# Curvature-free upper bounds for the smallest area of a minimal surface.

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**Abstract**

In this paper we will present two upper estimates for the smallest area of a possibly singular minimal surface in a closed Riemannian manifold $M^n$ with a trivial first homology group. The first upper bound will be in terms of the diameter of $M^n$, the second estimate will be in terms of the filling radius of a manifold, leading also to the estimate in terms of the volume of $M^n$. If $n = 3$ our upper bounds are for the smallest area of a smooth embedded minimal surface. After that we will establish similar upper bounds for the smallest volume of a stationary $k$-dimensional integral varifold in a closed Riemannian manifold $M^n$ with $H_1(M^n) = ... = H_{k-1}(M^n) = \{0\}$, $(k > 2)$. The above results are the first results of such nature.

## 1 Main results

In this paper we prove an effective version of results of J. Pitts ([P] , section 4) establishing the existence of stationary integral varifolds of all dimensions $\leq i$ corresponding to a homology class $h \in H_i(M^n)$ for closed Riemannian manifold $M^n$. Namely, we are able to give a priori upper bounds for the smallest volume of a stationary integral varifold of dimension $k$ in $M^n$ in terms of the diameter or $M^n$ or in terms of the volume of $M^n$. However, we are able to do this only in the situation when all homology groups of $M^n$ of dimensions $< k$ vanish. Moreover, we will need the homological filling functions in these dimensions for our estimates. (These homological filling functions will be defined below.) The results of Pitts as well as our results are especially strong in the codimension one case, when results of [SSY] and [SS] imply extra regularity of the stable stationary varifolds. For example, if $k = n-1$ and $n \leq 7$, then our upper bounds are for the smallest volume of a smooth embedded minimal hypersurface.

For readers who are not familiar with Geometric Measure Theory note that stationary integral $k$-dimensional varifolds are (singular) minimal submanifolds. Here is a formal definition of an integral varifold. An integral varifold consists of a countably $k$-rectifiable Borel set $M$ and a locally $H^k$-summable positive integer-valued function $\theta$ on $M$. Here everything is defined up to sets of measure zero. This function $\theta$ determines a measure $d\mu = \theta dH^k$, where $H^k$ is the Hausdorff measure. $\mu(M)$ is called the mass of the varifold. Equivalently, we can consider (disjoint) sets $M_i = \{x \in M | \theta(x) = i\}$, and to think about the integral varifold as $\Sigma_{j=1}^{\infty} n_j M_{n_j}$, where $n_j$ denote the sequence of all values of $i$ such that $M_i$ is non-empty. So, informally speaking, integral $k$-varifolds consist of countably many disjoint "nice" $k$-dimensional pieces with positive integer multiplicities, where adding or subtracting sets of $H^k$-measure zero does not change the varifold.

The stationarity of a varifold $v$ has the following formal meaning: Let $X$ be a smooth vector field on $M^n$. Consider the corresponding 1-parametric flow of diffeomorphisms $\Phi_t$. Apply these diffeomorphisms to $v$ and consider the mass of the resulting varifold as a function of $t$. By definition, $v$ is stationary, if $t = 0$ is a critical point of this function.

We would like also to note that there exists an oriented analog of integral varifolds, namely, integral cycles. Since one can integrate $k$-forms over integral $k$-cycles, they can be regarded as a subset of the dual space to the space of $k$-forms. The dual space norm of $k$ cycles is called their mass. F. Almgren proved ([A]) that the $i$th homotopy group of the space of integral $k$-cycles is isomorphic with the $(i+k)$th homology group of the ambient manifold. This fact plays a crucial role in the geometric measure theory. Yet integral cycles have the following technical disadvantage in comparison with integral varifolds: the mass is not a continuous function on the space of cycles. Indeed, consider a translate of a cycle $z$ in $R^n$ by a small vector. Consider the sum of this translate and $-z$ (that is, $z$ taken with the opposite orientation). If $\varepsilon \neq 0$ then the mass of the cycle is equal to twice the mass of $z$. However, when $\varepsilon$ become equal to zero the cycles cancel each other and the mass is equal to zero. If one would forget about the orientations and consider the corresponding varifolds, then this phenomenon cannot take place: the mass is a continuous functional on spaces of varifolds.

Following an earlier work of F. Almgren J. Pitts developed a version of Morse theory for spaces of cycles. He described how one can start from a non-trivial homology class of the manifold and to assign to it non-trivial stationary integral varifolds in each dimension lower than the dimension of the homology class. The Almgren isomorphism between homology groups of the manifold and homotopy groups of the space of cycles plays an important role here. Note that one gets here merely stationary varifolds instead of minimal cycles precisely because of the mentioned above technical problem with the possible vanishing of the mass of cycles in the limit.

In this paper we provide a quantitative version of Pitts's results. We combine his technique with our technique from [NR1] to derive upper bounds for the volume of the stationary integral cycles. Our results could, thus, be considred as a multidimensional generalization of results of [NR1]. Yet, they are different from the results of [NR1] in the following three aspects: 1. Here we need to assume that first $(k-1)$ homology groups of the Riemannian manifold vanish, whenever we did not need any assumptions about the ambient Riemannian manifold in [NR1]; 2. In [NR1] stationary 1-dimensional varifolds had only finitely many pieces (and the number of pieces was controlled in terms of the dimension of the manifold); and 3. Our present estimates involve not only the diameter or the volume but also defined below homological filling functions. However, we believe that the last limitation is unavoidable and the minimal volume of, say, a minimal hypersurface in a three-dimensional Riemannian manifold diffeomorphic to $S^3$ cannot be majorized only in terms of the diameter of the manifold. Our present results can be also compared with the results of [NR2]. There we considered only the case when the ambient manifold is diffeomorphic to $S^3$. Moreover, our estimates required a two-sided bound for the sectional curvature of the manifold. However there we got an upper bound for the area of a minimal surface diffeomorphic to $S^2$, whereas here we provide an upper bound for the area of a minimal surface of an unknown topological type.

In order to state our results we need the notion of homological filling functions. In the definition below we will be considering only singular chains with smooth simplices. The volume of a singular simplex will be defined as the volume of the standard simplex endowed with the pullback measure; the mass $m$ of a singular chain $\Sigma_i a_i \sigma_i$, where $a_i \in R$ and $\sigma_i$ are singular simplices, is defined as $\Sigma |a_i|\ vol(\sigma_i)$.

**Definition 1.1 (Homological filling function.)** *Let $M^n$ be a closed Riemannian manifold of dimension $n$ with the trivial first homology group. Let $\gamma$ denote a closed piecewise differentiable curve. Then the homological filling function $FH = FH_1 : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ will be defined as follows:*

*$FH(x) = \max_{\{\gamma | length(\gamma) \leq x\}} \min_{\{\Sigma_2 | \partial\Sigma_2 = \gamma\}} Area(\Sigma_2)$, where $\Sigma_2$ is a singular 2-chain, and $Area(\Sigma_2)$ denotes its area.*

One can generalize this definition and to introduce homological filling functions $FH_k$ for all dimensions, so that $FH = FH_1$:

**Definition 1.2 (Homological filling function in dimension $k \geq 1$.)** *Let $M^n$ be a closed Riemannian manifold of dimension $n$ with the trivial $k$-th homology group. Then the homological filling function of order $k$, $FH_k : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ will be defined as follows: $FH_k(x) := \max_{\{\Sigma_k | m(\Sigma_k) \leq x\}} \min_{\{\Sigma_{k+1} | \partial\Sigma_{k+1} = \Sigma_k\}} m(\Sigma_{k+1})$, where $\Sigma_k$ runs over all $k$-dimensional Lipschitz singular cycles in $M^n$, and $\Sigma_{k+1}$ runs over all $(k+1)$-dimensional Lipschitz singular chains.*

In practice, it is not difficult to majorize homological filling functions. For example, the following idea was succesfully exploited in [NR0]: Assume that the ambient manifold is endowed with a simplicial or cell subdivision where all cells have simple geometry. Then we can fill a $k$-dimensional singular cycle in two steps. First, we find a homologous simplicial (or cell) approximation to the given $k$-cycle. If we have a sufficient control over the geometry of the cells, we can majorize the volume of a $(k+1)$-dimensional chain cobounded by the two $k$-cycles. (For example, the knowledge of an upper bound for Lipschitz distances between each simplex of the considered triangulation of the manifold and a regular simplex in $\mathbf{R^n}$ will be sufficient for this purpose.) On the second stage we are looking for an optimal simplicial (or cell) chain that fills the simplicial aproximation a priori knowing that it exists. We need only to majorize the sum of absolute values of coefficients of such a chain. Such an estimate follows from the following easy fact from linear algebra (cf. [NR0], section 4): If a system of linear equations with integer coefficients has an integer solution, then it has an integer solution with the norm that does not exceed a certain explicit expression involving the dimensions and upper bounds for the norms of the right hand side and the matrix of the system. We can apply this observation to the boundary operator regarded as a linear operator from the space of simplicial (cell) $(k+1)$-dimensional chains to the space of $k$-chains.

