Algorithmic Aspects of Homeomorphism Problems

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In this paper, we will describe some results regarding the algorithmic nature of homeomorphism problems for manifolds. Very well known is the result of Novikov that for no manifold of dimension at least five is the recognition problem solvable. This is a consequence of the undecidability of recognition of triviality for finite presentations of groups (see section 10 of [**VKF**] or the appendix to [**N**]). For instance, by the Poincaré conjecture, a PL manifold in these dimensions is the sphere iff its homology is that of a sphere and its fundamental group is trivial. Similarly, Kervaire and Milnor's [**KM**] work on smooth structures shows that one can also produce an algorithm to recognize the sphere up to diffeomorphism among simply-connected manifolds.

Indeed, there has been a certain literature on the recognition and classification problems for restricted classes of manifolds: see e.g. [**B**], [**W1**], [**W2**], [**Z**]. As is standard we will view PL manifolds as given by triangulations, and smooth manifolds via smooth triangulations or polynomial equations. Barden had speculated that modulo the problem of ascertaining whether or not fundamental groups were isomorphic, the homeomorphism problem could be decidable. However that is too optimistic as the following simple construction shows:

PROPOSITION 0.1. *Let π be a torsion-free group where $K(\pi, 1)$ has a finite 3-skeleton and such that the word problem for π is unsolvable*[1]*. Then one cannot decide homotopy equivalence nor homeomorphism for manifolds with fundamental group π.*

PROOF. Let X be a finite 3-skeleton, and let M be the boundary of a regular neighborhood in $\mathbb{R}^n$ for $n > 6$. Then $\pi_2(M) = 0$. Consider $M' = M\#S^2 \times S^{n-3}$. $\pi_2(M') \simeq \mathbb{Z}[\pi]$, with generator the core S^2. If we now surger $(g - 2e)S^2$ to obtain $M''(g)$, then $\pi_2(M''(g)) \simeq \mathbb{Z}[\pi]/(g - 2e)$; so $\pi_2 \simeq 0$ if $g = e$ and contains $\mathbb{Z}[1/2]$ (by torsion freeness) if $g \neq e$. Thus being able to decide homotopy type is the same as being able to decide diffeomorphism for these manifolds which boils down to solving the word problem. $\square$

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[1] These are produced by the usual proofs of the unsolvability of the word problem.

1. Some positive results

Nonetheless we would like to point out that something is true in the direction suggested by Barden, namely that by combining the works of [**B**],[**S**],[**Wi**] and [**GS**] (and [**T**]) one can prove:

THEOREM 1.1. *Every PL or smooth simply-connected manifold M^n of dimension n at least 5 can be recognized among simply-connected manifolds. That is, there is an algorithm to decide whether or not another simply-connected manifold is Top, PL or Diff isomorphic to M.*

ADDENDUM 1.2. Moreover, an analogous statement is true for embeddings in codimension not 2: one can algorithmically recognize any given embedding of one simply-connected manifold in another up to isomorphism of pairs, or up to isotopy, if the codimension of the embedding is not two. In [**NW**] it is shown that in codimension two knot triviality is undecidable for spheres in spheres.

The proof is by concatenation of known results. Firstly for the homeomorphism problem. [**Br**] enables one to give a description of the first n stages of a Postnikov system for M ($n = \dim M$). The problem is to decide whether the analogous system for some other N is homotopy equivalent. If so, obstruction theory shows that M and N are homotopy equivalent and we are most of the way done.

Deciding this can be done by induction on n. The bottom part of the Postnikov tower is the first nonzero homotopy(=homology) group. Matching up the next stage involves knowing whether the Postnikov k-invariants of M and N can be moved to one another by a self-homotopy equivalence of the first stage. This group of self homotopy equivalences is clearly an arithmetic group (it is commensurable to $\mathrm{SL}_m(\mathbb{Z})$ for some m) and we are asking how whether two integer vectors of some symmetric power of the defining representation lie in the same orbit. The main result of [**GS**] provides an algorithm for deciding whether two elements of a lattice in a vector space acted on by a given arithmetic group lie in the same orbit. (Note that to prove the existence of an algorithm, rather than to exhibit one, it suffices to give a commensurability result, not an explicit description of the self-homotopy equivalence.) For the general inductive step, note that the autohomotopy equivalences of any finite Postnikov piece form an arithmetic group by [**S**], [**Wi**] so one can compare the k-invariants for the two spaces. (Note that in terms of the complexity of this proof, to complete the induction one must go from the algorithm that shows that one Postnikov stage is homotopy equivalent to another to a specific homotopy equivalence that achieves this in the next stage of comparison of k-invariants. This, in principle, can increase the complexity of the problem enormously.)

