some neighborhood of S (contained in U). "Computable" means here the Turing computability of the restriction of ϕ on the set of Nash submanifolds defined by algebraic equations and inequalities with rational coefficients. (Note, that although it is quite possible that none of elements of S can be represented by a Nash submanifold *exactly*, every Riemannian structure on a compact manifold can be approximated to any accuracy by isometry classes of such Nash submanifolds.) We used Nash submanifolds of Euclidean space only as a way to approximately represent a Riemannian manifold by a finite set of data. Alternatively, one can use other ways to represent Riemannian manifolds by a finite set of rational or algebraic numbers in this definition. For example, one can represent Riemannian manifolds as was done in the proof of Theorem 1.

THEOREM 3. *Let $n \geq 5$ be fixed. Assume that for every compact manifold of dimension n $Nice(M)$ is a non-empty set of Riemannian structures of volume one on M, and $\bigcup_M Nice(M)$ is recognizable. Then for every compact n-dimensional manifold M $Nice(M)$ is an infinite set.*

Assume that for some $n \geq 5$ every compact n-dimensional manifold M admits an Einstein metric. Then we can regard the set of Einstein structures of volume one on M as $Nice(M)$. Theorem 3 immediately implies that:

THEOREM 4. *For any $n \geq 5$ at least one of the following three statements is false:*

(1) *Every compact n-dimensional manifold admits an Einstein metric;*
(2) *There exists a compact n-dimensional manifold admitting only a finite number of Einstein structures of volume one;*
(3) *The set of Einstein structures on compact n-dimensional manifolds is recognizable.*

Similarly to the proof of Theorem 2, one can alternatively replace statement (3) in the text of Theorem 4 by the analogue of statement (D) formulated above, but where g is allowed to be a Riemannian metric on any compact manifold of dimension n and not just only on S^n.

Appendix. A Smooth Version of S. Novikov's Theorem

Here we are going to give the proof of a smooth version of S. Novikov's theorem on the algorithmic unrecognizability of S^n, $n \geq 5$, in the class of non-singular algebraic hypersurfaces in $\mathbb{R}^{n+1}$. Our exposition is based on Novikov's proof sketched in [VKuF, ch. 10] and uses some technicalities from [BooHPo] and some ideas from semialgebraic geometry (cf. [Co], [BoCoR]).

THEOREM . *For any $n \geq 5$ there is no algorithm which for a given d and a vector of coefficients of a polynomial $p : \mathbb{R}^{n+1} \to \mathbb{R}$ with rational coefficients such that the zero set $Z(p)$ of p is non-empty, compact and non-singular, decides whether or not $Z(p)$ is diffeomorphic to the sphere S^n.*

Proof: The general idea of Novikov is to prove the algorithmic unsolvability of the triviality problem in the class of finitely presented groups with the trivial first and second homology groups. Such groups can be realized as fundamental groups of homology spheres of any dimension $n \geq 5$ (see [K]). The problem of recognizing S^n among such homology spheres will be obviously equivalent to the triviality problem for their fundamental groups and, thus, unsolvable.

One starts from a finite presentation of a group G with an unsolvable word problem. Using this presentation one can effectively construct a sequence of finite presentations of groups $\{G_i\}$ such that there is no algorithm deciding which of them are finite presentations of the trivial group. (In other words the set of indices i for which groups G_i are trivial is not recursive.) (This construction was originally found independently by M. Rabin and S. Adyan.) An explicit construction of such a sequence from a given finite presentation of a group with an unsolvable word problem can be found in [M, pp. 13-14]. One can easily see from formulae on pp. 13-14 that groups with these finite presentations have trivial first homology groups. Now Novikov explicitly constructs for every group G_i a finite presentation of another group $\tilde{G}_i$, which turns out to be the universal central extension of G_i. This finite presentation can be described as follows. Let F be the free group generated by all generators $f_1, \ldots, f_k$ of G_i and R be the normal subgroup of F generated by all relators $g_1, \ldots, g_m$. Since $H_1(G_i) = G_i/[G_i, G_i]$ is trivial, every generator f_j of G_i can be written as the product $\tilde{g}_j c_j$, where $\tilde{g}_j$ is a product of some powers of the relators of G_i and $c_j \in [F, F]$. Define $\tilde{G}_i$ as the group generated by $f_1, \ldots, f_k$ with relations $\tilde{g}_j = e, j = 1, \ldots, k$ and $f_l g_j f_l^{-1} g_j^{-1} = e$, $l = 1, \ldots k$, $j = 1, \ldots, m$. To show that the universal central extension of G_i is, indeed, isomorphic to $\tilde{G}_i$, note that the universal central extension of G_i is isomorphic to $[F, F]/[R, F]$ (cf. [Mi2, Corollary 5.8]). Consider the homomorphism ψ of $[F, F]/[R, F]$ onto $\tilde{G}_i$ which is the composition of the inclusion of $[F, F]/[R, F]$ into $F/[R, F]$ and the natural surjection of $F/[R, F]$ onto $\tilde{G}_i$. Obviously, this homomorphism is surjective. (The image of c_j in $[F, F]/[R, F]$ goes to the image of f_j in $\tilde{G}_i$.) To show that ψ is injective note that $\psi(u) = e$ in $\tilde{G}_i$ implies that u lifts to an element $\bar{u}$ in F which is equal to a product of powers of $\tilde{g}$'s modulo $[R, F]$. Permuting $\tilde{g}$'s we see that $\bar{u} = \prod_{j=1}^{k} \tilde{g}_j^{n_j} mod[R, F]$. But $\bar{u} = e\, mod[F, F]$. Since the images of $\tilde{g}_j$ in $F/[F, F]$ form a system of linearly independent generators of $F/[F, F]$, all exponents n_j are equal to zero, and $\bar{u}$ is in $[R, F]$. Therefore, $u = e$ in $[F, F]/[R, F]$, and ψ is an isomorphism.