Further, recall that M. Gromov ([Gr]) defined the filling radius of a closed Riemannian manifold $X^n$ embedded in a metric space $Y$ as the minimal radius of a neighborhood of $X^n$ in $Y$ such that $X^n$ bounds in this neighborhood. The filling radius of an abstract closed Riemannian manifold

$X^n$ is the filling radius of its Kuratowski embedding into $L^\infty(X)$ (Recall that the Kuratowski embedding assigns to each $x \in X^n$ the distance function to $x$. See a more formal and detailed definition of the filling radius below in the next section). In the same paper Gromov proved that the filling radius of a closed Riemannian manifold $M^n$ does not exceed $c(n)\ vol(M^n)^{\frac{1}{n}}$, where $c(n) = (n+1)n^n\sqrt{(n+1)!}$.

In this paper we will prove the following theorems:

**Theorem 1.3** *Let $M^n$ be a closed Riemannian manifold of dimension $n$ with the trivial first homology group. Then the smallest area $A(M^n)$ of a possibly singular minimal surface in $M^n$ satisfies the following inequalities:*

$$(i) A(M^n) \leq \frac{(n+1)!}{2} FH(2d(M^n));$$

$$(ii) A(M^n) \leq \frac{(n+2)!}{6} FH(6FillRad(M^n));$$

$$(iii) A(M^n) \leq \frac{(n+2)!}{6} FH(6(n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}}).$$

*Here $d(M^n)$ denotes the diameter, $FillRad(M^n)$ denotes the filling radius, and $vol(M^n)$ denotes the volume of $M^n$. If $n = 3$, then there exists a non-singular embedded minimal surface such that its area satisfies the inequalities (i)-(iii).*

**Remarks.**

**0.** Formally speaking, "possibly singular minimal surface" means here "2-dimensional stationary integral varifold". More generally, in the present paper the term "surface" always means a two-dimensional surface.

**1.** M. Katz proved that $FillRad(M^n) \leq \frac{1}{3}diam(M^n)$ ([K]). Therefore (ii) implies (i) with a worse value of the constant. Yet we prefer to prove (i) independently of (ii) not only because our proof of (i) gives a better value of the constant but also because it is somewhat simpler than the proof of (ii).

**2.** If $n = 3$ we are able to prove the inequalities (i)-(iii) with better values of constants:

$$A(M^3) \leq 6FH(2d(M^3)),$$

$$A(M^3) \leq 12FH(6FillRad(M^3)) \leq 12FH(648\sqrt{24}vol(M^3)^{\frac{1}{3}}).$$

However, we will omit the details of the proof of this improvement.

Theorem 1.3 can be generalized for higher dimensions of minimal submanifolds as follows:

**Theorem 1.4** *Let $M^n$ be a closed Riemannian manifold of dimension $n$ with $H_1(M^n) = H_2(M^n) = ... = H_{k-1}(M^n) = \{0\}$, where $k \geq 2$. Then there exists a non-trivial stationary integral varifold of dimension $k$, such that its mass $V_k$ is bounded from above by*

$$(i) V_k \leq \frac{(n+1)!}{k!} FH_{k-1}(k(FH_{k-2}(...(3FH_1(2d))...))).$$

$$(ii) V_k \leq \frac{(n+2)!}{(k+1)!} FH_{k-1}((k+1)FH_{k-2}(...(4FH_1(6FillRadM^n))...)).$$

$$(iii) V_k \leq \frac{(n+2)!}{(k+1)!} FH_{k-1}((k+1)FH_{k-2}(...(4FH_1(6(n+1)n^n\sqrt{(n+1)!}vol(M^n)^{\frac{1}{n}})...))).$$

*If $k = n-1$ and $n \leq 7$ then we can ensure that a non-trivial stationary integral varifold satisfying the inequalities (i)-(iii) is a smooth embedded hypersurface; if $k = n-1$ and $n = 8$, then we can ensure that it has only isolated singularities, if $k = n-1$ and $n \geq 9$ we can ensure that the Hausdorff dimension of its singular set does not exceed $k-7$.*

**Remarks.**
**1.** The third inequalities in Theorems 1.3, 1.4 follow from the second inequalities and the above mentioned upper bound for the filling radius in terms of volume proven by M. Gromov, [Gr].
**2.** Theorems 1.3, 1.4 provide effective versions of the results by J. Pitts [P], Theorems 4.10, 4.11. Therefore our proofs provide the existence of not merely stationary integral varifolds with volume bounded as stated in Theorems 1.3, 1.4 but stationary integral varifolds that are almost minimizing in a small annular neighborhood of every point precisely as in Theorems 4.10, 4.11 in [P]. (See 3.1(2) of [P] for the definition of almost minimizing varifolds.) In particular, our stationary varifolds are stable in a neighborhood of every point but finitely many, and the cardinality of the set of points, where stability is not guaranteed does not exceed some $N(n)$ depending only on the dimension of $M^n$.
**3.** In view of the last assertion of Theorem 1.3 one can ask for an upper bound of the smallest area of am embedded non-singular minimal hypersurface *diffeomorphic to* $S^2$ in a three-dimensional Riemannian manifold diffeomorphic to $S^3$. The existence of such a surface was first proven by F. Smith and L. Simon ([S], see also [CD]). In [NR2] we obtained explicit upper bounds for the area of such a surface. These estimates are given in terms of an upper bound for the diameter of $M^3$, a positive lower bound for its

volume and a two-sided bound for the sectional curvature. The methods of [NR2] have very little in common with the methods of the present paper.

Our results can be compared with the study of $k$-systoles, ($k > 1$), initiated by M. Berger and continued, first, by M. Gromov and then by M. Katz, I. Babenko and others (see surveys [Ber], [Gr2], [CK]). One of the central questions in that area was the following: Assume that $H_k(M^n) \neq 0$. Define the $k$-systole $sys_k(M^n)$ as the infimum of $k$-volumes of all homologically non-trivial integral $k$-cycles in $M^n$. Is there a constant $c$ depending only on the diffeomorphism type of $M^n$ such that $sys_k(M^n) \leq c\ vol(M^n)^{\frac{k}{n}}$? It was hoped that the answer for this question is positive for at least some topological types of manifolds, e.g. for $CP^n$. Yet it was discovered that, in general, there are no such upper bounds.

In the present paper we consider all minimal surfaces and not only those that are non-nullhomologous. Instead of requiring that $H_k(M^n)$ is non-trivial, we assume that all lower dimensional homology groups of $M^n$ vanish. Finally, we obtain an upper bound that involves only the homology filling functions and the volume (or the diameter), but not any information about the curvature of the ambient manifold.

## 2 Ideas of the proofs

This paper extends our earlier paper, (see [NR1]) in which we have found two curvature-free upper bounds for the minimal length of a strongly stationary 1-cycle. Strongly stationary 1-cycles can be considered as homological equivalents of closed geodesics and also as especially nice 1-dimensional equivalents of minimal surfaces with singularities. The proofs in the present paper generalize the proofs in [NR1]. Also they heavily use the results of J. Pitts ([P]), and can be regarded as a quantitative version of his work. In paricular, J. Pitts proved that if one has a non-contractible map of a sphere in the space of $k$-dimensional cycles such that the masses of this cycles do not exceed some $x$, then there exists a stationary integral $k$-varifold of mass that does not exceed $x$ (see the proof of Theorem 4.10 in [P]). We will repeatedly use this assertion in the following equivalent form:

**Lemma 2.1** *If there is no stationary inegral $k$-varifold of mass $\leq x$ in $M^n$, then all maps of spheres to the space of $k$-dimensional cycles of mass $\leq x$ in $M^n$ can be contracted via $k$-cycles of mass $\leq x$.*

First we will present an informal explanation of the proof of the first estimate of Theorem 1.3.

