COMBINATORICS OF THE SPACES OF RIEMANNIAN STRUCTURES AND LOGIC PHENOMENA OF EUCLIDEAN QUANTUM GRAVITY

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ABSTRACT

We survey some relationships between geometry of the space of Riemannian structures on closed manifolds, Euclidean Quantum Gravity, combinatorics of triangulations of a fixed closed manifold and computability theory.

The center of our discussion is a certain class of functions that includes entropies of sets of Riemannian structures on M^n with bounded geometry, number of triangulations of M^n with $\leq N$ simplices regarded as a function of N and partition functions in Euclidean Quantum Gravity. If the dimension n is greater than three, than these functions are known to be non-computable. However, they do not grow too fast, and one can ask how closely can they be approximated by computable functions. (For example, one of these functions measures the "size" of the set of isometry classes of Riemannian metrics on a given closed Riemannian manifold of dimension ≥ 4 with the injectivity radius ≥ 1 and volume $\leq x$.)

We prove first results asserting that such partition functions cannot be approximated by computable functions too closely and state much stronger conjectures in this direction. We prove that specific values of some such functions have peculiar logic properties. We discuss possible implications of our results for Euclidean Quantum Gravity.

0. INTRODUCTION

Let M^n be a closed differentiable manifolds. By definition, *Riemannian structures* on M^n are isometry classes of Riemannian metrics on M^n. Consider the set of all Riemannian structures on M^n, $Riem(M^n)$. It can be endowed with the Gromov-Hausdorff metric. It is well-known that the subset $I_{M^n}(V)$ of $Riem(M^n)$ formed by all Riemannian structures with injectivity radius ≥ 1 and volume $\leq V$ is precompact for any V, cf. [Pet]. Similarly, one can consider sets of Riemannian structures on M^n, where the sectional (or Ricci) curvature is bounded from both sides (or only below) and the diameter bounded above, and all these sets will be precompact. It had been shown in [N2], [N3], [NW1], [NW2] and [NW3] that the geometry of these subsets of $Riem(M^n)$ is very complicated. (We refer the reader to the survey paper [NW3] for an introduction.) Moreover, it became clear that for $n \geq 4$ the geometry of sets $I_{M^n}(V)$ as $V \longrightarrow \infty$ and analogous sets defined by imposing constraints on curvature and diameter is closely related with computability theory and algorithmic information theory.

Yet many questions remain open. The most obvious question is what is the "size" of $I_V(M^n)$ or, say $Al_D(M^n) = \{\mu \in Riem(M^n) | \ |K_\mu| \leq 1, diam(\mu) \leq D\}$. More precisely, let $c > 1$ be a constant. Two Riemannian structures on M^n are called *c-close*, if the corresponding metric spaces are bi-Lipschitz equivalent by means of a bi-Lipschitz homeomorphism f such that Lipschitz constants of both f and f^{-1} do not exceed c. Otherwise, the two Riemannian structures are called *c-distant*. How many pairwise c-distant Riemannian structures are there in $I_{M^n}(V)$ (or in $Al_{M^n}(D)$)? Alternatively, we can generalize the notion of c-distant Riemannian structures in $I_{M^n}(V)$ by choosing a specific increasing function τ and replacing c by $\tau(V)$. Denote the "size" of $I_{M^n}(V)$ measured as the maximal number of pairwise τ-distant points in $I_{M^n}(V)$ by $S_{M^n,\tau}(V)$. Note that $[N2]$ implies, in particular, that if $n \geq 5$, then for any *computable* function τ $S_{M^n,\tau}(V) \geq const^V$ for some $const = const(n) > 1$ and all sufficiently large values of V. But is it true that for an appropriate computable τ and some constant $Const(n)$ $S_{M^n,\tau} \leq Const(n)^V$ for all sufficiently large V? If not, will this inequality be true for the minimal cardinality of a c-net in the subset of $I_{M^n}(V)$ formed by all Riemannian structures with $|K| \leq 10^{10}$, where one can either take c equal to, say, 10^{10}, or make it an increasing computable function of V?

The study of large scale features of geometry of subsets of $Riem(M^n)$ such as $I_{M^n}(V)$, $Al_{M^n}(D)$, etc. is important for so-called Euclidean Quantum Gravity introduced by S. Hawking at the end of 70's. The functions similar to $S_{M^n}(V)$ appear as *partition functions* and play a central role in Hawking's attempt to unify Quantum Gravity and General Relativity. Euclidean Quantum Gravity becomes a source of many interesting purely mathematical problems about geometry of sets such as $I_{M^n}(V)$, $Al_{M^n}(D)$, etc. It also suggests the following idea: One can consider the (discrete) space of all triangulations of a closed manifold M^n as a "toy" combinatorial model of spaces of Riemannian metrics on M^n with bounded geometry. (By a triangulation of M^n we mean a simplicial complex that is PL-homeomorphic to one fixed smooth triangulation of M^n. Here one does not distinguish between simplicially isomorphic triangulations. Formally speaking, we are talking about the set of all similicial isomorphism classes of triangulations of M^n. If T_1 is a triangulation of M^n, then T_2 is a triangulation of M^n if and only if T_1 and T_2 have a common subdivision. Note also that there exists a set of $(n+1)$ types of local elementary operations on simplicial complexes called bistellar transformations or Pachner moves, that do not change PL-homeomorphism class of a simplicial complex and have the following important property: Every two triangulations of the same n-dimensional manifold can be connected by a finite sequence of bistellar transformations. Therefore one can define a distance on the set of all triangulations of M^n as the minimal number of bistellar transformations required to transform one triangulation into the other. Thus, the set of all triangulations of M^n becomes a (discrete) metric space. The chosen distance has some similarities with the Lipschitz distance on the space of Riemannian structures on M^n.) To see the analogy one can endow each simplex of a triangulation of M^n by the metric of a standard simplex with sides of length 1 and then endow the triangulation with the length metric. As the result one obtains an infinite set of (singular) metric spaces that are located close to $Riem(M^n)$ (in the space of isometry classes of compact metric spaces endowed by the Gromov-Hausdorff metric). This set can be regarded as a net in a neighborhood of a subspace of $Riem(M^n)$ made of Riemannian structures with "nice" local geometries at small distances. Its subsets formed by all triangulations with $\leq N$ n-simplices are expected to have the same large scale properties as $I_{M^n}(V)$, when N and V are large. Therefore, it is expected that results about large scale geometry of the space of triangulations will have more or less straightforward analogs for the space of Riemannian structures. In particular, there exists an analog of Hartle-Hawking Quantum Gravity, where one replaces the space of Riemannian structures by the space of triangulations (so-called simplicial model of Euclidean Quantum Gravity). This model was of a considerable interest to physicists - see [ADJ] and references there. The number of simplicial isomorphism classes of triangulations of M^n with $\leq N$ simplices, $T_{M^n}(N)$, is a direct analog of functions measuring entropies of families of precompact subsets of $Riem(M^n)$ such as $S_{\tau,M^n}(V)$. In the simplicial model of Euclidean Quantum Gravity these functions appear as a particular case of partition functions. The fundamental unsolved question about T_{M^n} due to J. Frölich ([F]) is whether or not T_{M^n} is bounded above by c^N for an appropriate constant $c = c(M^n) > 1$. (This problem is also discussed in [Gr2].) Note that although $T_{M^n}(N) < N^{cN}$ for an appropriate $c = c(n)$, the function T_{M^n} is not computable for all manifolds of dimension $n \geq 5$ as well as for at least a large class of 4-dimensional manifolds (and probably for all of them). (This fact was first observed in [NB]. The same assertion is true for a wide class of partition functions. See [ADJ] for a discussion of its relevance for Euclidean Quantum Gravity.) Therefore the following question is of interest for both mathematics and physics: How closely $T_{M^n}(N)$ can be approximated by a computable function? This question is discussed in section 1.4 of the present paper, where we use the algorithmic information theory to prove first results indicating that T_{M^n} cannot be closely approximated by a computable function. In the same chapter we state several further conjectures in this direction. All these results can be immediately generalized to functions measuring the minimal cardinalities of nets in precompact subsets of $Riem(M^n)$ defined above. In section 1.5 we discuss sums of convergent series over triangulations of M^n (or over nets in $Riem(M^n)$). For example, one can consider $s = \Sigma_T 2^{-2^{N_5(T)}}$, where the summation is performed over all triangulations T of S^5, and $N_5(T)$ denores the number of 5-simplices in T. The number s cannot be evaluated within to an accuracy smaller than a certain explicit $\delta > 0$ (see Theorem 1.5.1 for the exact statement), because of the following phenomenon: There is a triangulation T_* of a non-simply connected homology sphere such that there is no proof in the Zermelo-Fraenkel system of axioms that this triangulation is not a triangulation of S^5. Therefore there is no proof in ZF that $2^{-2^{N_5(T_*)}}$ does not enter the series for s. A similar phenomenon will take place for any recursive system of axioms including ZF. In chapter 3 we discuss some possible physical implications of impossibility to compute good approximations to T_{M^n} and other similarly behaving partition functions in Euclidean Quantum Gravity. We also observe that if some plausibly looking (and interesting for their own sake) purely mathematical conjectures about the partition functions hold, then it might be possible to verify the validity of mathematical assertions by means of physical measurements. Chapter 2 provides a bridge between Chapters 1 and 3. It contains a very brief introduction into some mathematical aspects of Euclidean Quantum Gravity.

