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IRRATIONALITY OF LIMITS OF QUICKLY CONVERGENT ALGEBRAIC NUMBERS SEQUENCES

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(Communicated by Larry J. Goldstein)

ABSTRACT. We present criteria for irrationality of limits of convergent sequences of rational numbers, algebraic numbers of the same degree and of strictly increasing degrees.

The criterion for irrationality of limits of a sequence of rational numbers has the form of an infinite system of inequalities on successive differences between elements. These inequalities are not strict. If these inequalities are satisfied, then the limit is rational if and only if all inequalities but a finite set of them are satisfied as equalities and the sequence becomes monotonous beginning from some element. So, the criterion permits to see a "border" between rationality and irrationality for some class of quickly convergent sequences.

Quickly convergent rational numbers sequences tend to converge to irrational limits. Many special cases of this phenomenon are known (e.g. [**1**, **4**]). In this paper we present sufficient conditions on the convergence of a rational numbers sequence, which are also necessary if some infinite system of inequalities on elements of the sequence is satisfied. The limit of this sequence is irrational, if and only if the infinite set of these inequalities are strict.

Similar criteria for irrationality of limits are obtained for sequences of algebraic numbers of the same degree and for sequences of algebraic numbers with strictly increasing degrees. To apply these criteria, it is sufficient to know only coefficients of minimal polynomials of elements of the sequence. These results are interesting also because it is little known about the approximation of an algebraic number by algebraic numbers of higher degrees (cf. [**2**] and its exposition in [**3**]).

Lemma 1 contains the main idea of our proofs. Henceforth, Z denotes the set of integers and N denotes the set of positive integers.

LEMMA 1. *Let $\{Q_n(x)\}_{n=1}^{\infty}$ be a sequence of polynomials with integral coefficients and let a be a real number, such that $Q_n(a) \neq 0$.*

(a) *Let all Q_n be of the same degree. If there exists an n_0, such that for $n \geq n_0$*

$$|Q_{n+1}(a)/Q_n(a)| \leq 1, \tag{1}$$

and if the inequality (1) *is strict for an infinite number of indices, then the number a is irrational.*

(b) *Let* $\deg Q_{n+1} > \deg Q_n$ *for sufficiently large values of n. If*

$$\lim_{n\to\infty} |Q_{n+1}(a)/Q_n(a)|^{1/(\deg Q_{k+1} - \deg Q_n)} = 0, \tag{2}$$

then the number a is irrational.

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