For the sake of simplicity of the explanation assume that $M$ is diffeomorphic to a round 3-sphere $S^3$. We are going to show that there exists a minimal imbedded surface in $S^3$ of area satisfying the upper bounds of Theorem 1.3

Let $f : S^3 \longrightarrow M$ be any diffeomorphism. Assume that $S^3$ was triangulated into simplices of diameter smaller than $\delta$. Let the standard 4-disc $D^4$ be triangulated as a cone over $S^3$. A $k$-simplex of $D^4$ will be denoted $[v_{i_0}, ..., v_{i_k}]$, or $\sigma_i^k$ (for an appropriate $i$), or sometimes $\sigma^k_{i_0,...,i_k}$, where $k \in \{1, 2, 3, 4\}$.

The proof will be by contradiction. Denote the diameter of $M$ by $d$. Suppose $A(M) > 12FH(2d+\delta)+5\delta$. We will show that in that case $f_*([S^3])$ bounds producing a contradiction. Here $[S^3]$ denotes the fundamental class of $S^3$.

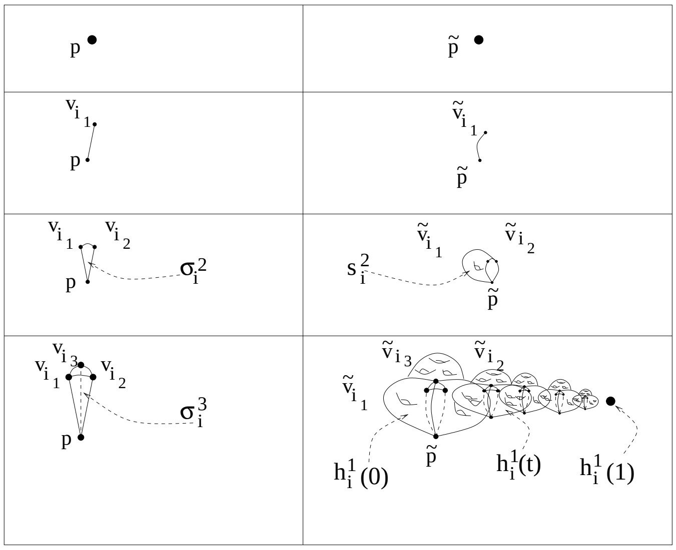



Figure 1: Extension process.

We will begin by extending the map $f : S^3 \longrightarrow M$ to 1-skeleton of $D^4$ and then by assigning to each 2-simplex of $D^4$ a singular 2-chain on $M$, (see fig. 1). To extend to 0-skeleton, we will assign to the center of the disc $p$, an arbitrary point $\tilde{p} \in M$. Next we extend to 1-skeleton by assigning to each arbitrary edge $[p, v_{i_1}]$ connecting the center of $D^4$ with a vertex $v_{i_1}$

and directed from $p$ to $v_{i_1}$, a minimal geodesic that connects $\tilde{p}$ with a vertex $\tilde{v}_{i_1} = f(v_{i_1})$, directed from $\tilde{p}$ to $\tilde{v}_{i_1}$ and denoted $[\tilde{p}, \tilde{v}_{i_1}]$. Now, consider a simplex $\sigma_i^2 = [v_{i_0}, v_{i_1}, v_{i_2}]$, where $v_{i_0} = p$. Its boundary is mapped to the curve $[\tilde{v}_{i_1}, \tilde{v}_{i_2}] - [\tilde{v}_{i_0}, \tilde{v}_{i_1}] + [\tilde{v}_{i_0}, \tilde{v}_{i_1}]$ of length $\leq 2d + \delta$. Let $s_i^2$ (also denoted as $s^2_{i_0,i_1,i_2}$) be a singular 2-chain of the smallest area, such that $\partial s_i^2 = f(\partial \sigma_i^2)$. Then its area is $\leq FH(2d + \delta)$. We will assign to $\sigma_i^2$ this surface $s_i^2$.

Now we will slightly change our tactics. Consider a 3-simplex $\sigma_i^3$. There is a preassigned chain $S_i^2 = \Sigma_{j=0}^3 (-1)^j s^2_{i_0,\dots,\hat{i}_j,\dots,i_3}$ of area $\leq 3FH(2d+\delta)+\delta$ that corresponds to the boundary of this simplex. (We can assume without loss of generality that the area of $s^2_{i_1,i_2,i_3} \leq \delta$, where $\sigma^2_{i_1,i_2,i_3}$ is a simplex in the fine triangulation of $S^3$). $S_i^2$ is an element in the space $Z_2(M^n, \mathbf{Z})$ of integral 2-cycles on $M^n$. Since by our assumption there are no minimal surfaces of "small" area, $S_i^2$ can be connected with the zero cycle by a curve that passes through cycles of area $\leq 3FH(2d + \delta) + \delta$, i. e. there exists $h_i^1 : [0, 1] \longrightarrow Z_2(M, \mathbf{Z})$, (sometimes denoted $h^1_{i_0,i_1,i_3}$), such that $h_i^1(0) = S_i^2$ and $h_i^1(1)$ is the 0-cycle. We will assign the above path to the simplex $\sigma_i^3$. Finally, take a 4-simplex $\sigma_i^4$, (see fig. 2). Consider its boundary $\partial \sigma_i^4 = \Sigma_{j=0}^4 (-1)^j [v_{i_0}, \dots, \hat{v}_{i_j}, \dots, v_{i_4}]$. Each of its faces $(-1)^j [v_{i_0}, \dots, \hat{v}_{i_j}, \dots, v_{i_4}]$ corresponds to the map $(-1)^j h^1_{i_0,\dots,\hat{i}_j,\dots,i_4}(t)$.

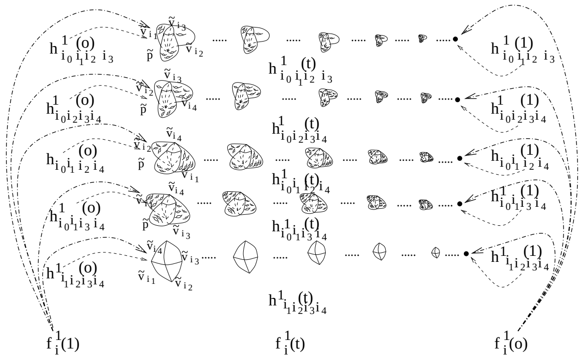



Figure 2: The loop $f_i^1(t)$.

**Now we will perform the trick that we will use throughout the paper:** We are going to consider the map $f_i^1 : [0, 1] \longrightarrow Z_2(M, \mathbf{Z})$, such that $f_i^1(t) = \Sigma_{j=0}^4 (-1)^j h_{i_0,\dots,\hat{i}_j,\dots,i_4}(1 - t)$. Note that $f_i^1(0)$ is the zero cycle and that $f_i^1(1)$ is also the zero cycle represented by 10 pairs of singular chains, where each pair contains two copies of one chain with opposite orientations. Thus, $f_i^1(t)$ is a loop in the space of 2-cycles. Note also that the area of

$f_i^1(t)$ is bounded from above by $12FH(2d+\delta)+5\delta$. By our assumption $f_i^1(t)$ is contractible in $Z_2(M,\mathbf{Z})$ along the cycles of area $\leq 12FH(2d+\delta)+5\delta$. (Otherwise Lemma 2.1 implies the existence of a minimal surface (=a stationary integral varifold) of area $\leq 12FH(2d+\delta)+5\delta$, which is impossible because of our assumption.) Thus, we obtain a disc $h_i^2 : D^2 \longrightarrow Z_2(M,\mathbf{Z})$. We will assign this disc to the simplex $\sigma_i^4$.

From the geometric measure theory we know that

$$\pi_k(Z_2(M,\mathbf{Z})) = H_{k+2}(M).$$

So, informally speaking, the disc of the form $h_i^2 : D^2 \longrightarrow Z_2(M,\mathbf{Z})$ corresponds to a singular chain $s_i^4$ on $M$. (However any formal construction of the correspondence will be technical. Our intention to avoid the technicalities of this construction is responsible for the fact that the proofs of Theorems 1.3 and 1.4 given in the next sections are more awkward than this sketch). Now consider $S^4 = \Sigma_{i=1}^{Q} s_i^4$, where $Q$ is the number of simplices of dimension 4 in the triangulation of $D^4$. $\partial S^4 = f_*([S^3])$. Thus, we obtain a contradiction. Therefore, we can conclude that $A(M) \leq 12FH(2d+\delta)+5\delta$. Now let $\delta$ go to zero.