1. ALGORITHMIC UNSOLVABILITY IN TOPOLOGY, GEOMETRY OF SPACES OF RIEMANNIAN STRUCTURES AND TRIANGULATIONS, AND LOGIC PROPERTIES OF PARTITION FUNCTIONS

Let M be a closed manifold of dimension at least four. Consider the space of all isometry classes of Riemannian metrics (=Riemannian structures) on M. We can scale the considered Riemannian structures by imposing the constraint $inj \geq 1$, or $|K| \leq 1$, or we can impose both these constraints, or some other constraints involving the injectivity or convexity radius or a curvature. Then we can consider the diameter or the volume as a functional on this space. We will be assuming that our scaling constraints guarantee the precompactness of the sublevel sets of the considered functional. For example, we can consider sublevel sets $Al_M(x)$, where diameter is $\leq x$ on the set of all Riemannian structures with $|K| \leq 1$, or sets $I_M(x)$ of Riemannian structures with injectivity radius ≥ 1 and $vol \leq x$. In each of these sublevel sets we consider a minimal net $Net(x)$ with respect to the Gromov-Hausdorff metric or the Lipschitz metric. The density ϵ of the net can be constant or, if we use the Lipschitz metric, can be expressed by a computable increasing function on x. Let F be a non-negative functional defined on the space of Riemannian structures on M that might depend on a finite-dimensional parameter p. We are intereted in sums $Z_M(x,p) = \Sigma_{\mu \in Net(x)} F(x,p)$, or more generally, $Z(x,p) = \Sigma_M c(M) Z_M(x,p)$, where $c(M) \geq 0$ depends only on the topological type of M. We call such sums as well as their limit as $x \longrightarrow \infty$ (if it exists) *partition functions*. The simplest examples of these functions are entropies of sublevel sets $Al_M(x), I_M(x)$, etc. (By definition, the entropy of a set is the minimal number of elements in an ϵ-net. Of course, it depends on ϵ.) We are interested in properties of partition functions as $x \longrightarrow \infty$ and ϵ is either a fixed sufficiently small number or tends to zero, that do not depend on a particular choice of a minimal ϵ-net.

We discovered that these functions have interesting properties that follow from the non-existence of an algorithm deciding whether or not a given manifold N is diffeomorphic to M. In particular, they usually cannot be too closely approximated by a "computable" function. Since this phenomenon is of a combinatorial nature, it is less awkward to study it, when one considers the discrete space of all triangulations of M instead of the space of Riemannian structures. (One does not distinguish between simplicially isomorphic triangulations). And this phenomenon appears already when one studies the simplest of the partition functions for the space of triangulations, namely, $T_M(N)$ defined as the number of distinct triangulations of M with $\leq N$ simplices. Our results and problems for $T_M(N)$ can be immediately generalized for many other partition functions on the space of triangulations and on spaces of Riemannian structures (where they become, however, more awkward).

Another phenomenon involves properties of particular values of partition functions. Let M be any closed manifold of dimension > 4 or a manifold from a large class of 4-dimensional manifolds defined below. Consider $\Sigma_{all\ triangulations\ of\ M} f(T)$, where $f(T) > 0$ is a computable rational-valued function on the set of triangulations of M such that the series converges. Denote its sum by s. Assume that we would like to determine a (rational) approximation t to s such that $|s-t| < \delta$ for some acceptable for us positive rational δ. By "determine" we mean 1) to find such t and 2) to prove that it has the desired property. In order to formally state what do we mean by 2) we need to fix some system of axioms A, say the Zermelo-Fraenkel system or some its acceptable extension. It turns out that there exists an explicit positive $\delta = \delta_{M,f,A}$ depending on M, f and A such our task is impossible: There is no rational t such that we can prove that $|s-t| \leq \delta$ in the system of axioms A! Of course, one can immediatelty state an analog of this assertion for the space of Riemannian structures instead of the space of triangulations.

Our interest to these phenomena was partially motivated by their apparent relevance for Quantum Gravity that will be explained in Chapter 2. Yet there is another, even more natural and purely mathematical, reason to be interested in the partition functions: They immediately appear if one would like to “average” a functional defined on a space of triangulations (or Riemannian structures) over a subset of this space. The (non-)computability properties of such averages is a topic for a future research.

1.1. Algorithmic unsolvability in topology

Recall that according to the classical Markov theorem there exists a closed four-dimensional manifold M_0 such that there is no algorithm deciding whether or not a given closed manifold M is diffeomorphic to M_0. In paticular, this assertion is true for M_0 diffeomorphic to the connected sum of 46 copies of $S^2 \times S^2$. Note that for any closed manifold M_0 of dimension greater than four there no algorithm deciding whether or not a given closed manifold M is diffeomorphic to M_0 (S.P. Novikov theorem).

The formal proof of these theorems goes as follows. The P.S.Novikov-W.Boone construction of groups with the algorithmically unsolvable word problem provides an explicit way to assign to any Turing machine T and its input w a finite presentation of a group $G(T)$ and a word $v(T, W)$ in $G(T)$ such that $v(T, w)$ represents the trivial element if and only if T halts on w. The Rabin-Adyan witness construction (cf. [M]) yields now an explicit finite presentation of the group $G(T, w)$ such that $G(T, w)$ is the trivial group if and only if $v(T, w)$ is the trivial element of $G(T)$ (if and only if T halts on w).

To prove Markov theorem one first starts from an easy algorithm described at the beginning of the twentieth century by Dehn constructing for a given finite presentation of a group a closed manifold of any prescribed dimension greater than or equal to four such that the given group is the fundamental group of this manifold. Markov introduced a set of elementary operations called Markov moves on finite presentations of groups which 1) do not change the isomorphism class of the group; 2) correspond to certain geometric operations on manifolds built from the finite presentations. (These geometric operations do not change the diffeomorphism type of the manifold.); 3) Any two finite presentation of the same group can be transformed one to the other by a finite sequence of Markov operations. Using these moves one can immediately see that the manifold with the fundamental group $G(T, w)$ constructed using the Dehn algorithm will be diffeomorphic to a connected sum of a certain number of copies of $S^2 \times S^{n-2}$ if and only if T halts on w. The number of copies in this connected sum depends only on T but not on w. This reduces the halting problem for T to the diffeomorphism problem for M, where M is the connected sum of several copies of $S^2 \times S^{n-2}$ and n is any number greater than or equal to 4. See [BHP] for more details.

To prove the S. Novikov theorem we first consider the most difficult case when M_0 is the n-dimensional sphere. In this case in order to reduce the halting problem for T to the diffeomorphism problem we must learn to construct for any input w a manifold $M(T,w)$ such that $M(T,w)$ is diffeomorhic to the sphere if and only if T halts with w. Since homology groups of closed manifolds are computable, manifolds $M(T,w)$ must have the same homology as the sphere. Which finitely presened groups are realizable as the fundamental groups of homology spheres? In dimensions greater than four there exists a simple answer for this question found by M. Kervaire ([K]): the necessary and sufficient condition for a finitely presented group to be the fundamental group of a homology sphere is the vanishing of the first two homology groups of the group. The groups $G(T,w)$ explained above and constructed using the witness construction have trivial first homology group but non-trivial second homology group. But S. Novikov observed that one can make the second homology group of these groups vanish by performing on them a construction known from homology algebra, namely, replacing them by their universal central extensions. A finite presentation of the universal central extension can be written down in terms of a finite presentation of a perfect group. The universal central extension is trivial if and only if the original group is trivial. Finally, the universal central extensions always have trivial first two homology groups. So, one can construct the universal central extensions of groups $G(T,w)$ and realize them as the fundamental groups of homology spheres. In the general case when M_0 is not necessarily the sphere we just consider the connected sums of M_0 with the constructed homology spheres. Such a connected sum is diffeomorphic to M_0 if and only if the fundamental group of the homology sphere is trivial. This completes the proof of the S. Novikov theorem. See the Appendix to [N1] for more details.

Note that both Markov's theorem and Novikov's theorem are true also in PL and topological categories. In particular, if M_0 either has dimension > 4 or is the connected sum of 46 copies of $S^2 \times S^2$ with a closed 4-dimensional manifold, then there is no algorithm deciding whether or not a given closed PL manifold M (given by its triangulation) is PL-homeomorphic to M_0.

The number 46 in Markov's theorem comes from a specific construction of a sequence of finitely presented groups G_i such that the set $\{i : G_i \text{ is trivial }\}$ is non-recursive. All groups in this sequence have 19 generators and 27 relations, and $46 = 19 + 27$.

We believe that there is no algorithm that recognizes S^4. This result would follow from the affirmative solution of the following well-known Magnus problem: There is no algorithm deciding triviality for finite presentations of groups such that the number of generators is equal to the number of relations. (It is known that such groups can be constructively realized as the fundamental groups of homology spheres.) In order to explain why such finite presentations can be enormously logically complicated mention that S. Gersten constructed a finite presentations of a group with two generators and one relation such that for a certain sequence of words of length n representing the trivial element one needs to apply the relation $2^{2^{\cdots^{2}}}$ (n times) in order to show their triviality. This result seem to imply that for a certain sequence of finite presentations of the trivial group with two generators and two relations (one of which is the same for all groups in the sequence and the other has variable length) one needs to apply a number of Tietze moves that grows like the tower of exponentials of the length in order to show the triviality of these groups. Thus, even groups with two generators and two relations seem to be quite complicated.

Let us finish this section by stating the following conjecture that elaborates on the algorithmic unrecognizability of S^4:

Conjecture 1.1. There is an algorithm that constructs for a given Turing machine τ a triangulation $T(\tau)$ of a four-dimensional manifold so that $T(\tau)$ is a triangulation of S^4 if and only if τ halts when it starts its work with the empty tape.

