Algorithmic unsolvability of the triviality problem for multidimensional knots

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Abstract. * *We prove that for any fixed $n \geq 3$ there is no algorithm deciding whether or not a given knot $f : S^n \longrightarrow \mathbb{R}^{n+2}$ is trivial. Some related results are also presented.*

The classical result of W. Haken ([H]) is that there exists an algorithm deciding whether or not a given knot in $\mathbb{R}^3$ is trivial. Our main result (Theorem 1 below) establishes that this result is not true for multidimensional knots. Our proof is in the same spirit as the algorithmic unsolvability of the homeomorphism problem for manifolds of dimension ≥ 4, but there is still a subtlety that prevents us from dealing with knots in 4-space as the discerning reader will see.

Theorem 1. For any fixed $n \geq 3$ there is no algorithm deciding whether or not a given knot $f : S^n \longrightarrow \mathbb{R}^{n+2}$ is trivial. Here f is either a PL-embedding of the boundary of the standard $(n+1)$- dimensional simplex into $\mathbb{R}^{n+2}$ or a smooth embedding of S^n into $\mathbb{R}^{n+2}$ given by a $(n+2)$-dimensional vector of trigonometric polynomials with rational coefficients of the spherical angles.

Remark. Formally speaking, the term "algorithm" is used in this paper as a synonym of the term "Turing machine". However, according to the Church-Turing thesis the class of Turing computable functions coincides with the class of functions computable in an intuitive sense. Although this general version of the Church-Turing principle is just an empirical principle, the following more restricted version of the Church-Turing principle can be rigorously proven: Take a programming language such as FORTRAN, PASCAL, C, etc. Strip it of all data type but the integer type. (Of course, the possibility to perform arithmetic computations with integers implies also the possibility to perform arithmetic computations with rational and algebraic numbers.) The class of Turing computable functions coincides with the class of functions computable by programs written in this slightly restricted version of the chosen programming language (cf. [BJ]). Therefore, when we need to prove the existence of a certain algorithm in our paper, we demonstrate the existence of the algorithm in an intuitive sense making obvious the existence of a program in one of the mentioning programming languages implementing this algorithm.

Proof. To prove this result we are going to show that the halting problem for Turing machines is many-to-one reducible to the triviality problem for multidimensional knots. Then Theorem 1 becomes an immediate corollary of the algorithmic unsolvability of the halting problem.

Let G be a finitely presented group and $w \in G$ be its element such that
(i) $H_1(G) = \mathbb{Z}$;
(ii) $H_2(G) = \{0\}$;
(iii) G is the normal closure of w.

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M. Kervaire proved ([K]) that the conditions (i)-(iii) imply that for any $n \geq 3$ there exists a knot $f : S^n \longrightarrow S^{n+2}$ such that its group is G.

Assume that K is a finitely presented group such that:
(a) $H_1(K) = H_2(K) = \{0\}$;
(b) There exists an element $h \in K$ such that $K/[h, K] = \{0\}$.
Then the group $G = K \times \mathbb{Z}$ and $w = (h, 1) \in G$ satisfy the mentioned above conditions (i)-(iii). (Here 1 denotes the generator of $\mathbb{Z}$.) Indeed, (i) immediately follows from the perfectness of K, and (ii) follows from the Künneth formula. To prove (iii) observe that in $\tilde{G} = G/\textit{the normal closure of } (h,1)$ $(h, 0) = (e, -1)$, where e denotes the identity in K. Hence the image of $(h, 0)$ in $\tilde{G}$ commutes with the image in $\tilde{G}$ of any element of G of the form $(g, 0)$, $g \in K$. Now the property (b) of K implies that $\tilde{G}$ is trivial, whence (iii) follows.