The proof of Theorem 1.3 will be given in Section 3. The proof of Theorem 1.4 will be given in Section 4. As we have mentioned before, the arguments of the proofs will be somewhat more technical than the explanations above. Indded, we sketched an extension process where simplices are being filled by chains. These chains are obtained from discs in spaces of cycles: Intuitively, we can think about a filling chain as being "made" of images of all points of the disc. Whenever this construction can be made rigorous when the dimension of cycles is equal to one (see [NR1]), we do not know any straightforward way to make it rigorous for cycles of a higher dimension. Therefore in the actual proof we perform an inductive extension process in the space of $k$-cycles in $M^n$, using the Almgren isomorphism between homotopy groups of the space of cycles and homology groups of the manifold. The actual proof will run as follows:

Once again, for the sake of simplicity of the explanation, assume that our manifold $M$ is diffeomorphic to the standard 3-sphere. Let $f : S^3 \longrightarrow M$ be a diffeomorphism, and suppose that the triangulation of $S^3$ and the induced triangulation on $M$ is very fine, (i.e. the diameter of simplices smaller than $\delta$). Let $D^4$ be triangulated as a cone over $S^3$.

The proof will consist of the following three steps:

**Step 1 (Easy)**. Corresponding to the map $f : S^3 \longrightarrow M$ one can construct a non-contractible map $\tilde{f} : S^1 \longrightarrow Z_2(M,\mathbf{Z})$, a loop in the space of the integral 2-cycles on a manifold $M$. We do not have any control over the masses of $f(t), t \in S^1$. (This loop corresponds to the fundamental class of the manifold under the Almgren isomorphism between $H_3(M, Z)$ and $\pi_1(Z_2(M, \mathbf{Z}))$.)

**Step 2 (Main Step)**. The loop $\tilde{f}$ is homotopic to the sum of $Q$ loops $\tilde{g}_i$, where the number $Q$ equals to the number of simplices in the triangulation of $S^3$, and the masses of all 2-cycles $\tilde{g}_i(t), t \in S^1$ satisfy upper bounds as in the right hand side of the inequality (i) in the text of Theorem 1.3.

**Step 3 (An application of [P])**. We show that if there is no minimal surface with "small" area than each loop in step 2 can be contracted to a point, thus obtaining a contradiction.

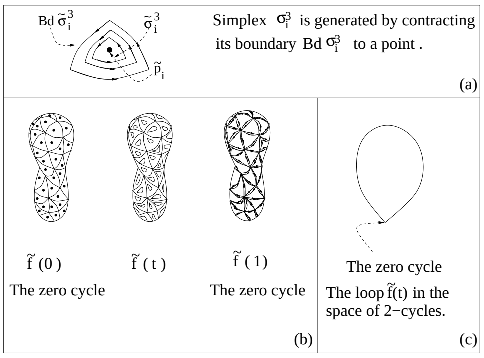
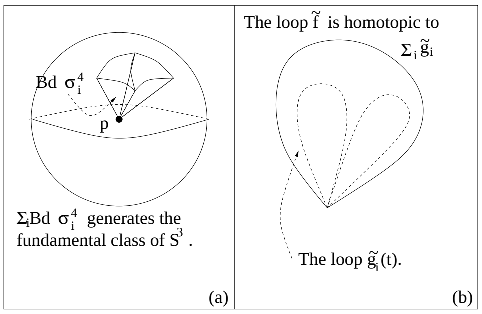



Figure 3: The loop $\tilde{f}$.

**Step 1.** Let $\{\sigma_i^3\}_{i=1}^Q$ be the set of simplices that constitute the fundamental class $[S^3]$ of $S^3$. Then $\{\tilde{\sigma}_i^3\}_{i=1}^Q$ are the corresponding simplices of $M$.

Without any loss of generality we can assume that each $\tilde{\sigma}_i^3$ can be obtained from its boundary $\partial\tilde{\sigma}_i^3$ by contracting it to the "center" of the simplex, the point $\tilde{p}_i$ by a homotopy $\tilde{h}_i : [0,1] \longrightarrow M$ along the 2-spheres $\tilde{h}_i(t)$, (see fig. 3 (a). This figure schematically depicts $\tilde{\sigma}_i^3$ as a simplex of dimension 2).

Then $\tilde{f}$ can be constructed as follows. We will begin with the zero cycle that consists of the sum of the "centers" of simplices: $\Sigma_{i=1}^Q \tilde{p}_i$. We will then follow the images of $\partial\tilde{\sigma}_i^3$'s under the area decreasing homotopies $\tilde{h}_i(t)$'s, that is $\tilde{f}(t) = \Sigma_{i=1}^Q \tilde{h}_i(1-t)$. Note that $\tilde{f}(1) = \Sigma_{i=1}^Q \partial\tilde{\sigma}_i^3$, which is also the zero cycle. Thus, we obtain a (non-contractible) loop in the space of 2-cycles, (see fig. 3 (b). There $f(S^3)$ is schematically depicted as a 2-dimensional sphere).




The fundamental class of $S^3$ corresponds to the loop $\tilde{f}$ (t).
The boundary of $\sigma_i^4$ corresponds to the loop $\tilde{g}_i$ (t).

Figure 4:

**Step 2.** Each loop $\tilde{g}_i$ is constructed as follows. Consider a disc $D^4$, such that $\partial D^4 = S^3$, triangulated as a cone over $S^3$. Let $p \in D^4$ be the center of this disc. We can assign to this point an arbitrary point $\tilde{p} \in M$, thus extending the map $f : S^3 \longrightarrow M$ to the 0-skeleton of $D^4$. Next consider a line segment of the form $[p, v_i]$ directed from $p$ to $v_i$. We can assign to it a minimal geodesic segment of length smaller than the diameter $d$ of $M$ joining the point $\tilde{p}$ with the vertex $\tilde{v}_i = f(v_i)$. This segment will be directed from $\tilde{p}$ to $\tilde{v}_i$ and denoted as $[\tilde{p}, \tilde{v}_i]$. This extends $f$ to 1-skeleton of $D^4$. Next consider a 2-simplex $\sigma_i^2 = [p, v_{i_1}, v_{i_2}]$. Its boundary is mapped to a closed curve $[\tilde{v}_{i_1}, \tilde{v}_{i_2}] - [\tilde{p}, \tilde{v}_{i_2}] + [\tilde{p}, \tilde{v}_{i_1}]$. Let $s_i^2$ (sometimes denoted as $s^2_{i_0,i_1,i_2}$) be a singular 2-chain of the smallest area filling this curve. We can assign $s_i^2$ to simplex $\sigma_i^2$, thus we obtain a map from the space of simplicial 2-chains of $D^4$ into the space of integral 2-currents on $M$. Now let us take an arbitrary 3-simplex $\sigma_i^3 = [p, v_{i_1}, v_{i_2}, v_{i_3}]$. Its boundary $\partial \sigma_i^3 = \sum_{j=0}^3 (-1)^j [v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_3}]$, where $v_{i_0} = p$. For each face $(-1)^j [v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_3}]$ we have pre-assigned a singular 2-chain $(-1)^j s^2_{i_0,...,\hat{i}_j,...,i_3}$. Consider $\Sigma_{j=0}^3 (-1)^j s^2_{i_0,...,\hat{i}_j,...,i_3}$. This is a 2-cycle of area smaller than $3FH(2d + \delta) + \delta$, (without loss of generality we can assume that the area of $s^2_{i_1,i_2,i_3}$ is smaller than $\delta$). Thus, it is an element in $Z_2(M, \mathbf{Z})$. Assuming there is no minimal 2-cycle that is locally area-minimizing and has area less than $3FH(2d + \delta) + \delta$, this cycle can be connected with the zero cycle by a mass non-increasing homotopy, that is a map $h^1_{i_0,...,i_3} : [0, 1] \longrightarrow Z_2(M, \mathbf{Z})$ that begins with our cycle and ends with the zero cycle. Finally, let us construct $\tilde{g}_i$. Consider a 4-simplex $\sigma_i^4 = [v_{i_0}, v_{i_1}, ..., v_{i_4}]$. Its boundary, $\partial \sigma_i^4 = \Sigma_{j=0}^4 (-1)^j [v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_4}]$. For each face in the boundary $(-1)^j[v_{i_0}, v_{i_1}, ..., \hat{v}_{i_j}, ..., v_{i_4}]$ there was constructed a map $(-1)^j h^3_{i_0,...,\hat{i}_j,...,i_4} : [0,1] \longrightarrow Z_2(M, \mathbf{Z})$. We will define the map $\tilde{g}_i(t)$ as the sum $\Sigma^4_{j=0}(-1)^j h^1_{i_0,...,\hat{i}_j,...,i_4}(1-t)$. Note that $\tilde{g}_i(0)$ is the zero cycle represented by $\tilde{p}_{i_0,...,\hat{i}_j,...,i_4}$ and that $\tilde{g}_i(1)$ is also the zero cycle, because of the cancellations due to the fact that each surface that corresponds to a 2-dimensional face of the simplex $\sigma^4_i$ enters twice with the opposite orientation. Thus, $\tilde{g}_i(t)$ is a loop. Area of $\tilde{g}_i(t)$ is bounded from above by $12FH(2d+\delta)+5\delta$. Moreover, $\tilde{f}(t)$ is homotopic to the sum of the above loops.