Groups $\tilde{G}_i$ have the following properties: (i) The first and the second homology groups of $\tilde{G}_i$ are trivial; (ii) $\tilde{G}_i$ is trivial if and only if G_i is trivial. Property (ii) is obvious. It implies that the triviality problem for the sequence $\{\tilde{G}_i\}$ is unsolvable. The triviality of $H_1(\tilde{G}_i)$ follows from Theorem 5.3 in [Mi2] or, alternatively, can be easily proven using the finite presentation of $\tilde{G}_i$ given above. Now Novikov notes that using homological algebra one can show that $H_2(\tilde{G}_i)$ is trivial. This can be done for example as follows (and the author wants to thank Vladimir Hinich who helped him to reconstruct this part of the proof): To show the triviality of $H_2(\tilde{G}_i)$ note that $\tilde{G}_i$ and the identical homomorphism constitute the universal central extension of $\tilde{G}_i$. (This immediately follows from Theorem 5.3 in [Mi2]). But the second homology group of a group H, such that $H_1(H)$ is trivial, is the kernel of the homomorphism of the universal central extension of H onto H (cf. Corollary 5.8 in [Mi2]). Therefore $H_2(\tilde{G}_i)$ is trivial. Thus, we have an effectively constructed sequence $\{\tilde{G}_i\}$ of finite presentations of groups with trivial first and second homology groups such that the triviality problem for this sequence is unsolvable. Now we are going to construct a sequence of compact non-singular algebraic hypersurfaces S_i in $\mathbb{R}^{n+1}$ in such a manner that S_i are homology spheres and $\pi_1(S_i) = \tilde{G}_i$. Observe that because S_i are *hypersurfaces* in $\mathbb{R}^{n+1}$, S_i will be diffeomorphic to S^n if and only if $\tilde{G}_i$ is trivial (cf. [Mi1, ch. 9, Theorem A]). Thus, the construction of such a sequence will complete the proof of the theorem.

First, for every i we use the Dehn construction and then smooth out the corners in order to build a smooth hypersurface Q_i in $\mathbb{R}^{n+1}$ such that $\pi_1(Q_i) = \tilde{G}_i$ and all homology groups of Q_i but the second and the $(n-2)$th are trivial (and the second homology group is the direct sum of several copies of $\mathbb{Z}$). The details can be found in [BooHPo]. Q_i is the smoothed out boundary of a sufficiently small neighborhood of a two-dimensional simplicial complex K_i, embedded in $\mathbb{R}^{n+1}$, with fundamental group $\tilde{G}_i$. Now we use the fact that $H_2(\tilde{G}_i)$ is trivial. By virtue of H. Hopf's theorem this fact implies that the Hurewicz homomorphism $\pi_2(Q_i) \to H_2(Q_i)$ is surjective. Thus, it is possible to realize the generators of $H_2(Q_i)$ by spheroids. Moreover, using a general position argument these generators can be realized by pairwise non-intersecting embedded spheres. These spheres will have trivial normal bundles. (This fact easily follows from Lemma 3.5 in [KMi]. These spheres can be effectively found by a trial and error algorithm which will be described below.) Now we are going to kill one by one these generators by surgeries, and we would like to perform these surgeries inside $\mathbb{R}^{n+1}$. At the beginning the unbounded connected component U_i of the complement of Q_i in $\mathbb{R}^{n+1}$ is homotopy equivalent to the complement of the two-dimensional complex K_i in $\mathbb{R}^{n+1}$ and, thus, is 2-connected. So, any embedded sphere σ in Q_i, realizing a generator of $H_2(Q_i)$, will be null homotopic in U_i. If $n+1 \geq 7$, then the standard general position argument implies that we can always realize this homotopy by a 3-disc embedded in U_i, meeting Q_i transversally along σ. If $n+1=6$, then one must also apply the Whitney trick to get the embedded disk. Thus, we can perform the first surgery inside $\mathbb{R}^{n+1}$. After several surgeries we get a smooth hypersurface Q_{ij}. To show that the next surgery can be done inside $\mathbb{R}^{n+1}$ we need to demonstrate that the first and the second homotopy groups of the outer connected component U_{ij} of the complement of Q_{ij} are trivial. Q_{ij} is the boundary of a tubular neighborhood of a 3-dimensional complex K_{ij} embedded in $\mathbb{R}^{n+1}$. Thus, U_{ij} is homotopy equivalent to the complement of K_{ij}. If $n+1 \geq 7$ this implies that U_{ij} is 2-connected. If $n+1=6$, this immediately implies that U_{ij} is simply connected. In order to show that $\pi_2(U_{ij}) = H_2(U_{ij})$ is trivial, note that $H^3(K_{ij})$ is trivial (this follows from the fact that $H_2(K_i)$ was free abelian and K_{ij} was obtained from K_i by adding several 3-cells killing several linearly independent generators of $H_2(K_i)$), and apply the Alexander duality theorem.