Observe, that if this conjecture is true, then it will be true for any four-dimensional manifold M_0^4 instead of S^4: One just needs to take an appropriate connected sum. (Recall, that the proof of Markov's theorem implies that this conjecture is true for all manifolds of the form $N^4\#\#_{i=1}^{46}S^2\times S^2$.)

1.2. Quantitative measures of algorithmic unsolvability.

The quantitative measures of algorithmic unsolvability are provided by the algorithmic information theory. The notion we need here is the time-bounded Kolmogorov complexity. Informally, it can be explained as follows. In decision problems one has two sets: the set L of all instances of the problem and its subset A. The decision problem is the problem of recognition when a given element of L is an element of A. Elements of L are called instances of the problem. The set L is countable (at least in all decision problems considered here) and there is usually a natural complexity function f which measures the complexity of instances. For example, one can regard the length of words as the complexity of instances of the word problem, the length of the given finite presentation of a finite group as complexity of instances of the triviality problem for finitely presented groups, the number of simplices in a triangulation as the complexity of the homeomorphism problem for manifolds (if they are presented by triangulations). If L is the set of positive integer numbers then we can regard either the number itself or the number of digits in the number (say, in the binary representation) as its complexity. The number of instances of bounded complexity must be finite for any value of the bound.

Now assume that the decision problem is algorithmically unsolvable (that is, the characteristic function of A is not computable). However if one is allowed to use an arbitrary oracle information then every decision problem becomes solvable: one can just request the list of all answers. Now what is the minimal amount of oracle information required to an algorithm to solve the problem for all instances of complexity $\leq N$? The algorithm should be the same for all N; the information is written in the binary form, and by "the amount of information" we mean the length of the bit sequence regarded as the function of N. This function is defined only up to a finite summand since we can "hide" several bits of oracle information inside the algorithm. As before, one can understand an algorithm as a program in one of computer languagues such as C, FORTRAN, etc. using only the integer type of data. It is easy to see that this function does not depend on the choice of the programming language. This function is called the Kolmogorov complexity of the decision problem. Of course it can be defined within the realm of conventional mathematics (i.e. without refering to computer programs), cf. [M]. (See also [C] and other books by G. Chaitin for a very nicely written introduction to the algorithmic information theory.)

A similar notion arises when one not only wants to solve the decision problem but to do so in time bounded by a computable function of N. For any computable function t one can define the time-bounded Kolmogorov complexity of the decision problem as the minimal number of bits of oracle information required to solve the problem for all instances of complexity N in time not exceeding $t(N)$. There is a fundamental difference between the time-bounded Kolmogorov complexity and "usual" Kolmogorov complexity: time-bounded Kolmogrov complexity of finite binary sequences is computable whenever "usual" Kolmogoriv complexity is not. As a consequence Barzdin ([B]) proved that there exists a recursively enumerable decision problem $A \subset L = \mathbb{N}$ such that for any computable time-bound t its time-bounded Kolmogorov complexity for all positive integer numbers $\leq N$ is not less than $N/const(t) - const$. In other words there is no essentially better way to ask for the information than just to ask for the list of all answers. (We could do better by not more than a constant factor.) Recall that a decision problem is recursively enumerable if there exists an algorithm that always determines that a given element of L is in A if it is indeed in A, but possibly works infinite time if a given element is not in L. Note that usual Kolmogorov complexity of a recursively enumerable decision problem $A \subset \mathbb{N}$ does not exceed $\log_2 N + const$, and for some recursively enumerable decision problems the Kolmogorov complexity is equal to $\log_2 N - const$ ([B]), so this upper bound cannot be improved.

In [N2] (Theorem 3.1) the author proved that the mentioned result of Barzdin has its topological counterparts: In the situation of Markov and S. Novikov theorems one has not only the algorithmic unsolvability of the diffeomorphism (or PL-homeomorphism) problems but an exponential growth of the time-bounded Kolmogorov complexity as a function of the number of simplices of a given triangulation for any computable time bound. In the smooth situation one has a similar exponential lower bound for the time-bounded Kolomogorov complexity providing that the manifold is given as a Riemannian manifold (see [NW1], Theorem 1.1 for the exact statement). These results are of importance for understanding the geometry of spaces of Riemannian structures and spaces of triangulations as explained in the next section.

1.3. Geometry of spaces of Riemannian structures and spaces of triangulations.

As a corollary the geometry of graphs of various interesting functionals on spaces of Riemannian structures is very complicated. (This class of functionals includes the curvature-pinching $\sup |K| diam^2$, isoembolic ratio vol/inj^n, etc.) The graphs of these functionals look like very mountaineous landscapes with a lot of valleys, mountain passes, etc. In more technical terms one looks at the number of connected components of sublevel sets of these functionals, Recall that a sublevel set of a functional ϕ is the set of isometry classes of Riemannian metrics on a considered manifold such that the value of ϕ does not exceed x, where x is variable. It turns out that for all sufficiently large x sublevel sets of these functionals are not connected. Furthermore, the number of connected components of these functionals grows exponentially with x. Moreover, the number of basins that are very deep in comparison with x also grows exponentially. (By the depth of a basin we mean here how much one should increase the value of ϕ before one is able to get out of the basin. For any computable function l of x there will be exponentially many basins, where one must increase the value of the functional to more than $l(x)$ before one is able to reach another basin.) See [N2], [N3], [NW1], [NW3], [W] for more details.

This phenomenon takes place for every closed manifold of dimension > 4 as well as for many (conjecturally for all) closed four-dimensional manifolds. Whenever we were unable to demonstrate that the graph of the Einstein-Hilbert action

$\frac{\int_M \; scal \; d\mu_M}{vol(M)^{\frac{n-2}{n}}}$ has this kind of behaviour, we believe that it does have this behaviour, and that our methods can be applied to prove this assertion. Note that if we "regularize" this functional by adding to it a term $\epsilon|K|diam^2$ for an arbitrary small positive ϵ, then our methods produce the proof of this picture (for the regularized functional) - see [NW1], [NW3].

In order to give to the reader an idea how representatives from these basins look like consider the case when the manifold is S^n. If one takes a representative from these non-trivial very deep basins and writes down a finite presentation of its fundamental group (this group is, of course, trivial), then one gets the finite presentation that is comparatively short, but if one wants to transform it to the trivial finite presentation of the trivial group using a sequence of elementary operations, e.g. Markov moves, then one needs a sequence containing enormously long finite presentations. The existence of such short but non-trivial finite presentations of the trivial group is an immediate corollary of the algorithmic unsolvability of the triviality problem for finitely presented groups.

Roughly speaking, the idea of the proof of results described at the beginning of this section is to assume that the geometry of the graph of the Riemannian functional on a closed manifold M is nice and to use this assumption to construct an algorithm recognizing M thus obtaining a contradiction with cited above results of A. Markov and S. Novikov. Very roughly speaking this algorithm looks for a path from a given Riemannian manifold to a fixed Riemannian structure on M in the union of sublevel sets of the considered functional on all closed manifolds of the same dimension as M.

Another way to explain the existence of non-trivial basins is to observe that there exists an effective coding of Turing machines and their inputs by finite presentation of groups such that Turing machine halts with the input if and only if the corresponding finitely presented group is trivial. As it had been explained above, this finitely presented group can be chosen to be superperfect which makes possible its realization as the fundamental group of a Riemannian manifold with the same homology as M ("homology M"). Thus, one can start with a Turing machine T and its input ω and to get a Riemannian manifold $M(T,w)$ which is diffeomorphic to M if and only if T halts with ω. The geometric complexity of $M(T,w)$ is related with the number of states of T + the length of ω, and the computation of T with ω corresponds to a path from $M(T,w)$ that leads to a fixed Riemannian metric on M, if T halts with ω. This path can be shown to be close to the optimal and the geometric complexity of metrics in this path is closely related with the halting time of T on ω. On the other hand any two points in the same connected component of the space of Riemannian structures of bounded geometry can be connected by a not very long sequence of small "jumps". A juxtaposition of these two observations can be used to prove some of the results mentioned at the beginning of the present section.

The exponential growth of the number of basins can be proven using the algorithmic information theory (namely, the Barzdin theorem mentioned above).

From this brief discussion we see that the discussed phenomena are essentially of combinatorial nature and take place already for (discrete) spaces of triangulations that can be considered as a discrete analog of spaces of Riemannian structures ([N2]). In the Riemannian case such results were first discovered in [N3], followed by [NW1] (where the difficulties caused by the lack of compactness of sublevel sets of considered there Riemannian functionals were overcame using a powerful machinery from topology of manifolds, number theory and representation theory), [NW2] and [NW3].

At this point we would like to refer the reader to the survey paper [NW3] and to the monograph [W], where he will also find some other results on geometry of sublevel sets of Riemannian functionals not mentioned here and obtained using methods of computability theory and algorithmic information theory.

1.4. Non-computability of partition functions

It was noted in [NB] that partition functions in Simplicial Quantum Gravity are usually non computable. The simplest of these partition functions is the number of triangulations of a fixed manifold M with $\leq N$ simplices. Denote this number by $T_M(N)$. (Two simplicially isomorphic triangulations are regarded as identical.) If the dimension of the manifold is greater than four, or if the dimension of the manifold is four and it can be represented as the connected sum of another manifold and a sufficiently large number of copies of $S^2 \times S^2$ then this function is not computable. (Conjecturally, this result extends to all closed 4-dimensional manifolds.) Here is the explanation: If $T_M(N)$ would be computable then one could decide whether or not a given Riemannian manifold is M using the following algorithm. Let a triangulation T of some manifold with N simplices be given. Compute $T_M(N)$. Now start from a fixed triangulation of M and apply all possible sequences of k bistellar transormations (a.k.a Pachner moves; cf. [ADJ]) , where $k = 1, 2, 3, ...$ One keeps track of the number of different triangulations of M obtained in this way until one gets all $T_M(N)$ triangulations. It is important here that one can decide whether or not two given triangulations are simplicially isomorphic by merely trying all ways to match their vertices. Now one can just check if T is on the list of all triangulations of M obtained in this way.