**Step 3.** Since $\tilde{f}(t)$ is homotopic to the sum of the loops $\tilde{g}_i$, (see fig. 4), at least one of those loops in not contractible. Therefore, if we try to contract it using a mass decreasing flow described in ch. 4 of [P] (and introduced earlier by F. Almgren), it should get stuck on a critical point, which, as it was shown by J. Pitts in chapter 4 of [P] corresponds to a stationary integral varifold. In ch. 7 of [P] J. Pitts uses the results of [SSY] to show that in the three-dimensional case this stationary varifold will turn out to be an embbedded minimal surface.

Similarly, one can show that $A(M^n) \leq \frac{(n+2)!}{6} FH(6FillRad(M^n))$. Recall that the filling radius of a Riemannian manifold $M^n$ was introduced by Gromov in [Gr] as follows:

**Definition 2.2 (Filling Radius)** *Let $M^n$ be a Riemannian manifold topologicaly imbedded into an arbitrary metric space $X$. Then its filling radius, denoted $FillRad(M^n \subset X)$, is the infimum of $\varepsilon > 0$, such that $M^n$ bounds in the $\varepsilon$- neighborhood $N_\varepsilon(M^n)$, i.e. homomorphism $H_n(M^n, \mathbf{Z_2}) \longrightarrow H_n(N_\varepsilon(M^n), \mathbf{Z_2})$ induced by the inclusion map vanishes. Let $M^n$ be an abstract manifold. Then its filling radius, denoted $FillRad(M^n)$ will be $FillRad(M^n \subset X)$, where $X = L^\infty(M^n)$, i.e. the Banach space of bounded Borel functions $f$ on $M^n$ and the imbedding of $M^n$ into $X$ is the map that assigns to each point $p$ of $M^n$ the distance function $f_p(q) = d(p,q)$.*

In the same paper Gromov proved the following important inequality relating the filling radius and the volume:

**Theorem 2.3 ((M. Gromov))**

$$FillRad(M^n) \leq \sqrt{(n+1)!} n^n (n+1) vol(M^n)^{\frac{1}{n}}.$$

Now, suppose once again that $M^n$ is diffeomorphic to $S^3$. The definition of the Filling Radius implies that $M^n$ bounds in the $(FillRad(M^n)+\delta)$ - neighborhood of $M^n$ in $L^\infty(M^n)$. Let $W$ fill $M^n$ in the $(FillRad(M^n)+\delta)$-neighborhood of $M$, (that is $M^n = \partial W$). Without loss of generality we can assume that $W$ is a polyhedron. Suppose that $W$ together with $M^n$ is endowed with a very fine triangulation. As in (i) the proof will consist of the three steps: constructing a non-contractible loop $\tilde{f}: S^1 \longrightarrow Z_2(M^n, \mathbf{Z})$; constructing a family of loops $\tilde{g}_i : S^1 \longrightarrow Z_2(M^n, \mathbf{Z})$; concluding that one of those loops must be non-contractible, and therefore there exists an almost minimizing integral varifold of area smaller than the area of integral cycles through which this loop passes.

**Step 1.** The first step is analogous to that of (i).

**Step 2.** Let $v_i$ be a vertex of $W$. Assign to it a closest point in $M^n$, denoted $\tilde{v}_i$. Without any loss of generality we can assume that $\tilde{v}_i$ are vertices of the considered triangulation of $M^n$. Thus, $d(v_i, \tilde{v}_i) \leq FillRad(M^n)+\delta$. This extends the identity map $Id$ on $M^n$ to 0-skeleton of $W$. Now to any 1-simplex $[v_{i_1}, v_{i_2}] \subset W \setminus M^n$ we can assign a minimal geodesic segment connecting $\tilde{v}_{i_1}$ and $\tilde{v}_{i_2}$ of length smaller than $2FillRad(M^n)+3\delta$. This extends $Id$ to 1-skeleton of $M^n$. Next consider an arbitrary 2-simplex $\sigma_i^3 = [v_{i_1}, v_{i_2}, v_{i_3}]$. Its boundary is mapped to a curve of length smaller than $6FillRad(M^n)+9\delta$. Let $s^2_{i_1,i_2,i_3}$ be a singular 2-chain of the smallest area that has this curve as its boundary. Its area is going to be smaller than $FH(6FillRad(M^n)+9\delta)$. The rest of the procedure (including Step 3) is the same as in the proof of (1).

# 3 The proof of Theorem 1.3

In this section we will prove 1.3. We will begin by proving statements (i) and (ii). Statement (iii) follows from (ii) and Theorem 2.2. Since we assumed that the first homology group of $M^n$ vanishes, $M^n$ is orientable.

*Proof.* We are going to prove the theorem by contradiction. Assume that there is no stationary integral varifold as in the text of the theorem and then prove that it exists.

Let $q$ be the minimal value of $i$ such that $H_i(M^n) \neq 0$. Assume that $M^n$ is endowed with a very fine triangulation. In the case (i) choose a $q$-dimensional singular cycle $S$ in $M^n$ representing a non-trivial element of $H_q(M^n)$ with singular simplices satisfying the Lipschitz condition. In case (ii) chose a (singular) $W$ that fills $M^n$ (in $L^\infty(M^n)$) such that for each $x \in W$

$dist(x, M^n) \le FillRad(M^n) + \delta$ (see [Gr]). Since we can always subdivide all simplices of $S$, we can assume that $S$ is a sum of singular simplices $\sigma_i^q$ with diameter $d(\sigma_i^q) \le \delta$ for some small $\delta$ in case (i). In case (ii) assume that $W$ (and $M^n = \partial W$) has been triangulated into simplices $\sigma_i^n$ of diameter at most $\delta$. In both cases let $Q$ denote the number of simplices. The proof will consist of three steps:

**Step 1.** We will begin by constructing a non-contractible map $\tilde{f} : S^{q-2} \longrightarrow Z_2(M^n, \mathbf{Z})$ that corresponds to the original $q$-cycle $S$ under the Almgren isomorphism in case (i) and the map $\tilde{f} : S^{n-2} \longrightarrow Z_2(M^n, \mathbf{Z})$ that corresponds to the fundamental class of $M^n$ in case (ii).

**Step 2.** We will construct maps $\tilde{g}_i$ from $S^{q-2}$ in case (i) and from $S^{n-2}$ in case (ii) to the space $Z_2(M^n, \mathbf{Z}), i \in \{1, ..., Q\}$, where in case (i) $Q$ is a number of $q$-simplices of $S$ (counted with their multiplicities which are always assumed to be non-negative). (In other words, in this case $Q$ is the $l_1$-norm of $S$.) In case (ii) $Q$ is a number of simplices of dimension $(n+1)$ in the considered triangulation of $W$. It will turn out that $\tilde{f}$ is homotopic to the sum of $\tilde{g}_i's$. Therefore, at least one of the maps $\tilde{g}_i$ is not contractible. The most important feature of $\tilde{g}_i$ is that the mass of $\tilde{g}_i(t)$ does not exceed the right hand side of (1) (or (2)) for every $t$ and $i$.

**Step 3.** Applying Lemma 2.1 we can conclude that there exists an almost minimizing integral varifold of area smaller than that of the cycles through which passes the non-contractible map of Step 2.