This same arithmeticity applies to the action of $\mathrm{Aut}(M)$ on the structure set $S(M)$ computed by the surgery exact sequence (see [**B**], [**W3**], [**R**]). More precisely, letting

$$S(M) = \{(M', h) | h : M' \to M \text{ is a homotopy equivalence}\}/\text{Cat isomorphism},$$

one has an isomorphism of abelian groups $S^{\mathrm{Top}}(M) \simeq [M^\circ : F/\mathrm{Top}]$, where $M^\circ =$ M with a point deleted, and F/Top is a very simple generalized cohomology theory, mostly detected by rational Pontrjagin classes. Thus one can decide first whether the rational Pontrjagin classes can be matched up, since this is a question about an action of an arithmetic group on a vector space, and then Browder-Novikov theory asserts that what remains is a finite order problem, which clearly presents no algorithmic difficulties. Q.E.D. (Theorem).

In order to prove the results about embeddings one needs to understand the relevant homotopy theory and the relevant surgery. The latter can be found in Wall's book [**W2**] (see also [**We**]) where these issues are related to the theory of "Poincaré embeddings". In other words, one has a similar surgery exact sequence which computes the set of embeddings that realize a given homotopical model in terms of maps into various classifying spaces. Oddly enough, as far as we know, the relevant arithmeticity theorems for autohomotopy-equivalences of the relevant diagrams of spaces (in the sense of, say, [**DK**],[**DS**]) do not appear explicitly in the literature. However, these results are implicit in [**T**] which deal with arithmeticity of equivariant self-homotopy equivalences. As a warm-up, let's first prove arithmeticity for self homotopy equivalences of pairs (A, B), where A and B are finite simply-connected CW complexes. Consider the space with involution $X = A \times E\mathbb{Z}_2 \cup B \times c(E\mathbb{Z}_2)$, where c denotes "cone"; this space is homotopy equivalent to A and has a homotopically trivial involution whose fixed-point set is B. One can easily see (by obstruction theory) that the self-homotopy equivalences of (A, B) form a subgroup of finite index in the equivariant self-homotopy equivalences of X[2], which is itself arithmetic as verified in [**T**].

From an embedding, one can build an auxiliary space as above, now with $\mathbb{Z}_6$ action, whose equivariant self homotopy equivalences are commensurable with the stratified self-homotopy equivalences of the pair (M, N). (One constructs a $\mathbb{Z}_6$ action whose fixed set is the boundary of a regular neighborhood of the submanifold, and whose fixed sets of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are the submanifold and its complement, respectively.) An argument (involving "diagram Postnikov systems" formally similar to equivariant Postnikov systems) similar to the case of closed manifolds completes the proof. Q.E.D. (Addendum).

REMARK 1.3. Along the way we also verified the theorem for simply-connected manifolds with simply-connected boundary.

REMARK 1.4. Although group actions were used in the proof of the relevant arithmeticity statements in the theorem, one can reverse the process and use these arithmeticity statements to get information about *connections* between group actions and embeddings; see [**We2**].

REMARK 1.5. By way of contrast, one can show, using the unsolvability of Hilbert's tenth problem, that the problem of the existence of a smooth embedding of M in N is, in general, unsolvable.

[2]Strictly speaking, one should choose a finite CW invariant subspace of X containing A and having the same rational homotopy type. The rational version of Wall's finiteness theory easily provides such. (See e.g. [**We3**].)

2. Some negative results

We now return to nonsimply-connected manifolds. Since the homotopy problems are apt to be undecidab1e[3], it is interesting to consider instead the issue of deciding whether (simple) homotopy equivalent manifolds are isomorphic. There are two variants of the question: One asks if one can decide whether a given homotopy equivalence is homotopic to an isomorphism. The second asks whether, with the knowledge that the manifolds are homotopy equivalent, one can decide whether they are isomorphic. The trouble is that one does not specify here what the underlying homotopy equivalence should be.

We will soon see that the answer to both questions is no. The first is easier to settle. Roughly speaking the issue is that considered by surgery theory, so one needs only to concoct a situation where the surgery exact sequence can be shown to be uncomputable[4]. The second problem runs into the problem of self-homotopy equivalences[5], and here we know nothing like arithmeticity, so conceivably the whole obstruction to isomorphism can be absorbed into some self homotopy equivalence. We will get around this by using $H\rho$ defined in [**We4**].

THEOREM 2.1. *There are manifolds which cannot be distinguished algorithmically even among manifolds assumed to be (tangentially simple) homotopy equivalent to them.*

We start with the work of [**BDM**] which produces finitely presented groups with fairly general recursively presentable group homology.

Let A denote the assembly map studied in surgery theory, see e.g. [**R**],[**We**].