Note that this argument not only implies the non-computability of $T_M(N)$ but provides us with an explicit algorithm that reduces the halting problem for a Turing machine τ to the problem of computing $T_M(N)$ for $N = [const(n)\ length(\tau)] + Const(M)$, where $const(n)$ is an explicit constant depending only on N, and $Const(M)$ is the minimal number of simplices in a triangulation of M. (Of course, one needs to combine this argument with a construction of a manifold $M(\tau)$ such that $M(\tau)$ is PL-homeomorphic to M if and only if τ halts. This construction was outlined in section 1.1.)

Here we discuss how closely partition functions can be approximated by computable functions.

Theorem 1.4.1. Let M be a closed manifold of dimension $n \geq 5$ or a closed four-dimensional manifold representable as a connected sum of a closed four-dimensional manifold and 46 copies of $S^2 \times S^2$. There exists a constant $c(n) > 1$ such that for any (Turing) computable function $a(N)$ $|T_M(N) - a(N)| > c(n)^N$.

Proof: The Kolmogorov complexity $K(T_M, N)$ of $T_M(N)$ does not exceed $\log_2(|T_M(N) - a(N)|)$. (Indeed, we can use the binary expansion of $T_M(N) - a(N)$ as the oracle information and calculate $T_M(N)$ as $a(N) + (T_M(N) - a(N))$.) But the problem of recognition of M can be solved by means of an algorithm if $T_M(N)$ is known (see above or [NB] for the details). Therefore the Kolmogorov complexity of the problem of recognizing M among manifolds with $\leq N$ simplices can be majorized by $\log_2(|T_M(N) - a(N)|)$. On the other hand the proof of the already mentioned Theorem 3.1 in [N2] implies that the Kolmogorov complexity $K(N)$ of the recognition problem is bounded from below by the Kolmogorov complexity of the halting problem of any Turing machine T on inputs (binary strings) of length $\leq N/const(T, n)$. According to a theorem proven by Barzdin ([B]) there exists a Turing machine T such that the Kolmogorov complexity of its halting problem on inputs of length $\leq N$ is not less than $N/Const$. Combining these inequalities together we obtain the required inequality. $\square$

Important Remark: The result of the theorem will be true not only for $T_M(N)$ but for a wide class of partition functions. For example, if we replace $T_M(N)$ by $F(N) = \Sigma_M \Sigma_{\{T:|T| \leq N\}} c(M) f(T)$, where the first summation is performed over all PL-isomorphism type of closed simply connected 4-dimensional manifolds M, the second summation is performed over all triangulations T of M with $\leq N$ simplices, a non-negative $c(M)$ depends only on the topological type of M, $c(\#_{i=1}^{46} S^2 \times S^2) \neq 0$, and $f(T)$ is a computable non-negative function of a triangulation, then for any computable function $a(N)$ $|F(N) - a(N)| \geq$

$(\min_{triangulations\ with \leq N\ simplices} f(T))2^{const\ N}$, where $const > 0$, and the minimum is taken over all triangulations of $\#_{i=1}^{46} S^2 \times S^2$ with $\leq N$ simplices. For simplicity we can assume that f and c assume values in the set of rational numbers. In this way the notion of "computability" does not require additional explanations. In principle, we could assume only that c, f are real-valued, but there is an algorithm finding arbitrarily close approximations to their values. The proof of this assertion is analogous to the proof of Theorem 1.4.1. It is very easy to modify this assertion so that it will involve partition functions defined as sums over nets in spaces of Riemannian structures with bounded geometry , and we leave finding such a modification as an exercise for the reader. In order to state such a result one needs a notion of computable real-valued function defined on a set of Riemannian structures. This notion will be introduced in the next section.

If we believe that for every M $T_M(N)$ grows exponentially and, moreover, $\frac{T_M(N+1)}{T_M(N)} < C$ for some C then one can prove a stronger theorem not using the notion of Kolmogorov complexity. Assume for simplicity that $M = S^n$. The idea is to combine the following three observations: First, if we know a good computable approximation to $T_M(N)$ then it is easy to find a slightly worse computable approximation $a_1(N)$ minorizing $T_M(N)$. Secondly, if we know a computable lower bound $a_1(N)$ of $T_M(N)$ then we can find a list of $a_1(N)$ triangulations of M with $\leq N$ simplices. Third, for every triangulation of S^n with N_0 simplices we can take its connected sum with all triangulations of the sphere S^n with $N - N_0$ simplices, thereby obtaining $\tilde{T}_{S^n}(N - N_0) \geq T_{S^n}(N)/const$ triangulations of S^n with N simplices. (The connected sum of two triangulations is defined as follows: Remove a simplex of the highest dimension from both triangulations and glue the boundaries of the "holes". This operation is, of course, non-uniquely defined.) Assume that an algorithm detects at least one of these triangulations (that is, one of this triangulations is on the list of $a_1(N)$ triangulations that were found). Then we will know that the original triangulation of S^n with N_0 simplices is, indeed, a triangulation of S^n. So, if there exists an algorithm detecting for any positive rational ϵ and any $N \geq L(\epsilon)$ $[(1-\epsilon)T_{S^n}(N)]$ triangulations of S^n, where L is a computable function (of $[\frac{1}{\epsilon}]$), then we can decide whether or not any given triangulation τ is a triangulation of S^n as follows: Find $[(1-\epsilon)T_{S^n}(N)]$ triangulations of S^n with N simplices for $N = L(\epsilon)$ that can be computed by means of this algorithm and check whether or not one of them can be represented as a connected sum of the given triangulation τ with another simplicial complex.

Combining these observations together we obtain the following theorem:

Theorem 1.4.2. Let $n > 4$. Assume that there exist a constant $C > 1$ such that for any N $\frac{T_{S^n}(N)}{T_{S^n}(N-1)} < C$. Then there are no (Turing) computable functions $a(N)$ and $L(k)$ such that $\frac{T_{S^n}(N)}{a(N)} \in [1-\frac{1}{k}, 1+\frac{1}{k}]$ for all $N \geq L(k)$. (In other words, even if $a(N)/T_{S^n}(N) \longrightarrow 1$, this convergence must be non-computably slow.)

Again, it is easy to generalize this result for more general partition functions on the space of triangulations and on spaces of Riemannian structures with bounded geometry.

These results motivate the following questions:

Problems: Let M be a closed manifold of dimension greater than three. Is it true that:

1.([NB]) For every computable function $a(N)$

$$\lim \sup_{N \longrightarrow \infty} \max\{T_M(N)/a(N), a(N)/T_M(N)\} > 1?$$

2. (A stronger conjecture) There exists $c > 1$ such that for every computable function $a(N)$ for infinitely many values of N

$$\max\{T_M(N)/a(N), a(N)/T_M(N)\} > c^N?$$

3. For every computable function $b(N)$ with values in the set of rational numbers there exists $c > 0$ not depending on b such that for infinitely many values of N

$$|\frac{T_M(N)}{T_M(N-1)} - b(N)| > c?$$

4. Consider functions $\phi(c) = \Sigma_T f(T, c)$, where $c \in U \subset R^k$, U is open, the summation in the right hand side is performed over all triangulations of a manifold M as in Theorem 1.4.1, the series in the right hand side converges for every $c \in U$, f is a non-negative computable function. Study (non-)computability of a such functions. How closely non-computable ϕ can be approximated by computable functions of c?

All these problems can be posed for general partition functions on spaces of traingulations and Riemannian structures.

1.5. Numbers that cannot be even approximately evaluated.

Theorem 1.5.1. Let M be a closed manifold of dimension ≥ 5, or a 4-dimensional manifold representable as the connected sum of a closed four-dimensional manifold M_1 with $\#_{i=1}^{46} S^2 \times S^2$. If Conjecture 1.1 is true, then M can be any closed four-dimensional manifold. Let f be a function defined on the set of triangulations of all closed manifolds of the same dimension as M with values in the set of positive rational numbers such that $\Sigma_{all\ triangulations\ T\ of\ M}\ f(T)$ converges. Denote the sum of this series by s. Then there exists a triangulation T_0 of a manifold with the same homology groups as M that can be, in principle, explicitly constructed with the following property: Let $\delta = \delta(M, f) = f(T_0)/4 > 0$. Then for every rational number t the assertion that $|s - t| < \delta$ is unprovable in the system of axioms ZFC (Zermelo-Fraenkel system of axioms + the axiom of choice). The result will be true for any consistent recursive system of axioms A as strong as ZFC (for a possibly different triangulation T_0.) In this more general case T_0 will depend on A and, in principle, can be explicitly found from a recursive description of A.

Remark. ZFC is regarded as a de facto system of axioms for the whole mathematics. This theorem tells us that the sum of the considered series cannot be evaluated within to accuracy $\delta > 0$. Here "evaluated" definitely does not mean that a desired rational approximation is not computable! (Any rational number is computable by the trivial computer program that just prints this number.) Rather it means "found" in the intuitive sense of the word, when one is supposed not only to guess an answer, but also to prove that the answer is, indeed, correct. The requirement that f is rational-valued is unnecessary and was introduced only to make the notion of computability easier. This theorem can be easily generalized for a general class of partition functions on spaces of triangulations and Riemannian structures (see below). It is partially motivated by its applications to Quantum Gravity in the last section.