Here is a detailed description of these three steps:

**Step 1.** Let us begin by considering simplices $\tilde{\sigma}_i^3 = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_3}]$ of dimension 3, where $\tilde{v}_{i_0}, \ldots, \tilde{v}_{i_3}$ are vertices of one of the $q$-dimensional simplices forming $S$ (in case (i)) or one of the $n$-dimensional simplices from the considered fine triangulation of $M^n$ (in the case (ii)). Without loss of generality we can assume that this simplex can be generated by contracting its boundary $\partial\tilde{\sigma}_i^3$ to the "center" of the simplex $\tilde{p}_i^3$, (sometimes denoted $\tilde{p}_{i_0,...,i_3}$ ) with a homotopy $\tilde{h}_i^1(t) = \tilde{h}^1_{i_0,...,i_3}$. This homotopy can be chosen, for example, to be the radial homotopy of the boundary of the (very small and, therefore, almost Euclidean) simplex to its center. Note that we can consider this homotopy as a path in the space of integral 2-cycles that starts with the boundary of a simplex and ends with the zero cycle. Next consider a 4-dimensional simplex $\tilde{\sigma}_i^4 = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_4}]$. Its boundary $\partial\tilde{\sigma}_i^4 = \Sigma_{j=0}^4 (-1)^j [\tilde{v}_{i_0}, ..., \hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_4}]$. For each 3-face $(-1)^j[\tilde{v}_{i_0}, ..., \hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_4}]$ we have constructed a path $(-1)^j \tilde{h}^1_{i_0,...,\hat{i}_j,...,i_4} : [0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$. Now, let us construct a map $\tilde{f}_i^1 : [0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$ as follows: let $\tilde{f}_i^1(t) =$

$\Sigma_{j=0}^4(-1)^j\tilde{h}^1_{i_0,\dots,\hat{i}_j,\dots,i_4}(1-t)$. Then, note that $\tilde{f}^1_i(0)$ is the zero cycle that corresponds to the sum of the "centers" of the 3-faces of $\tilde{\sigma}^4_i$ and that $\tilde{f}^1_i(0)$ is also the zero cycle due to the cancellation of each pair of the two faces, that have the same geometric image, but are taken with the opposite orientation. Thus, this map is really a "small" loop in the space of the integral 2-cycles. We can contract it there and obtain a "small" disc $\tilde{h}^2_i : D^2 \longrightarrow Z_2(M^n, \mathbf{Z})$, (this map will also be sometimes denoted as $\tilde{h}^2_{i_0,\dots,i_4}$). Geometrically, this disc corresponds to the considered 4-dimensional simplex regarded as the chain filling its boundary. The construction of this disc can be made absolutely explicit: It is just a slicing of the simplex into 2-surfaces that extends the slicing of its boundary. It can be obtained just by coning of the slicing of the boundary. Now we can just proceed by induction until we will get a slicing of each $q$-dimensional singular simplex of $S$ (case (i)) or $n$-dimensional simplex of $M^n$ (case (ii)). As the result, we obtain maps $\tilde{h}^q_i$ of $D^{q-2}$ into $Z_2(M^n, \mathbf{Z})$ (or $\tilde{h}^n_i$ of $D^{n-2}$ into $Z_2(M^n, \mathbf{Z})$, where the mass of each 2-cycle in the image of each of these maps can be made arbitrarily small.

We finish by defining the map $\tilde{f}$ as the sum of those maps over all simplices in the triangulation. Note that this map turns out to be a map from the $(q-2)$- (Case (i)) or $(n-2)$- (Case (ii)) dimensional sphere, since the boundary of the disc is being mapped to the zero 2-cycle. Note that by defining $\tilde{f}$ in that way we lose any control over the mass of 2-cycles in the image of $\tilde{f}$ (since we have very little control over $Q$).

**Step 2.** We will now construct maps $\tilde{g}_i$s, such that $\tilde{f} = \Sigma_{i=1}^Q \tilde{g}_i$. The beginning of procedure of constructing those maps is somewhat different for case (i) and for case (ii).

**Case (i).** Choose any point $\tilde{p}$ of $M^n$. For each vertex $v$ of each $q$-dimensional simplex of $S$ fix a minimal geodesic $[\tilde{p}, v]$ between $\tilde{p}$ and $v$.

**Case (ii).** Let $v_i$ be an arbitrary vertex of $W$. Let $\tilde{v}_i$ be a point of $M^n$ that minimizes the distance between $v_i$ and $M^n$. Then $d(v_i, \tilde{v}_i) \leq FillRadM^n + \delta$. Without any loss of generality we can assume that $\tilde{v}_i$ is one of the vertices of the considered triangulation of $M^n$. We will assign $\tilde{v}_i$ to the vertex $v_i$. Now consider an arbitrary edge of $W$ of the form $[v_i, v_j]$. We can assign to this edge a minimal geodesic segment joining the corresponding vertices $\tilde{v}_i$ and $\tilde{v}_j$. It will be denoted $[\tilde{v}_i, \tilde{v}_j]$. This segment will have the length of at most $2FillRadM^n + 3\delta$.

The rest of the procedure will be almost the same in the two cases. Consider an arbitrary 2-simplex $\sigma^2_i = [v_{i_0}, v_{i_1}, v_{i_2}]$ in the triangulation of $W$ in the case (ii), or consider an abstract 2-simplex $[v_{i_0}, v_{i_1}, v_{i_2}]$, where $v_{i_0} = \tilde{p}$

and $[v_{i_1}, v_{i_2}]$ for an 1-dimensional face of one of the $q$-simplices forming $S$ in the case (i). (Below $v_{i_0}$ will always denote $\tilde{p}$ in the case (i)).

The boundary of $\sigma_i^2 = [v_{i_0}, v_{i_1}, v_{i_2}]$ has been already mapped to a closed curve of length at most $2d + \delta$ in Case (i) (of $6FillRadM^n + 9\delta$ in Case (ii)). Let $s_i^2 = s^2_{i_0,i_1,i_2}$ be a singular surface (=singular 2-chain) of area not greater than $FH(2d+\delta)+\tilde{\delta}$ in Case (i) $(FH(6FillRadM^n+9\delta)+\tilde{\delta}$ in Case (ii)) filling the closed curve. (Here $\tilde{\delta}$ can be made arbitrarily small.) We will assign this surface to the simplex $\sigma_i^2$.

Next consider a 3-simplex $\sigma_i^3 = [v_{i_0}, ..., v_{i_3}]$. (Again, in case (i) we consider an abstract simplex $\sigma_i^3$, but we are assuming that $v_{i_0} = \tilde{p}$ and $[v_{i_1}, \ldots, v_{i_3}]$ is a two-dimensional face of one of the $q$-simplices forming $S$. In case (ii) $\sigma_i^3$ is a 3-dimensional simplex of $W$.) Its boundary corresponds to the singular surface $\Sigma_{j=0}^3(-1)^j s_{i_0,\ldots \hat{i}_j,\ldots,i_3}$. Its area is $\leq 3FH(2d + \delta) + 4\tilde{\delta}$ in the case (i). (Here and below we can assume that volumes of faces of all dimensions of all simplices of $S$ are less than $\tilde{\delta}$.) It is $\leq 4FH(6FillRadM^n + 9\delta) + 4\tilde{\delta}$ in Case (ii). This surface can be considered as an element in the space of integral 2-cycles on $M^n$. Now, either there exists a minimal 2-cycle that locally minimizes the mass and has a non-zero mass smaller than that of this surface, or this surface, viewed as a point in $Z_2(M^n, \mathbf{Z})$ can be connected with the zero cycle by a path $h_i^1 : [0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$ without an increase of its mass. In the first case we obtain the desired contradiction, so we can assume that the second case holds.

Therefore, we can establish a correspondence between the 3- simplices and the above maps $h_i^1$.

Now consider a 4-simplex $\sigma_i^4 = [v_{i_0}, ..., v_{i_4}]$. For each face $(-1)^j[v_{i_0}, ...., \hat{v}_{i_j}, ...., v_{i_4}]$ there was preassigned a map $(-1)^j h^1_{i_0,\ldots,\hat{i}_j,\ldots,i_4}$ : $[0,1] \longrightarrow Z_2(M^n, \mathbf{Z})$. Now, let $f_i^1(t) = \Sigma_{j=0}^1(-1)^j h^1_{i_0,\ldots,\hat{i}_j,\ldots,i_4}(1-t)$. Each $f_i^1(t)$ is an integral 2-cycle of area $\leq 4 \cdot 3FH(2d + \delta) + 17\tilde{\delta}$ in Case (i) ( $\leq 5 \cdot 4FH(6FillRadM^n + 9\delta) + 20\tilde{\delta}$ in case (ii)). Note that $f_i(0)$ is the zero cycle and that $f_i(1)$ is also the zero cycle due to the cancellations of the pairs of surfaces that enter with the opposite orientation. Thus, $f_i^1(t)$ can be regarded as a loop in the space of the integral 2-cycles. Since we have assumed that there are no almost minimizing integral varifolds of "small" area the map $f_i^1(t)$ is contractible over a disc $h_i^2(s)$ such that for each $s$ the mass of $h_i^2(s)$ does not exceed 12 $FH(2d+\delta)+17\tilde{\delta}$. Therefore, we can assign this map of the disc to a 4-simplex $\sigma_i^4$. (Here we are using Lemma 2.1).