LEMMA 2.2. *For groups that are obtained from the trivial group by a series of amalgamated free products and HNN extension, the assembly map $A : H_*(B\pi; L) \to L_*(\pi)$ is an isomorphism away from the prime 2.*

This is very close to the result of [**C**] except that he only allows amalgamation along already constructed subgroups in his inductive scheme and we allow arbitrary subgroups in ours. However, if one adds in the fact that both the domain and range of the assembly map commute with direct limits, one is allowed to have arbitrary unions of constructible groups in Cappell's sense to be constructible. This slightly larger class is clearly closed under taking subgroups as "tree theory" shows: i.e. define a group of type n to be one that acts on some (perhaps infinite) simplicial tree with type $(n-1)$ groups as vertex and edge stabilizers. Clearly, this property is preserved under passing to subgroups, and the direct limit argument shows that Cappell's proof of the above lemma for "constructible" defined in terms of finite trees implies the version defined here. Q.E.D.

Now take a finitely presented group π from [**BDM**] whose homology, while torsion free, is not algorithmic in dimension $n+5$. More precisely, assume that this

[3]It is, of course, very interesting to investigate how algebraicizable the homotopy theory of nonsimply-connected spaces with given fundamental group is as an algorithmic "invariant" of a group. Finite groups do not present much difficulty, but even free abelian groups would reward a careful analysis.

[4]In general the surgery sequence is a long exact sequence for computing $S(M)$ and involves deep information about quadratic forms over $\mathbb{Z}[\pi, M]$.

[5]As elements (M', h) of $S(M)$ can be nontrivial because of either $M' \not\approx M$ or because h just happens not to be homotopic to a homeomorphism.

homology group has infinitely many generators x_i, $i = 1, 2, 3....$ and infinitely many relations, all of the form $x_i = 0$ for $i \in I$, where $I \subset \mathbb{N}$ is a recursively enumerable nonrecursive set. Using this lemma, and examination of the construction of [**BDM**], one can assume that A maps all of this split injectively into $L_*(\pi)$. Let M be an antisimple manifold with fundamental group π; concretely, choose M to be the boundary of a regular neighborhood of a 2-complex with fundamental group π embedded in $\mathbb{R}^{n+1}$. According to [**We4**] one can define $H\rho$ for manifolds homotopy equivalent to M. By definition, $H\rho(M) = 0$. We can act (in the sense of the surgery exact sequence[6]) on $S(M)$ by an arbitrary element of of $L_{n+1}(\pi)$ to obtain a new manifold M_α tangentially simple homotopy equivalent to M (essentially M_α comes with a cobordism to M and α is the obstruction to surgering this cobordism rel $M_\alpha \cup M$ to a product $M \times I$). In particular we can choose α to be the $A(x_i)$ for $i \in \mathbb{N}$ defining for us a sequence $M_i = M_{A(x_i)}$. (To give an algorithmic definition of A sufficient for our purposes, one can realize elements of homology by manifolds, by a theorem of Thom, and thus constructively by a trial and error algorithm. Then one computes the symmetric signature of that manifold by a simplicial method, and rolls it up by Ranicki's algebraic surgery to be an explicit quadratic form.)

Claim 2.3. *$H\rho(M_i) = 0$ iff $A(x_i) = 0$ and hence iff $i \in I$ and iff $M_i \simeq M$.*

Since the homology is torsion free, $x_i = 0$ iff $A(x_i) = 0$ modulo the image of $\oplus H_{n+1-4j}(\pi; L)$, since by assumption, $x_i \in H_{n+5}(\pi)$ and A is an isomorphism away from 2. (Note that since A is injective the image of $\oplus H_{n+1-4j}(\pi; L)$ is disjoint from the image of $H_{n+5}(\pi; L)$.) By definition, the latter is $H\rho(M_n)$. Thus $H_\rho(M_i) = 0$ implies $x_i = 0$ which then implies $M_i = M$. Note that although $H\rho$ depends on a choice of identification of fundamental group with π, the *vanishing* of it is independent of this, so no automorphism, even moving π_1 around, helps in producing an isomorphism if $H\rho(M_i) \neq 0$. Since the M_i are constructively built manifolds tangentially simple homotopy equivalent to M, an algorithm to decide homeomorphism within this restricted class would also give an algorithm to decide membership in I, which by hypothesis is imposible, proving the theorem. Q.E.D.

Remark 2.4. It is possible to prove the theorem using only a special case of the definition of $H\rho$, that is actually a bit simpler. We can assume that π above is the fundamental group of an acyclic three complex (this just requires assuming that $H_1(\pi) = H_2(\pi) = 0$) and use this complex in building an antisimple homology sphere. If $M' \to M$ is a homotopy equivalence, then $[M'] = [M]$ in $\Omega(M)$ — where Ω denotes smooth bordism, (since Ω is a homology theory, and $\Omega(*)$ is detected by Stiefel-Whitney and Pontrjagin numbers, all of which vanish for homology spheres either because they lie in trivial groups or using the Hirzebruch signature theorem) and hence in $\Omega(B\pi)$. This enables one to eliminate the use of intersection homology and Witt bordism theory from from [**We4**], which define the appropriate context for the bordism between homotopy equivalent manifolds, in general. However, one still needs the truncation algebraic nullcobordism trick from there to get a closed algebraic Poincaré complex and to see the independence of the cobordism used.

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