Proof: According to the classical Gödel theorem the consistency of the considered system of axioms is unprovable. Consider a Turing machine τ that looks for a statement B such that B and its negation are both provable in the considered system of axioms. The consistency of the considered system of axioms implies that this Turing machine never stops, but this fact is unprovable. Now we can construct as above a triangulated manifold $M(\tau)$ such that $M(\tau)$ is PL-homeomorphic to M if and only if τ halts. Since τ does not halt, $M(\tau)$ is not PL-homeomorphic to M, but this fact is not provable. Denote the constructed triangulation of $M(\tau)$ by T_0. Let $\delta = f(T_0)/4$. Let t be a rational number, such that $|s - t| < \delta$. Then $t < s + f(T_0)/4$. Now we are going to prove the assertion by contradiction. Assume that the assertion that $|t - s| < \delta$ is provable. We are going to demonstrate that there exists a proof in the considered system of axioms of the fact that $f(T_0)$ does not enter the sum for s, and therefore T_0 is not a triangulation of M, thus, arriving to a contradiction. Note, that for each specific triangulation of M we can prove that this triangulation is, indeed, a triangulation of M. Looking for more and more triangulations of M we can compute a lower bound for s that is arbitrarily close to s, and to prove that this is, indeed, a lower bound for s. (But the theorem that we are proving asserts that one will not be able to prove that this is a very close lower bound.) Since it is provable (and therefore true) that $s > t - \delta$, we will be able to find a provable lower bound u for s such that $u > t - 2\delta = t - f(T_0)/2$. (This u will be the sum of values of f over a certain finite set of triangulations of M.) But since $u + f(T_0) > t + f(T_0)/2 > s - \delta + f(T_0)/2 = s + f(T_0)/4 > s$, and all these inequalities are provable in the considered system of axioms, one obtains a proof that $f(T_0)$ cannot enter the expression for s. QED

Example. Let $s = \Sigma_T 2^{-2^{N_5(T)}}$, where the summation is performed over all triangulations T of S^5, and $N_5(T)$ denotes the number of 5-dimensional simplices in T. Then s satisfies the conditions of the theorem.

Important Remark. In order to find T_0 (and, thus, δ) one needs to write a program (or, more precisely, a Turing machine) looking for for a proof of an assertion B and its negation in ZFC, then use the witness construction to find an explicit group which is trivial if and only if this Turing machine halts, Then one constructs an explicit homology M which will be PL-homeomorphic to M if and only if this group is trivial. The last two constructions are pretty straightforward. If one would like to make T_0 as small as possible, then one needs to “optimize” the Turing machine that does not halt but this fact is unprovable in ZFC. To achieve this goal one can use one of the unprovable in ZFC Π^1_0 formulae from [Fr]. The required Turing machine will look for a (non-existing) counterexample to one of these formulae. I believe that this approach can be used to produce a four-dimensional manifold with not more than one thousand handles such that this manifold is different from $M^4 = \#_{i=1}^{46} S^2 \times S^2$ but this fact cannot be proved in ZFC. Moreover, the actual minimal number of handles in such manifolds is probably much smaller. It is conceivable that this number could be as low as $40 - 50$ or even less than that.

Note, that for any non-computable real number s there exists $\delta = \delta(s) > 0$ such that for every rational t there is no proof in ZFC that $|s - t| < \delta$. (This fact had been pointed out to me by Steve Simpson.) Indeed, otherwise one can compute arbitrarily close approximations to t by taking $\delta = \frac{1}{N}$ for $N = 1, 2, 3, \ldots$ and checking all proofs in ZFC until a proof of the assertion $|s - t_N| < \frac{1}{N}$ for an appropriate t_N is found. But, of course, this argument does not yield a specific value of $\delta(s)$. The importance of Theorem 1.5.1 is in the fact that one can use its proof (outlined above) to find an explicit positive lower bound for δ.

Problem. Assume that the conjecture that $T_{S^5}(N) \leq C^N$ for some constant C is correct. Let $C_* = \limsup_{N \longrightarrow \infty} \frac{\ln T_{S^5}(N)}{N}$. Is the conclusion of Theorem 1.5.1 true for C_*? Of course, one can ask the same question for any closed manifold of dimension greater than three instead of S^5.

To state an analog of this theorem for the space of Riemannian structures consider, for example, the space R_{n,v_0} of all Riemannian structures on n-dimensional manifolds with $|K| \geq 1, vol \geq v_0 > 0$ for some $n \geq 4$ and some v_0. It is well-known that for every $D > 0$ there exists an explicit positive $\epsilon = \epsilon(D, v_0)$ such that if two such Riemannian structures with diameters $\leq D$ are ϵ-close (in Gromov-Hausdorff metric), then the underlying manifolds are diffeomorphic. Let NET be an ϵ-net in R_{n,v_0}, such that for $\mu, \nu \in NET$ the Gromov-Hausdorff distance $d_{GH}(\mu, \nu) \geq \epsilon/2$. Assume that NET is computable in the following sense: There is an algorithm computing for every given rational D and δ δ-approximations to all elements of NET with diameter $\leq D$. (There will be only finitely many such elements.) Denote by $NET(M)$ the subset of NET formed by Riemannian structures on M. Now we would like to define the notion of computable real-valued function on R_{n,v_0}. First, note that we constructed in [NW1] an everywhere dense subset of R_{n,v_0} formed by "algebraic over rationals" Riemannian structures. Each element of this set is representable in a finite form (by a finite collection of rational numbers). The underlying manifolds are presented by atlases, where all transition functions are semi-algebraic over the field of algebraic numbers. The Riemannian metrics are given by matrices, where all entries are smooth semi-algebraic functions over the field of algebraic numbers, too. We say that f is computable, if its restriction on this set is computable in the following sense: There is an algorithm computing an approximation to the value at a prescribed argument within to any prescribed positive rational accuracy. Of course, this notion can be extended to functions defined on other sets of Riemannian structures (not necessarily on R_{n,v_0}), once one knows how to construct sufficiently dense computable nets in the domain(s).

Theorem 1.5.2. Let a manifold M be as in Theorem 1.5.1. Let f be a positive computable real-valued function on R_{n,v_0} such that the series $\Sigma_{\mu \in NET(M)} f(\mu)$ converges to a number s. Let A be any consistent recursive system of axioms as strong as ZFC. Then there is $\delta = \delta(M, f, A) = f(T_0)/4 > 0$ such that for no rational t there is a proof in A that $|s - t| < \delta$. Here T_0 is an element of NET that can, in principle, be explicitly constructed.

Remark. Theorems 1.5.1, 1.5.2 follow from Gödel incompleteness theorem asserting that for every sufficiently strong consistent recursive system of axioms A its consistency is not provable in A (and from the observation that the consistency of A is equivalent to the assertion that a certain specific Turing machine never stops). Yet there are plenty of other Turing machines that never stop, but this assertion is not provable in ZFC. For example, Harvey Friedman constructed examples of Turing machine that do not stop, but the assertion that such a machine does not stop is equivalent to consistency of ZFC + some large cardinal axioms (see [Fr]).

2. EUCLIDEAN QUANTUM GRAVITY.

2.1. The Hartle-Hawking approach

A fundamental problem of physics is to unify Quantum Mechanics and Gravity. One possible approach to doing this is the Hartle-Hawking approach ([HH]). This approach leads to the appearance of formal expressions ("partition functions") looking like

$$Z = \Sigma_M c(M) \int_{Riem(M)} \exp(-I) d[g], \quad (*)$$

where the summation is taken over all diffeomorphism types of closed four-dimensional manifolds M, the integral is taken over the space $Riem(M)$ of all isometry classes of Riemannan metrics, on a closed four-dimensional manifold M, $[g]$ denotes the isometry class of a Riemannian metric g, some constants $c(M) \geq 0$ can be zero, and I denotes the Euclidean Hilbert action $I = -\frac{1}{16\pi G}(\int \mathrm{scal(M,g)d}\mu_\mathrm{g} - 2\Lambda\mathrm{vol(M,g)})$. Here $\mathrm{scal(M,g)}$ denotes the scalar curvature, G is the gravitational constant, and Λ is the cosmological constant. One also adds to the action a term involving the matter fields, which will not be discussed here. In the simplest version constants $c(M)$ are equal to 1 or 0 depending on whether or not M is in the class of manifolds over which we would like to perform the summation. For example, [Ha] suggests performing the summation only over simply-connected manifolds. One can imagine a theory where the change of topology is not permitted and $c(M) = 1$ only in the case, when M, say, is the four-dimensional sphere. It is important for us that not only there are no reasons to think that all constants $c(M)$ are equal, but that studies in two-dimensional Quantum Gravity strongly suggest that $c(M)$ must depend on the topology of M and converge to zero when the topology of M becomes increasingly complicated (so that the series in the right hand side of (*) converges). (In the two-dimensional Quantum Gravity one chooses $c(M) = c^g$, where $c < 1$ is a constant and g is the genus of the surface M, cf. [ADJ]).

The underlying assumption here is that the probability density, in some sense, of a positive definite spacetime metric g on the manifold M is proportional to

$\exp(-I)$. (By adding to I a term involving matter fields we can interprete $\exp(-I)$ as the probability density of the metric g on M and matter fields.) According to [H] the probability of an observable O having the value A can be found by integrating $\Pi_A \exp(-I)$ over all Euclidean metrics (and fields) from some class C, where the value of $\Pi_A = 1$ if the value of O is equal to A and zero otherwise. For example, for any two 3-dimensional space-like surfaces X, Y one can consider a propagator ("Hartle-Hawking wave function") defined as the weighted sum over all 4-manifolds M with boundary diffeomorphic to the disjoint union of X and Y of the integral of $\exp(-I)$ over the subspace of Riem/Diff formed by all Riemannian metrics that coincide on the boundary with the given Riemannian metrics on X and Y. (Here one needs to add to the action a boundary term, that we do not discuss here (cf. [Car2], [ACRST]). Also a coupling with matter fields is possible.)