Now suppose that for each simplex of dimension $k$ we have constructed the corresponding map $h_i^{k-2}: D^{k-2} \longrightarrow Z_2(M^n, \mathbf{Z})$. Let us consider an arbitrary $(k+1)$-simplex $\sigma_i^k = [v_{i_0}, ..., v_{i_{k+1}}]$. Consider a face in the boundary of this simplex $(-1)^j [v_{i_0}, ..., \hat{v}_{i_j}, ..., v_{i_{k+1}}]$. To this face there corresponds a map $h^{k-2}_{i_0,...,\hat{i}_j,...,i_{k+1}}: D^{k-2} \longrightarrow Z_2(M^n, \mathbf{Z})$. Now consider a map $f_i^{k-2}: D^{k-2} \longrightarrow Z_2(M^n, \mathbf{Z})$, such that $f_i^{k-2}(r,\theta) = \Sigma_{j=0}^{k+1}(-1)^j h^{k-2}_{i_0,...,\hat{i}_j,...,i_{k+1}}(1-r,\theta)$. Here we are assuming that $D^{k-2}$ is the standard ball of radius one in $\mathbf{R^{k-2}}$, and $r \in [0,1], \theta \in S^{k-3}$ denote the polar coordinates. Note that $f_i^{k-2}(1,\theta)$ is the zero cycle due to the cancellations. Therefore, one can consider $f_i^{k-2}$ as a map from $S^{k-2}$ to the space of 2-cycles. By our assumption this map is contractible. Contracting it we obtain $h_i^{k-1}: D^{k-1} \longrightarrow Z_2(M^n, \mathbf{Z})$. (Otherwise Lemma 2.1 implies the existence of the stationary varifold as in Theorem 1.3, and we obtain the desired contradiction.) We will assign this map to simplex $\sigma_i^{k-1}$.

We will continue in the above manner until we construct maps $f_i^{q-2}$ in Case (i) and maps $f_i^{n-2}$s in Case (ii). We will call these maps $\tilde{g}_i$. The sum of these maps is homotopic to the map $f$ constructed in Step 1.

**Step 3.** We can conlude that one of the maps $\tilde{f}_i$ is not contractible. Therefore, another application of Lemma 2.1 implies that there exists a non-trivial stationary integral varifold with the mass as claimed in Theorem 1.3.

If $n = 3$, then the regularity assertion follows from Theorem 7.12 in [P]. □

# 4 Proof of Theorem 1.4

The proof is similar to that of Theorem 1.3. We perform an inductive extension process similar to the process described in the proof of Theorem 1.3. For dimensions up to dimension two the process is identical to the process in the proof of Theorem 1.3. For dimensions from one to $k$ we fill cycles in $M^n$ that correspond to boundaries of the considered simplices by chains of the minimal mass in $M^n$. Starting from dimension $k$ we work in the space of $k$-dimensional cycles in $M^n$ and perform "honest" extensions in this space. ("Honest" means that maps of boundaries of simplices are filled by maps of the simplices.) The only possible obstruction to this extension process is a stationary $k$-varifold as in Theorem 1.4. But since this extension process cannot be performed up to dimension $q+1$ in the case (i) (or $n+1$

in the case (ii)) such a stationary varifold must exist.

*Proof.* The proof is by contradiction. We will assume that there is no stationary varifold as in Theorem 1.4. As in the proof of Theorem 1.3 we are going to discuss cases (i) and (ii) at the same time. The theorem will be proved in three steps.
**Step 1.** In the case (i) denote the minimal dimension $i$ such that $H_i(M^n) \neq 0$ by $q$. Let $S$ be a triangulated singular $q$-cycle representing a non-trivial element of $H_q(M^n)$. As in the proof of Theorem 1.3 we are assuming that all simplices of $S$ have very small diameters. In the case (ii) let $W$ be a chain that fills $M^n$ in the $(FillRad(M^n)+\delta)$-neighborhood of $M^n$ in $L^\infty(M)$. We are going to construct a non-contractible map $\tilde{f} : S^{q-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in Case (i) and a map $\tilde{f} : S^{n-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in the case (ii). (Here and below $Z_k(M^n, \mathbf{Z})$ denotes the space of integral $k$-cycles on $M^n$.)
**Step 2.** We will construct maps $\tilde{g}_i$ from $S^{q-k}$ (or $S^{n-k}$) to the space of $k$-dimensional cycles on $M^n$, such that the map $\tilde{f}$ of Step 1 will be homotopic to their sum. The masses of $k$-cycles $\tilde{g}_i(t)$ do not exceed the right hand side in the inequality (i) (or (ii)) in Theorem 1.4.
**Step 3.** We will conclude that, one of the maps $\tilde{g}_i$ constructed on Step 2 is not contractible, and therefore the results of J. Pitts (namely, Lemma 2.1) imply the existence of a stationary varifold as in Theorem 1.4 thereby obtaining a contradiction.

The extra regularity of the stationary varifold in the hypersurface case asserted in Theorem 1.4 immediately follows from [P], ch. 7 and [SS].

Here is a detailed description of the first two steps of the proof.

**Step 1.** Consider an arbitrary simplex $\tilde{\sigma}_i^{k+1} = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_{k+1}}]$. In the case (i) this simplex is a $(k+1)$-dimensional face of one of singular simplices of $S$; in the case (ii) it is one of $(k+1)$-dimensional simplices of a smooth fine triangulation of $M^n$. Without any loss of generality we can assume that this simplex is generated by a mass decreasing homotopy $\tilde{h}_i^1 : [0,1] \longrightarrow Z_k(M^n, \mathbf{Z})$ that connects its boundary with a point $\tilde{p}_i^{k+1}$, (the homotopy will be sometimes denoted as $\tilde{h}^1_{i_0,\dots,i_{k+1}}$, and the point will be sometimes denoted as $\tilde{p}^{k+1}_{i_0,\dots,i_{k+1}}$). Now consider a $(k+2)$-dimensional simplex $\tilde{\sigma}_i^{(k+2)} = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_{k+2}}]$. Each oriented face $(-1)^j[\tilde{v}_{i_0}, ..., \hat{\tilde{v}}_{i_j}, ..., \tilde{v}_{i_{k+2}}]$ of its boundary corresponds to the map $(-1)^j \tilde{h}^1_{i_0,\dots,\hat{i}_j,\dots,i_{k+2}}$. Define a new map $\tilde{f}_i^1 : [0,1] \longrightarrow Z_k(M^n, \mathbf{Z})$ by letting $\tilde{f}_i^1(t) = \Sigma_{j=0}^{k+2}(-1)^j \tilde{h}^1_{i_0,\dots,\hat{i}_j,\dots,i_{k+2}}(1-t)$. Note that $\tilde{f}_i^1(0) = \tilde{f}_i^1(1)$ and is the zero cycle, thus the newly constructed map is a loop in the space of $k$-cycles. Since all the simplices in $S$ or in the considered triangulation of $M^n$ have small diameters, this loop is contractible via "small" cycles. We can choose a contraction that amounts to slicing the $(k+2)$-dimensional simplex into $k$-cycles, so that this slicing extends the slicing of the boundary. This contraction can be explicitly constructed using a coning of the slicing of the boundary. This produces a disc $\tilde{h}_i^2 : D^2 \longrightarrow Z_k(M^n, \mathbf{Z})$. Thus, we have a correspondence between the simplex $\tilde{\sigma}_i^{k+2}$ and the map $\tilde{h}_i^2$.