The property (a) guarantees that if $n \geq 4$ then K is the fundamental group of a $(n+1)$-dimensional homology sphere. In fact, one can find in [K2] a construction of such a homology sphere Σ^{n+1} starting from a given finite presentation of K. A detailed exposition of this construction can be found in the Appendix to [N1]. Σ^{n+1} is constructed in [N1] as a non-singular algebraic hypersurface in $\mathbb{R}^{n+2}$. Therefore, if K is trivial, then Σ^{n+1} is diffeomorphic to S^{n+1}. If $n = 3$, then the Markov modification of the classical Dehn construction (cf. [BHP]) allows one to construct a compact four-dimensional manifold M^4 such that its fundamental group is K and its homology groups are isomorphic to corresponding homology groups of the connected sum of several copies of $S^2 \times S^2$. (This manifold can be constructed as a non-singular algebraic hypersurface in $\mathbb{R}^5$ defined as the zero set of a polynomial with rational coefficients.) Moreover, if K is trivial (but the given finite presentation of K is arbitrary) this construction yields the manifold M^4 diffeomorphic to the connected sum of several copies of $S^2 \times S^2$. Also, the property (a) of K and the Hopf theorem imply that the Hurewicz homomorphism $\pi_2(M^4) \longrightarrow H_2(M^4)$ is surjective.

Now we are going to describe an algorithm constructing for a given finite presentation of a group K with properties (a), (b) a multidimensional knot in $\mathbb{R}^{n+2}$ such that its group is $K \times \mathbb{Z}$. The resulting knot will be presented in both forms (PL- and smooth polynomial) described in the text of Theorem 1. Moreover, if K is the trivial group, then this algorithm produces a trivial knot.

One first constructs a smooth hypersurface $X^{n+2} = \Sigma^{n+1} \times S^1$ in $\mathbb{R}^{n+3}$, if $n \geq 4$, and $M^4 \times S^1$, if $n = 3$, using the above-mentioned Kervaire construction from [K2] or the Markov construction described in [BHP]. If $n = 3$ then one realizes a basis of $H_2(M^4)$ by embedded in $M^4 \times S^1$ disjoint two-dimensional spheres and performs surgeries killing the corresponding elements of $\pi_2(M^4 \times S^1)$. Denote the resulting manifold by X^5. (If K is the trivial group, then the resulting manifold has the homotopy type of $S^4 \times S^1$. In this case Corollary on p.297 of [S] implies that X^5 is actually diffeomorphic to $S^4 \times S^1$. If $n \geq 4$ and K is the trivial group, then by virtue of the h-cobordism theorem $\Sigma^{n+1} \times S^1$ is also diffeomorphic to $S^{n+1} \times S^1$.) Note that for any K and $n \geq 3$ $H_2(X^{n+2}) = \ldots = H_n(X^{n+2}) = \{0\}$. Now one realizes the element $(h, 1) \in \pi_1(X^{n+2})$ by an embedded smooth closed curve γ and performs the surgery killing this element and, thus, by virtue of (iii) the whole group $G = K \times \mathbb{Z} = \pi_1(X^{n+2})$. Thus, the resulting manifold, which is constructed as a hypersurface in $\mathbb{R}^{n+3}$, will be diffeomorphic to S^{n+2}. It is easy to see (cf. [K]) that the axis of the handle $D^2 \times S^n$ attached during the last surgery will be the required knot in the constructed S^{n+2}. Denote this knot by $Kn(K \times \mathbb{Z})$. If K is the trivial group, then γ will be isotopic to the meridian of $X^{n+2} = S^{n+1} \times S^1$. Hence in this case $Kn(K \times \mathbb{Z})$ will be trivial.

By a PL- or a semialgebraic trial and error algorithm (as in [ABB] or [N1]; see also [BHP]) we can find a PL-homeomorphism of an appropriate triangulation of the constructed pair $(S^{n+2}, Kn(K \times \mathbb{Z}))$ with a pair $(\partial\Delta^{n+3}, KnPL(K \times \mathbb{Z}))$, where $\partial\Delta^{n+3}$ denotes the boundary of the standard $(n+3)$-dimensional simplex and $KnPL(K \times \mathbb{Z})$ denotes some its PL-submanifold; or a C^2-smooth semialgebraic homeomorphism of the pair $(S^{n+2}, Kn(K \times \mathbb{Z}))$ with a pair $(\partial B^{n+3}, KnSm(K \times \mathbb{Z}))$, where ∂B^{n+3} denotes the boundary of the unit ball centered at the origin in $\mathbb{R}^{n+3}$ and $KnSm(K \times \mathbb{Z})$ denotes some its smooth semialgebraic submanifold of codimension 2. (The relevant definitions from semialgebraic geometry can be found in [BCR].) A priori knowing that $KnPL(K \times \mathbb{Z})$ is PL-homeomorphic with S^n (respectively, diffeomorphic with S^n), we can find by a trial and error algorithm a PL-homeomorphism $\partial\Delta^{n+1} \longrightarrow KnPL(K \times \mathbb{Z})$ (respectively, a C^2-smooth semialgebraic homeomorphism $S^n \longrightarrow KnSm(K \times \mathbb{Z})$). It is easy now to modify the above-described construction in order to obtain the required knot in $\mathbb{R}^{n+2}$ (and not in S^{n+2}). (In the smooth case one can use the stereographic projection and a constructive version of the Weierstrass approximation theorem to obtain a polynomial map f.)