The justification for passing from considering the usual Lorentz metric to considering positively defined Riemannian metrics is so-called Wick rotation which essentially consists in considering the partition functions on the imaginary axis of the complex plane of values of time t (in other words performing the change of variable t for $-it$) and then returning to the real axis via analytic continuation. (One can adopt a more radical point of view ([Ha]) according to which the quantum theory is, in fact, defined only for the Euclidean space-time, and we interprete it as being Lorentz only due to pecularities of our perception.)

Note that one does not know which measure on the space of isometry classes of Riemannian metrics should be used in the right hand side of (*). This is one of the major open problems of this theory (but see [GHS]).

2.2. Geometry of the space of Riemannian structures on a compact manifold and Euclidean Quantum Gravity.

Consider the metric space of all isometry classes of closed Riemannian manifolds of a fixed dimension endowed with the Gromov-Hausdorff metric. (A lot of information about basic properties of this metric space can be found, for example, in [Pet] and [Gr]). The Gromov precompactness theorem implies that for any n, x, d the space of isometry classes of Riemannian structures with Ricci curvature not less than x and diameter not exceeding d on all compact n-dimensional Riemannian manifolds is a precompact subspace of the space of isometry classes of all compact metric spaces endowed with the Gromov-Hausdorff metric. Further, striking results of Colding [Col] and Perelman [Per] establish the continuity of volume regarded as a functional on the space of all Riemannian structures with Ricci curvature $\geq x$ for any x and the continuity of the total scalar curvature regarded as a functional on the space of all Riemannian structures with sectional curvature $\geq x$. Moreover, it seems very plausible that the total scalar curvature is continuous on the space of all Riemannian structures with Ricci curvature bounded below by x for any x. These results suggest an approach to construction of a "measure" in the right hand side of (*) by first defining it for spaces of Riemannian structures of bounded curvature and possibly diameter (and maybe also of volume bounded from below by a positive constant), and then defining the measure for the whole space of Riemannian structures by passing to the limit or, less naively, using a scaling (renormalization) approach (see [ADJ]). However, to the best of our knowledge, this natural idea had never been explored.

Another idea involves the saddle point approximation. In the case of a negative cosmological constant Λ the integral over $Riem(M)$ in the right hand side of (*) is being replaced by the sum $\Sigma c(M)\Delta_M \exp(\frac{-|\Lambda|}{8\pi G}\, vol(M^n))$ over all Einstein metrics of non-zero scalar curvature on all manifolds M. Here the scalar curvature of the considered Einstein metrics of positive or negative scalar curvature is being scaled so that its absolute value is equal to $n(n-1) = 12$. (Einstein metrics of zero scalar curvature are being considered separately.) Δ_M is the factor that comes from gauge fixing and small fluctuations around the extrema (cf. [Car 2], [ACRST]). Einstein metrics are critical points of the total scalar curvature normalized by an appropriate power of volume (so that the resulting functional is scale-invariant). On the basis of our work [NW1], [NW3] we believe that if one would allow some "mild" singluarities of Einstein metrics, then there will be exponentially many Einstein metrics with $Ricci = -(n-1)$ of volume $\leq x$ on every closed manifold of dimension ≥ 4 (as $x \longrightarrow \infty$). Moreover, we believe that these metrics will be scattered throughout the space of Riemannian structures in a more or less uniform way (cf. [NW3], [W] for more precise statements), so that the behaviour of this sum will have essentially the same properties as the summation over nets in the space of Riemannian structures. Although the proof of this picture is beyond our reach, we can prove it for the functional $\int_M scal/vol^{\frac{n-2}{n}} + \epsilon|K|diam^2$ for every positive ϵ providing that either $dimM > 4$, or $M = \#_{i=1}^{46} S^2 \times S^2 \# N_0^4$ for some closed N_0^4 (cf. [NW3]).

2.3. Simplicial Euclidean Quantum Gravity

One of possible ways to make sense of the expression (*) is Simplicial Quantum Gravity. The idea is to replace the integration over the space of Riemannian structures by the summation over the space of (simplicial isomorphism types of) triangulations of the manifold. Given a triangulation one can assign to every 1-dimensional simplex the same length and then endow it with the length metric, thus turning it into a metric space. Although these metric spaces are singular we still can regard them as a net in a small neighborhood of the space of isometry classes of Riemannian metrics. The partition functions in Euclidean Simplicial Quantum Gravity are obtained from the formula for partition fuctions in Quantum Gravity (e.g. (*)) by 1) replacing integration by summation over triangulations; 2) replacing the total scalar curvature by its simplicial analogue $\Sigma_{\sigma^{n-2}\in T_{n-2}}2\epsilon_{\sigma^{n-2}}V_{\sigma^{n-2}}$, where T_{n-2} is the set of $(n-2)$-dimensional simplices of the triangulation, σ^{n-2} denotes a $(n-2)$-dimensional simplex, $V_{\sigma^{n-2}}$ denotes the volume of σ^{n-2}, and

$$\epsilon_{\sigma^{n-2}}=2\pi-\Sigma_{\{\sigma^n|\sigma^{n-2}\in\sigma^n\}}\alpha(\sigma^{n-2},\sigma^n),$$

where $\alpha(\sigma^{n-2},\sigma^n)$ denotes the angle between two $(n-1)$-dimensional faces of σ^n meeting at σ^{n-2}. ($\epsilon_{\sigma^{n-2}}$ is called the deficit angle at σ^{n-2}.) This approach was suggested by T. Regge (see [CMS] concerning mathematical foundations of the Regge calculus). Note that in our situation all simplices are equilateral and therefore all angles $\alpha(\sigma^{n-2},\sigma^n)=\arccos(\frac{1}{n})$, and the volumes $V_{\sigma^{n-2}}$ also depend only on n. Thus the expression for the simplicial analogue of the total scalar curvature can be written as $(4\pi\ N_{n-2}(T)-\arccos(\frac{1}{n})n(n+1)N_n(T))V_{n-2}$, where $N_k(T)$ denotes the number of k-dimensional simplices, and V_k denotes the volume of the equilateral k-dimensional simplex with side one. Combining this expession with the obvious formula for the volume we obtain the formula for the Einstein-Hilbert action: $S_T(c_{n-2},c_n)=c_nN_n(T)-c_{n-2}N_{n-2}(T)$, where c_n,c_{n-2} are coupling constants. The partition function within a fixed topological type of the manifold M^n is given by the following formula:

$$Z_{M^n}(c_{n-2},c_n)=\Sigma_T\exp(-S_T(c_{n-2},c_n)),$$

where the summation is performed over all simplicial isomorphism classes of triangulations of M^n. Then, one defines

$$Z(c_{n-2},c_n)=\Sigma_Mc(M)Z_M(c_{n-2},c_n),\qquad (**)$$

where the summation is performed over all diffeomorphism types of closed n-dimensional manifolds M and $c(M)\geq 0$. The expression in the right hand side of (**) can be rewritten as $\Sigma_Tc([T])\exp(-S_T(c_{n-2},c_n))$, where the summation is performed over all simplicial isomorphism classes of all smooth triangulations of smooth n-dimensional manifolds and the topological weight $c([T])$ depends only on the diffeomorphism type of the manifold triangulated as T. Of course, the most natural choice of n here is $n=4$. We refer the reader to [ADJ] and references there for more details.

An important question for the viability of this approach is whether or not the number $T_M(N)$ of different simplicial isomorphism classes of triangulations of a fixed closed (four-dimensional) manifold M with $\leq N$ simplices grows exponentially or superexponentially. More precisely, is there $c(M) > 1$ such that $T_M(N) \leq c(M)^N$ for each N? (This question was posed in [F].) At the very least we need the existence for every l of a constant $c(M,l)$ such that $c(M,l)^N$ majorizes the number $T_M(N,l)$ of simplicial isomorphism classes of triangulations of M with $\leq N$ simplices such that not more than l n-dimensional simplices can be adjacent to a vertex of the triangulation. The not yet proven conjecture that $T_M(N)$ (or at least $T_M(N,l)$) has an exponential upper bound would make this approach much easier to implement.

3. LOGIC PHENOMENA OF QUANTUM GRAVITY

A well-known philosophical problem is whether or not it is possible to verify mathematical assertions by physical measurements. Is there some place in the physical world where the validity of, say, the Riemann Hypothesis is reflected, so that one can just perform the measurements confirming or refuting it (instead of trying to prove or disprove the conjecture)? Or, perhaps the mathematical world is quite independent from the physical reality? (A precise conjecture in this direction can be found in [L].) Until now, however, this question was almost metaphysical in view of the absence of meaningful ideas that could lead to its resolution in either direction. The only known to us exception is the paper [WZ], where it had been demonstrated that non-computability results of M. Pour-El and J. Richards in the theory of wave equations ([PR]) do not lead to even a hypothetical possibility of measuring non-computable quantities.

Here we suggest how it *might* be possible to check experimentally mathematical conjectures from a certain very wide class under certain assumptions about what does the (not yet existing) theory of Quantum Gravity look like. We assume that the approach suggested by Hartle and Hawking or a similar one is correct plus making some rather plausible technical assumptions about coefficients in this theory explained below. A crucial assumption is that different topologies do *not* enter with equal weights in the expression for partition function. This assumption seems to be in accordance with the views of many specialists in Euclidean Quantum Gravity (e.g. [BDGSS]) and is partially substantiated by the divergence of partition functions without this assumption ([ACRST]).