Proceeding by induction suppose we have constructed maps $\tilde{h}_i^{l-k} : D^{l-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ that correspond to $l$-dimensional simplices $\tilde{\sigma}_i^l$. Consider an arbitrary $(l+1)$-dimensional simplex $\tilde{\sigma}_i^{l+1} = [\tilde{v}_{i_0}, ..., \tilde{v}_{i_{l+1}}]$. Define the map $\tilde{f}_i^{l-k} : D^{l-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ by letting $\tilde{f}_i^{l-k}(r, \theta) = \Sigma_{j=0}^{l+1}(-1)^j \tilde{h}^{l-k}_{i_0,\ldots,\hat{i}_j,\ldots,i_{l+1}}(1-r, \theta)$. This map corresponds to the boundary of $\tilde{\sigma}_i^{l+1}$. Note that $\tilde{f}_i^{l-k}(1, \theta)$ is the zero cycle, and thus it is a map from the $(l-k)$-dimensional sphere to the space of integral $k$-cycles. (Here $r$ and $\theta$ denote polar coordinates in $D^{l-k} = \{x \in \mathbf{R}^{l-k} | \|x\| \leq 1\}$.) This sphere is contractible over the disc $\tilde{h}_i^{l+1-k} : D^{l+1-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in the space of "small" integral $k$-cycles. This contraction can be explicitly constructed by extending the slicing of the boundary of the considered $(l+1)$-dimensional simplex to its interior, and this can be explicitly done just by the coning. We continue in the above manner until we construct the maps $\tilde{h}_i^{q-k}$s in case (i) and the maps $\tilde{h}_i^{n-k}$s in case (ii). Take the sum of those maps over all the simplices in the triangulation to obtain sphere $\tilde{f} : S^{q-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in the space of integral $k$-cycles in Case (i) and $\tilde{f} : S^{n-k} \longrightarrow Z_k(M^n, \mathbf{Z})$ in Case (ii).

**Step 2. (The main step.)** In the case (i) let $\tilde{p}$ be an arbitrary point of $M^n$. Consider a "cone" over $S$ with the vertex at $\tilde{p}$: for any $(j-1)$-dimensional face $[v_{i_1}, \ldots, v_{i_j}]$ of one of $q$-simplices forming $S$ we consider a $j$-dimensional simplex $[v_{i_0}, \ldots, v_{i_j}]$, where $v_{i_0} = \tilde{p}$. Initially these simplices are abstract, but then we realize them by singular chains of $M^n$ as follows: First, we realize all simplices $[v_{i_0}, v_{i_1}]$ by some of the shortest closed geodesics between $\tilde{p}$ and $v_{i_1}$. If $v_{i_1}$ and $v_{i_2}$ are endpoints of a 1-dimensional simplex in the boundary of one of the $q$-simplices forming $S$, then we assign to $[v_{i_1}, v_{i_2}]$ this singular 1-dimensional simplex in $M^n$. Then, by induction, for each $l \leq k$ we take the singular cycle formed by considering the alternating sum of $(l+2)$ singular chains assigned on the previous step to $l$-dimensional faces of $[v_{i_0}, \ldots, v_{i_l}]$. Then we consider a $(l+1)$-dimensional singular chain of (almost) minimal volume filling this cycle. We assign this chain to the considered $(l+1)$-dimensional simplex. We assign to $[v_{i_1}, \ldots v_{i_l}]$ the corresponding face of the corresponding $q$-simplex in $S$. Procceding in this way eventually we assign singular chains to all simplices of dimensions up to $k$. In case (ii) we first map all vertices of $W$ to (some of) the closest points of $M^n$. Without any loss of generality we can assume that these closest points are vertices of the considered triangulation of $M^n$. Then we assign to every one dimensional simplex of $W \setminus M^n$ one of the shortest geodesics between the images of its endpoints. Then we inductively assign singular chains of $M^n$ to all simplices of $W \setminus M^n$ of dimensions up to $k$. To do this we fill the cycle formed by all chains that have already been assigned to simplices of the boundary in an optimal (or almost an optimal way). (Here "optimal" means that we are minimizing the volume.) Note that the length of the boundary of each considered 2-dimensional simplex will be bounded by $2d+\delta$ in the case (i) and by $6FillRad(M^n)+9\delta$ in the case (ii). Therefore the vollumes of all singular $k$-chains assigned to $k$-dimensional simplices will be bounded by $FH_{k-1}((k+1)FH_{k-2}(kFH_{k-3}(\ldots 4\ FH_1(6FillRad(M^n)\ldots))) + \varepsilon$ in case (ii) and by $FH_{k-1}((k+1)FH_{k-2}(kFH_{k-3}(\ldots 3FH_1(2d)\ldots)))+\varepsilon$ in case (i). Here $\varepsilon$ can be made arbitrarily small by choosing sufficiently fine triangulations and choosing filling chains sufficiently close to the optimal possible. The difference of constants between cases (i) and (ii) is explained by the fact that in case (i) for each $j$ one of the singular chains assigned to $(j+1)$ simplices in the boundary of each considered $j$-dimensional simplex is a face of one of the $q$-simplices forming $S$. Therefore it has a very small volume.

Now consider an arbitrary $(k+1)$-simplex $\sigma_i^{k+1} = [v_{i_0}, \ldots, v_{i_k}]$ coming from the coning of a $k$-dimensional simplex of $S$ in the case (i) or from $W \setminus M^n$ in the case (ii). Each oriented face in its boundary $(-1)^j[v_{i_0,\ldots,\hat{i}_j,\ldots,i_{k+1}}]$ corresponds to the already constructed singular chain that we denote $(-1)^j s_{i_0,\ldots,\hat{i_j},\ldots,i_{k+1}}$. Consider the following cycle $S_i^{k+1} = \Sigma_{j+0}^{k+1}(-1)^j s_{i_0,\ldots,\hat{i}_j,\ldots,i_{k+1}}$. This is an element of $Z_k(M^n,\mathbf{Z})$ of volume $\leq (k+1)FH_{k-1}(kFH_{k-2}((k-1)FH_{k-3}(\ldots 3FH_1(2d)\ldots))) + \varepsilon$ in Case (i) and of volume $\leq (k+2)FH_{k-1}((k+1)FH_{k-2}(kFH_{k-3}(\ldots 4FH_1(6FillRad(M^n)\ldots))) + \varepsilon$. (Here $\varepsilon$ depends on $\delta$ and can be made arbitrarily small.) By our assumption this cycle can be connected with the zero cycle with a path that will only pass through the cycles of smaller volume. Let us denote this path by $h_i^1 : [0,1] \longrightarrow Z_k(M^n,\mathbf{Z})$. We will assign this path to the above simplex $\sigma_i^{k+1}$. Now suppose we have constructed the maps $h_i^{l-k} : D^{l-k} \longrightarrow Z_k(M^n,\mathbf{Z})$ corresponding to simplices $\sigma_i^l$, where $l \leq q-1$ in Case (i) and $\leq n-1$ in Case (ii). Consider an arbitrary $(l+1)$-dimensional simplex $\sigma_i^{l+1}$. Each face $(-1)^j[v_{i_0},...,\hat{v}_{i_j},...,v_{i_{l+1}}]$ corresponds to a map $(-1)^j h^{l-k}_{i_0,\ldots,\hat{i}_j,\ldots,i_{l+1}}(r,\theta)$. (Here we are assuming that $D^{l-k}$ is the standard ball of radius 1, and $r\in[0,1],\theta\in S^{l-k-1}$ denote the polar coordinates in $D^{l-k}$.) Define a new map $f_i^{l-k}(r,\theta)=\Sigma_{j=0}^{l+1}(-1)^j h^{l-k}_{i_0,\ldots,\hat{i}_j,\ldots,i_{l+1}}(1-r,\theta)$. Note that $f(1,\theta)$ is the zero cycle, and thus, this map factorizes into the composition of the map identifying the boundary of the ball to a point, and a map from the resulting sphere $S^{l-k}$ to the space of $k$-cycles. So, $f_i^{l-k}$ can be regarded as this last map of $S^{l-k}$ into the the space of $k$-cycles. By our assumption this map is contractible. (Of course, here we again use Lemma 2.1.) We contract this map, and denote the resulting map by $h_i^{l-k+1}: D^{l-k+1}\longrightarrow Z_k(M^n,\mathbf{Z})$. We continue in the above manner until we construct maps $f_i^{q-k}$ in case (i) or the maps $f_i^{n-k}$ in case (ii). Those are our maps $\tilde{g}_i's$. Note that $\tilde{f}$ is homotopic to their sum. This completes the description of Step 2. Now we perform Step 3 (that is, apply Lemma 2.1 to a non-contractible $\tilde{g}_i$) and obtain a contradiction that completes the proof of the theorem. QED.

□

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