Now it is clear that Theorem 1 follows from the Lemma 2 below:

Lemma 2. There exists an algorithm which for any given Turing machine T and its input λ constructs a finite presentation of a group H with the following properties: (1) There exists an element $h \in H$ such that $H/[h, H] = \{e\}$; (2) The second homology group of H is trivial; (3) H is trivial if and only if T eventually halts when it starts its work with λ.

Proof: In many proofs of the algorithmic unsolvability of the word problem for finitely presented groups one actually describes an algorithm constructing for a given Turing machine T and its input λ a finite presentation of a group G and a word representing an element $w \in G$ such that $w = e$ in G if and only if T eventually halts, when it starts its work with the input λ (cf. [R], [AC]). The Adian-Rabin "witness" construction (cf. [M], pp. 13-14) enables one to construct a finite presentation of a group G_w such that G_w is trivial if and only if $w = e$ in G. Moreover, if $w \neq e$ in G then G embeds in G_w. To get this finite presentation one adds to the list of generators of G (denoted by $x_1, \ldots, x_k$) 3 new generators a, b, c and to the list of relations of G the following $(k+3)$ new relations:

$$a^{-1}ba = c^{-1}b^{-1}cbc \tag{1}$$

$$a^{-2}b^{-1}aba^2 = c^{-2}b^{-1}cbc^2 \tag{2}$$

$$a^{-3}[w,b]a^3 = c^{-3}bc^3 \tag{3}$$

$$a^{-(3+i)}x_iba^{(3+i)} = c^{-(3+i)}bc^{(3+i)},\ i=1,2,\ldots,k. \tag{4}$$

It is not difficult to see (and the proof is given in [M] on pp. 14-15) that the normal closure of w in G_w is G_w. (This fact is used in [M] to demonstrate that G_w is trivial if $w=e$ in G). The proof uses only the relations (1)-(4) and goes as follows: Obviously, $[w,b]$ belongs to the normal closure of w. Hence, (3) implies that b belongs to the normal closure of w. Now (4) implies that for any i x_i belongs to the normal closure of w, and (1) implies that c belongs to the normal closure of w. Now (2) implies that a belongs to the normal closure of w.

The relations (1)-(4) can be rewritten in the following equivalent form:

$$[bca^{-1},b] = cb^{-1} \tag{$1'$}$$

$$[bc^2a^{-2}b^{-1},a] = ca^{-1} \tag{$2'$}$$

$$[c^3a^{-3},[w,b]][w,b] = b \tag{$3'$}$$

$$x_i = [a^{(3+i)}c^{-(3+i)},b],\ i=1,2,\ldots,k. \tag{$4'$}$$

Using ($3'$) one can replace ($1'$) by the following equivalent relation:

$$c[b,w][[w,b],c^2a^{-3}][b,bca^{-1}] = e, \tag{$1''$}$$

and then replace ($2'$) by the following equivalent relation:

$$a[b,w][[w,b],c^2a^{-3}][b,bca^{-1}][bc^2a^{-2}b^{-1},a] = e. \tag{$2''$}$$

It is obvious that G_w is perfect. Hence there exists the universal central extension $\bar{G}_w$ of G_w (cf. [Mi], Theorem 5.7 or [Ros], Theorem 4.1.3). The universal central extension of a perfect group with generators $f_1,\ldots,f_n$ and relators $r_1,\ldots,r_m$ has the following finite presentation: The set of generators is the same set $f_1,\ldots,f_n$. The set of relators includes all words $[r_j,f_i]$, $j\in\{1,\ldots,m\}$, $i\in\{1,\ldots,n\}$ and, in addition, n relators R_{f_i}, corresponding to the generators f_i, and defined as follows: R_{f_i} is any product of powers of relators r_j such that $f_i = R_{f_i}comm_i$ in the free group F generated by $f_1,\ldots,f_n$, where $comm_i$ is a product of commutators of elements of F. (The existence of R_{f_i} immediately follows from the perfectness of the group. A proof of the fact that the above-described finite presentation is, indeed, a finite presentation of the universal central extension of the considered group is a part of the proof of the S. Novikov theorem on the algorithmic unrecognizability of S^n for $n \geq 5$ and can be found in the Appendix to [N1]).

In particular, if one wishes to write down such a finite presentation for $\bar{G}_w$, it is possible to take the left hand side of $(2'')$ as R_a, the product of b and the inverse of the left hand side of $(3')$ as R_b, the left hand side of $(1'')$ as R_c, the product of x_i and the inverse of the right hand side of $(4')$ as R_{x_i}. Now the fact that the relators $R_a, R_b, R_c, R_{x_i}, i \in \{1, \ldots, k\}$ are among the relators of $\bar{G}_w$ implies that formulae (1)-(4) will be true also in $\bar{G}_w$. Therefore one can prove that $\bar{G}_w$ coincides with the normal closure of $w \in \bar{G}_w$ exactly as it was done in [M] on p. 14-15 for G_w and $w \in G_w$. Now, the perfectness of $\bar{G}_w$ implies that $w \in \bar{G}_w$ can be represented as a product of commutators of products of conjugates of powers of w. Hence, $\bar{G}_w/[w, \bar{G}_w]$ is trivial. It is well-known that the second homology group of the universal central extension of a perfect group is trivial (cf. Lemme 2 in [K3] or Corollary 4.1.18 in [Ros]). Finally note that the universal central extension of a perfect group is trivial if and only if the group is trivial. Hence $w = e$ in G if and only if $\bar{G}_w$ is trivial. Thus, $\bar{G}_w$ has properties (1)-(3) introduced in the statement of Lemma 2. So we can take $H = \bar{G}_w$. □

Remark. An anonymous referee of this paper noticed that Lemma 2 can be proven without using the notion of the universal central extension. Instead he suggested to define the same group $\bar{G}_w$ as the group with the same generators as G_w, the relations $(1')$-$(4')$ and the further relations

$$[r, a] = [r, b] = [r, c] = [r, x_i] = e, \quad i = 1, \ldots, k, \tag{5'}$$

where r runs over the set of all relators of G. (It is not difficult to see that this finite presentation is a simplification of the finite presentation of $\bar{G}_w$ considered above.) One can prove that $\bar{G}_w = [w, \bar{G}_w]$ exactly as above. If $w = e$ in G then w freely equals to the product of conjugates by words in the x_i of the elements r and their inverses. By $(5')$ w commutes with all generators of $\bar{G}_w$ in $\bar{G}_w$. Therefore $\bar{G}_w = [w, \bar{G}_w] = \{e\}$. It was shown in [M] that if $w \neq e$ in G, then G embeds into G_w. Since G_w is a quotent of $\bar{G}_w$, the group $\bar{G}_w$ is not trivial in this case. It remains to prove that $H_2(\bar{G}_w) = 0$. The referee have indicated a direct argument (similar to the proof of Lemma 2.2 in [N3]) showing that this is, indeed, the case.

It is known that in the smooth category there exist non-trivial knots in codimensions > 2. However, the work of J. Levine [L] (see also [Hf]) implies the algorithmic solvability of the triviality problem for smooth knots of codimension ≥ 3. However, we shall see elsewhere that the general embeddability problem in dimension > 2 is algorithmically unsolvable even for simply connected manifolds (and for reasons not related with fundamental groups).