Here is our idea in a nutshell. The approach suggested by Hartle and Hawking relates physical observables with partition functions that are sums over all topologies of closed four-dimensional manifolds of certain integrals over the space of isometry classes of Riemannian metrics on the manifold. Each integral in the sum comes with a weight depending on the topology of the manifold. For example, in a version of the model where the only allowed topology is the topology of the 4-dimensional sphere, the weight corresponding to this topology is equal to one, and all other weights are equal to zero. More generally, the considered series are convergent only if the weights corresponding to manifolds with complicated topology (in particular, with a complicated fundamental group) are very small or just zero (cf. [ACRST]).

The classical theorem of Markov implies that there is no algorithm deciding whether or not two given 4-dimensional manifolds are isomorphic. Moreover, for $n \geq 5$ and any n-dimensional manifold M_0^n there is no algorithm that decides whether or not a given manifold M^n is diffeomorphic to M_0^n. It is highly plausible but not yet proven that this assertion is true also in dimension 4. The proof of this theorem is based on the classical Gödel and Turing result asserting the non-existence of an algorithm deciding whether or not a given Turing machine halts. (Informally, this means that there is no algorithm that decides if a given computer program eventually ends its work or runs infinitely long.) Moreover, the proofs of the mentioned algorithmic results in topology are constructive: We can find an explicit M_0^4 and an explicit way to construct $M^4(T)$ for each given Turing machine T (= a given computer program) so that $M^4(T)$ is diffeomorphic to M_0^4 if and only if T halts with the empty tape. If T does not halt then $M^4(T)$ has the same homology as M_0^4 but a very complicated fundamental group. We can make this fundamental group as large as we wish. We have a lot of freedom of choosing M_0^4. It is a plausible conjecture, that for any T M_0^4 can be an arbitrary closed four-dimensional manifold here (e.g. S^4).

So, the weights $w(T)$ and w_0 corresponding to $M^4(T)$ and M_0^4 coincide if T halts and are very different if T does not halt. For example, assume that only the topological type of S^4 has a non-zero weight in our model. Assuming the conjecture that there is no algorithm recognizing S^4 we can conclude that the weight equal to 1 or 0 depending on whether or not our manifold is S^4 is non-computable. Our main idea is that since observables depend on the values of the weights we can perform experiments determining the values of the observables and then determine the weights solving some systems of equations. Since w_0 and $w(T)$ either coincide or are very different it is sufficient to find $w(T)$ only approximately in order to find out whether or not T halts.

But it is well-known that many unsolved problems of mathematics can be rewritten as the assertion that a (very explicit) Turing machine looking for a counterexample never halts. Thus, there might exist an experimental method verifying the validity of a very wide class of mathematical conjectures.

To illustrate these ideas assume for simplicity that an observable is given by the series $\Sigma_T 2^{-2^{N_4(T)}}$, where the summation is performed over all triangulations T of $M^4=\#_{i=1}^{46}S^2\times S^2$, and $N_4(T)$ denotes the number of four-dimensional simplices in T. Assume that this observable can be measured within to any prescribed accuracy. This means that we will be able to obtain the value of the sum S within to any prescribed accuracy. The series can be rewritten as $\Sigma_{N=1}^{\infty}T_{M^4}(N)2^{-2^N}$, where $T_{M^4}(N)$ denotes the number of triangulations of M^4 with $\leq N$ simplices. Recall that the halting problem for any Turing machine T with description of length $\leq L$ can be effectively reduced to computation of $T_{M^4}([const(n)\ L])$ (see section 1.4). But since the series converges very rapidly we will be able to find $T_{M^4}(N)$ recursively as the nearest integer to $2^{2^N}(S-\Sigma_{i=1}^{N-1}T_{M^4}(i)2^{-2^i})$. Note that here we can use a sufficiently close approximation to S instead of S. (In other words we can use the result of a sufficiently precise measurement of S).

Our second idea is based on the fact that for any recursively axiomatizable first order theory there exists a Turing machine T that does not halt, but this fact cannot be proven within the considered theory (seee section 1.5). This fact might lead to the impossibility to predict a value of an observable with accuracy better than a certain positive constant. To put this observation into a proper perspective note that [GH] and [Ht2], ch. VII discussed the philosphical implications of non-computability of numbers that appear in Quantum Gravity. It had been noted in [Ht2], ch. VII, that this non-computability might have no serious implications since it does not preclude computations of these numbers within to any fixed prescribed accuracy. (The non-computability of a real number means only that the amount of computations required to compute the number within to a given accuracy grows uncontollably fast as the accuracy tends to zero. So, formally speaking, it does not manifest itself for any specific value of the desired accuracy.) Our observation means that this is not so: If we are interested not only in the existence of the computations but would like to know how to perform them (or, more formally, we would like to have a proof that the computations produce a desired approximate value), then there will be finite limit to the accuracy that one can reach. This undeterminancy can, in principle, be explicitly estimated and is not too small.

Here is a brief survey of previous work on interactions between computability theory and Quantum Gravity. It seems that the relevance of the algortithmic unsolvability issues for the Hartle-Hawking model was first discussed in [Ht] and [GH]. In particular, J. Hartle observed that at least in high dimensions there is no algorithm deciding whether or not a simplicial complex is a manifold and suggested to define partition functions using summation not only over all topologies of manifolds but also some of topologies of pseudomanifolds. This idea was later developed in papers by K. Schleicher and D. Witt ([SW]). Yet this issue does not arise in dimension four due to the fact that the three-dimensional sphere is algorithmically recognizable ([T]) and, therefore, there is an algorithm deciding whether or not a given 4-dimensional simplicial complex is a triangulated 4-manifold. Our paper [NB] contained an observation that the partition functions arising in Simplicial Quantum Gravity are not computable, and therefore cannot be evaluated by numerical methods as it was attempted by several groups of researchers (cf. [ADJ]). Yet ad hoc there is a possibility that these functions can in fact be "almost" computable: Assume for example, that $Z(N)$ is a strictly increasing non-computable integer-valued function such that there exists a computable function $a(N)$ such that $|Z(N)-a(N)|\leq 1$. Assume further that the values of a physical observable can be obtained by taking ratio of two such functions and passing to the limit. Then these values will be, in fact, computable. (The possibility that a physical quantity defined using non-computable functions might turn out to be computable was pointed out in [GH] and [Ht2].) So, one needs an analysis how closely the non-computable partition functions arising in Quantum Gravity can be approximated by computable functions. To the best of our knowledge questions of this type were never investigated before. We believe that these partition functions cannot be approximated by computable functions with a relative error converging to one. In section 1.4 we proved slightly weaker results in this direction (Theorems 1.4.1, 1.4.2). The algorithmic information theory turns out to be a useful tool in this investigation. To the best of owr knowledge these results are first results of such nature in Mathematics.

3.0. Basic assumptions.

The phenomena described in this chapter are based on the assumption that the physical world satisfies a version of the Hartle-Hawking Quantum Gravity.

We impose very few constraints on the cofficients $c(M)$ in formulae (*) in section 2.1 and (**) in section 2.3. For example, one can assume that $c(S^4)=1$ and $c(M)=0$ is equal to zero for all other manifolds M. In this case we need to assume the validity of Conjecture 1.1. But, if we will be assuming that $c(\#_{i=1}^{46}S^2\times S^2)\neq 0$, then we do not need this conjecture. The crucial assumption is that $c(M)$ do not have the same values, or, more generally, do not depend only on $H_*(M)$ (which is computable.) This assumption seem to follow from the assumption that the series in (*) or (**) converges: Since there are superexponentially many topological types of four-manifolds that can be triangulated with $\leq N$ simplices, this means that coefficients $c(M)$ are zero (or superexponentially small which is not natural from the point of view of physics) for almost all topological types of manifolds (but exponentially many) (cf. [F]). (We would like to attract the attention of the reader to the paper [Car] which discusses the implications for Quantum Gravity of the superexponential growth of the number of hyperbolic manifolds with volume $\leq V$

as well as to [ACRST]. See also [BDGSS] for a different physical argument that seems to imply that $c(M) = 0$ for all non-simply connected manifolds.)

The phenomena discussed in sections 3.1, 3.2 below are quite robust with respect to changes in the theory (e.g. inclusion of some topological types of singular manifolds in the formulae for partition functions).

3.1. Experimental Mathematics

Here we would like to describe how it might be theoretically possible to determine the validity of mathematical statements by physical measurements providing that our world is indeed described by a theory reasonably close to the theory suggested by Hartle and Hawking. (Moreover, one can even verify validity of some statements not provable in the usual ZFC system of axioms of set theory (which is a de facto system of axioms of the whole mathematics)!) We are going to explain our ideas in the framework of the simplicial version of Euclidean Quantum Gravity, but these ideas equally apply to the original Riemannian version. To give a very brief explanation of this mechanism assume that we want to know whether a certain mathematical statement S is true or false. We can explicitly construct a manifold M_S corresponding to this statement and a nice "model" manifold M_0 ($\#_{i=1}^{46} S^2 \times S^2$ or even S^4, if the Conjecture 1.1 is true) such that M_S and M_0 will be the same if and only if S is true. There is no algorithm to decide whether or not M_S and M_0 are "the same" (diffeomorphic). However, we might be able to answer this question by means of a measurement. The idea is that in the Hartle-Hawking version of Quantum Gravity all physical quantities theoretically can be calculated in terms of partition functions (see above). In the definition of these partition functions one has coefficients depending only on topology of various four-dimensional Riemannian manifolds. In particular, the value of one of these coefficients, c_S, depends on whether M_S and M_0 are diffeomorphic. One can regard these coefficients as unknowns, measure values of physical quantities and determine the values of coefficients from the equations relating partition functions and physical observables. As the result one will obtain an approximate value of c_S. Looking at this value we will see whether or not M_S is diffeomorphic to M_0 and therefore whether or not S is true.