When all generators of the second homology group of Q_i will be killed, we must smooth out the corners. The result will be a compact hypersurface S_i which is a boundary of a small neighborhood of a finite 3-dimensional acyclic complex $\bar{K}_i$ such that $\pi_1(\bar{K}_i) = \tilde{G}_i$. It is easy to see that the fundamental group of the constructed hypersurface is isomorphic to $\pi_1(\bar{K}_i)$ and, thus, to $\tilde{G}_i$. Using the Mayer-Vietoris exact sequence and the Alexander duality theorem one can easily see that the constructed hypersurface is a homology sphere.

All steps of the construction described above are, in fact, effective. But the shortest way to show the existence of an algorithm constructing homology spheres S_i is to use a semialgebraic trial and error algorithm (somewhat similar to the algorithm described in the proof of Theorem 1). For example, to perform the smoothing of the corners on the last stage, we look for a polynomial $p \in \mathbb{Q}[x_1, \dots, x_{n+1}]$ such that its gradient does not vanish at any point of its zero set $Z(p)$, and such that its zero set approximates the piecewise smooth (semialgebraic) hypersurface Q obtained at the previous stage in the following sense. Let $r(Z(p))$ denote the injectivity radius of the normal exponential map for $Z(p)$. (Informally, $r(Z(p))$ is the maximal radius of a non-selfintersecting open tube around $Z(p)$.) We require that on the normal to every point $x \in Z(p)$ there exists a single point $y(x) \in Q$ such that $|x - y(x)| \leq r(Z(p))/2$, and the map $x \to y(x)$ is a homeomorphism. It is not difficult to check that this condition is a first order formula in the theory of real closed fields. (Here we regard the coefficients of p as real variables.) By virtue of the Tarski-Seidenberg theorem (cf. [Co], [BoCoR]) this condition can be verified for every fixed vector of coefficients of a polynomial of $n+1$ variables. Thus, we can find p checking one-by-one all polynomials of degree d with rational coefficients with the numerator and denominator bounded by M (and gradually raising d and M).

Also, to find the disjoint embedded spheres realizing generators of $H_2(Q_i)$ (represented by simplicial chains in a triangulation of Q_i) we should look for a collection of disjoint polynomial (over $\mathbb{Q}$) embeddings of S^2 not to the manifold Q_i but to its neighborhood $N(Q_i)$ of a sufficiently small radius (say, half the injectivity radius of the normal exponential map for Q_i). For every such embedding we check whether or not its projection to Q_i is an embedded sphere (this is a semialgebraic condition, as well as the disjointness with the other embeddings) and whether or not this sphere represents the required element of $H_2(Q_i)$. The fact that we will find such a collection of embeddings in a recursive time follows from the fact that the algorithm must stop for every i by virtue of the already proven existence part of the construction. □

Remark: Observe that we could choose the original group G with unsolvable word problem as we like. In particular, we can assume that G has the exponential growth. Now Lemma 3.6 from [M], which we used to construct the groups G_i, implies that if G_i is not trivial, then the original group G embeds into G_i. So, in this case G_i is of exponential growth and has unsolvable word problem. One can easily see, that $\tilde{G}_i$ is also a group of exponential growth and has an unsolvable generalized word problem.

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