Theorem 1 can be used to obtain some information about the geometry of the space of *trivial* (in either PL or the smooth category) n-dimensional knots in $\mathbb{R}^{n+2}$ (or S^{n+2}) for any $n \geq 3$. Here is an example of such application. Let for any n, N $Knot_n(N)$ denote the space of all piecewise-linear maps f of S^n (regarded as the boundary of the standard $n+1$-dimensional simplex) to $\mathbb{R}^{n+2}$, made of not more than N linear pieces. Let $TrivKnot_n(N) \subset Knot_n(N)$ denote the subset of $Knot_n(N)$ formed by all maps f such that $f(S^n)$ has the trivial knot type. Obviously, $TrivKnot_n(N)$ is the union of a family of connected components of $Knot_n(N)$. We regard $TrivKnot_n(N_1)$ and $Knot_n(N_1)$ as subsets of, correspondingly, $TrivKnot_n(N_2)$ and $Knot_n(N_2)$ for any $N_2 \geq N_1$. We will demonstrate below that for any $n \geq 3$ and all sufficiently large N $TrivKnot_n(N)$ is disconnected. Still, it is easy to see that $TrivKnot_n(N)$ can have only finitely many connected components. Hence for any n, N there exists $M \geq N$ such that $TrivKnot_n(N)$ is comprised in a connected component of $TrivKnot_n(M)$. (Indeed, we can choose a representative f_i from every connected component of $TrivKnot_n(N)$. For every pair i, j $f_i(S^n)$ and $f_j(S^n)$ can be connected by an isotopy which can be approximated by an isotopy passing through piecewise-linear maps of S^n into $\mathbb{R}^{n+2}$ made of not more than M_{ij} linear pieces for some number M_{ij}. One can now take $\max_{i,j} M_{ij}$ for M.) Now for a fixed n consider the minimal of such numbers M as a function of N. Denote this function by $Tr_n(N)$.

Theorem 3. Let $n \geq 3$ be any fixed number. The function $Tr_n(N)$ cannot be majorized by any Turing computable function of N.

Scetch of the proof. Assume that $Tr_n(N)$ can be majorized by a Turing computable function of N. Now it is not difficult to construct an algorithm deciding whether or not a given PL-knot in $\mathbb{R}^{n+2}$ is trivial, thus obtaining a contradiction with Theorem 1. The idea is that for any k $Knot_n(k)$ can be regarded as a semi-algebraic set in an Euclidean space of a sufficiently large dimension. The possibility to compute an upper bound $\theta_n(N)$ for $Tr_n(N)$ enables one to replace the triviality problem for PL-knots by the equivalent problem of recognizing whether or not a given point in $Knot_n(N) \subset Knot_n(\theta(N))$, where N is a given number, belongs to the same connected component of $Knot_n(\theta_n(N))$ as the point, corresponding to the standard unknotted embedding of S^n into $\mathbb{R}^{n+2}$. But it is well-known that the problem of recognizing whether or not two given points are in the same connected component of a given semi-algebraic set is algorithmically solvable (cf. [BCR]). $\square$

Remark. The statement that $TrivKnot_n(N)$ is disconnected is obviously equivalent to the inequality $Tr_n(N) > N$. Hence Theorem 3 implies that *for any $n \geq 3$ and for an infinite set of values of N $TrivKnot_n(N)$ is disconnected.* Moreover, using Lemma 6 of [N4] instead of the mere algorithmic unsolvability of the halting problem for Turing machines in the foundation of the proof of Theorem 3 one can prove that not only Tr_n cannot be majorized by any Turing computable function, but that for any Turing computable function θ and any $n \geq 3$ $Tr_n(N) > \theta(N)$ for all sufficiently large N. As a corollary, we see that *for any $n \geq 3$ for all sufficiently large N $TrivKnot_n(N)$ is disconnected.* (It is quite plausible, however, that $TrivKnot_1(N)$ and $TrivKnot_2(N)$ are also disconnected for all sufficiently large N, and that this statement admits a constructive proof.)

The methods of [N3] show that for $n \geq 3$ the number of connected components of $TrivKnot_n(N)$ grows at least exponentially with N. In fact for any recursive function f the rank of the image of $H_0(TrivKnot_n(N))$ in $H_0(TrivKnot_n(f(N))$ grows exponentially.

Other results about non-computability in geometry and its applications in the spirit of Theorem 3 can be found in [N0]-[N5], [ABB], [G, section 5.C].

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