Let us explain this in more details.

We suggest that it might be possible to check experimentally if a given Turing machine halts with a given input. So, the first step is to reduce the mathematical problem P of interest to the question whether or not a specific Turing machine $T(P)$ halts for the empty tape. $T(P)$ should be explicitly constructed. This can be easily done for very many interesting mathematical problems.

Consider for example the Riemann Hypothesis. It is shown in [Lag] that the Riemann Hypothesis is equivalent to the assertion that for each integer $n \geq 2$ the sum of all divisors of n, $\sigma(n)$, satisfies the inequality

$$\sigma(n) < H_n + \exp(H_n) \ln H_n,$$

where $H_n = \Sigma_{i=1}^{n} \frac{1}{i}$ is the n-th harmonic number. Now we see that the Riemann Hypothesis is equivalent to the assertion that a Turing machine that is looking for a counterexample to this inequality trying one by one all integer values of $n \geq 2$ never halts.

For any mathematical assertion P we can easily represent the provability of P in ZFC (the Zermelo-Frankel system of axioms of the set theory+the axiom of choice, which is often regarded as a de facto system of axioms for the whole mathematics) as the statement that the Turing machine looking for a proof of P in the Zermelo-Frenkel system of axiom + the axiom of choice halts.

It is even more interesting from a philosophical point of view that we can represent the consistency of ZFC as the assertion that the Turing machine looking for proofs in ZFC of a well-formed formula and its negation does not halt. This observation might imply the possibility of verification of the consistency of ZFC by a physical experiment as described below.

The second step will be to represent the halting problem $T(P)$ as the triviality problem for a finitely presented group $G(T(P))$ and then to represent $G(T(P))$ as the fundamental group of either a triangulated manifold or a Riemannian manifold $M(P)$ depending on whether we believe in Simplicial Quantum Gravity or in the Hawking-Hartle version of the Quantum Gravity. $M(P)$ is PL-homeomorphic (or diffeomorphic) to the connected sum of 46 copies of $S^2 \times S^2$ if and only if $T(P)$ halts with the empty tape. (Again, it is quite plausible that we can construct $M(P)$ such that $M(P)$ is PL-homeomorphic (correspondingly, diffeomorphic) to S^4 if and only if $T(P)$ halts with the empty tape.) The second step is provided by the geometric witness construction sketched above. This construction is well-understood and can be actually performed in practice (using a computer). (See [N1], [N4], [NW1] on how to represent a Riemannian manifold as a finite set of data.) Note that we can perform this construction so that if $T(P)$ does not halt then $G(T(P))$ is not only non-trivial but contains any prescribed in advance group and, therefore, is as complicated as we want it to be.

So the original problem boils down to the question whether or not a given Riemannian manifold or a triangulated manifold is diffeomorphic (correspondingly, PL-homeomorphic) to a fixed manifold M_0 $(= \#_{i=1}^{46} S^2 \times S^2$ or even $S^4)$. This question is algorithmically unsolvable but might be solvable by means of an experiment as follows: Experimentally determine a number of values of physical quantities with a very high accuracy. Assume that we work in Simplicial Quantum Gravity. Oversimplifying the situation assume that expressions for all physical quantities are of the form $F(\Sigma_T c(\text{topology of } T)w(T)f(T))$, over all triangulations T of all compact four-dimensional manifolds, where the weights $c(\text{topology of } T)w(T)$ play the role of the measure on the space of triangulations. $c(\text{topology of } T)$ can be equal for example to zero or one depending on whether or not T is a triangulation of S^4 (or on whether or not T is a triangulation of a simply-connected manifold), but $w(T)$ is assumed to be computable (within to any prescribed accuracy). Now we can regard the values of $c(\text{topology of } T)$ as unknowns, formulas expressing physical quantities that can be measured as equations and to determine $c(T)$ for all (or for some) triangulations with a not very large number of simplices from these equations. Since we would like only to distinguish between M_0 and a homology M_0 with sufficiently complicated fundamental group we need to determine $c(T)$ only within to some fixed accuracy $\delta > 0$. If $c(M_0) = 1$ and all other $c(M) = 0$, then we can take $\delta = 0.49$. There are three sources of error: the error of experiment which hopefully can be made arbitrarily small, the error of computation of $w(T)$ which also can be made arbitrarily small, and the error due to the fact that it will be necessary to replace the infinite sum over the set of all triangulations by a finite sum over the set of all triangulations with the number of simplices less than some number N_* chosen by us and much larger then the number of simplices in the triangulation of interest. Hopefully making N_* sufficiently large we can make this part of the error arbitrarily small. (This involves some assumption about our infinite system of equations: the solutions of truncated finite systems with finitely many variables should approximate the solution of the original infinite system reasonably well. This assumption is of course just a conjecture, but we feel that this conjecture is a plausible one.)

The discussion above motivates the following conjecture:

Conjecture (Experimental Mathematics in Simplicial Quantum Gravity). Assume that at least one of the non-negative coefficients $c(M)$ in the right hand side of formula (**) in section 2.3 is not equal to zero, and the series in the right hand side of (**) (for $n = 4$) converges for all pairs (c_2, c_4) in an open domain U in the plane. Then the halting problem for Turing machines can be solved in a computable time by a Turing machine that uses an oracle providing for each given k, each given pair of rational numbers $(c_2, c_4) \in U$ the first k digits of the sum of the series in the right hand side of (**).

Of course, one can state various Riemannian versions of this conjecture. Also, one can state it for convergent series of the form $\Sigma_T c(T) f(\rho, T) \exp(c_2 N_2(T) - c_4 N_4(T))$ (instead of (**)), where the sum is taken over all triangulations T of closed 4-dimensional manifolds, $c(T)$ depends only on the PL-homeomorphism type of T, $\rho \in [a, b]$ is a parameter, the coefficients c_2 and c_4 are fixed, and f is a positive real-valued computable function such that for any T it is differentiable with respect to ρ, and $\frac{df(\rho,T)}{d\rho}$ has a discrete set of zeroes. (By computability of f we mean that there exists an algorithm computing its value within to a prescribed accuracy for any given T and a rational value of $\rho \in [a, b]$.) The conjecture is that having an oracle providing an approximation to the value of the series for any prescribed rational $\rho \in [a, b]$ within to any prescribed accuracy one can solve the halting problem for arbitrary Turing machines.

Note that our assumption that physical observables are given by expression similar to (*) or (**) is possibly too optimistic. It can be that they are obtained from expression for partition functions using a renormalization approach in which case the possibility of experimental determination of constants $c(T)$ could depend on whether or not the answers for problems at the end of chapter 1.4 are positive.

Finally, it might be interesting to note that the idea of "experimental mathematics" outlined above seems to be in a direct contradiction with the following philosophical postulate stated dy Leonid Levin ([L]): "Every "physically existing" sequence must be "finitary recursive", i.e. have approximately as short recursive expression as any its non-recursive ones".

3.2. Logic uncertainty principle in Quantum Gravity

In this section we we give a physical interpretation of Theorems 1.5.1, 1.5.2 in chapter 1. Recall, that the starting point of the proof of these theorems is the observation that for any consistent first order system at least as strong as the Peano aritmetic there exists a Turing machine T that does not halt when it starts to work with the empty tape, but this fact is not provable in this system. For example, T can look for a formula A and proofs within the considered formal system of A and of the negation of A.

The geometric witness construction can be used to construct a closed Riemannian manifold M_T such that M_T is not diffeomorphic to the connected sum of 46 copies of $S^2 \times S^2$ but this fact cannot be proven in the considered system of axioms (e.g. ZFC). In fact M_T can be chosen to have fundamental group of arbitrarily large complexity. (Note that we believe that it is possible to construct M_T such that M_T is not diffeomorphic to the sphere S^4 but this fact is not provable in the considered system of axiom. This fact would be true in dimensions greater than four.)

Assuming that the weight $c(M_T)$ depending on the topology of M_T at the integral over the set of Riemannian structures close to M_T in the expression (* *) for partition fuctions is much smaller than the weight for the connected sum of 46 copies of $S^2 \times S^2$ we see that we are not able to distinguish between two values of the partition function (* *): one where the weight depending on the topology of M_T is correct and the second one (incorrect), where this weight is equal to the weight for the connected sum of 46 copies of $S^2 \times S^2$. By "distinguish" we mean that for any rational number r in the interval between these two values (correct and incorrect) there is no proof in the chosen formal system that the difference between the correct value of the observable and r is less than the length of the interval.

Thus, it is natural to expect that the same "logic uncertainty principle" will be true for many physical observables: *For any recursively axiomatized first order system A there exists $\delta_A > 0$ such that for any rational number t there is no proof in A of the assertion that the difference between t and the actual value of the observable s is less than δ_A. (Of course δ_A depends on the observable and on its value as well as on A.) One can find an explicit positive lower bound for δ_A.* Of course, practically it is extremely important that δ_A is not too small. In Important Remark after Theorem 1.5.1 we explain how one can find an explicit positive lower bound for δ_A. On the basis of this discussion and assuming that s is given as a sum of a convergent series over the space of triangulations of S^4 with terms decreasing only slightly slower than $T_{S^4}(N)^{-1}$ we think that it is quite plausible that the relative error δ_A/s could turn out to be sufficiently high to make the discussed phenomenon relevant for physics.

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