Geometric And Functional Analysis 1016-443X/96/0x001-xx$1.50+0.20/0
Vol. 6, No. x (1996)

DISCONNECTEDNESS OF SUBLEVEL SETS OF SOME RIEMANNIAN FUNCTIONALS

A. Nabutovsky

Abstract

Let M be a compact manifold of dimension greater than four. Denote by $Riem(M)$ the space of Riemannian structures on M (i.e. of isometry classes of Riemannian metrics on M) endowed with the Gromov-Hausdorff metric. Let $Riem_\epsilon(M) \subset Riem(M)$ be its subset formed by all Riemannian structures μ such that $\mathrm{vol}(\mu) = 1$ and $inj(\mu) \geq \epsilon$, where $inj(\mu)$ denotes the injectivity radius of μ.

We prove that for all sufficiently small positive ϵ the space $Riem_\epsilon(M)$ is disconnected. Moreover, if ϵ is sufficiently small, then $Riem_\epsilon(M)$ is representable as the union of two non-empty subsets A and B such that the Gromov-Hausdorff distance between any element of A and any element of B is greater than $\epsilon/9$. We also prove a more general result with the following informal meaning: There exist two Riemannian structures of volume one and arbitrarily small injectivity radius on M such that any continuous path (and even any sequence of sufficiently small "jumps") in the space of Riemannian structures of volume one on M connecting these Riemannian structures must pass through Riemannian structures of injectivity radius "uncontrollably" smaller than the injectivity radii of these two Riemannian structures.

These results can be generalized for at least some four-dimensional manifolds. The technique used in this paper can also be used to prove the disconnectedness of many other subsets of the space of Riemannian structures on M formed by imposing various constraints on curvatures, volume, diameter, etc.

0. Main Result and the Idea of its Proof

Let M be a compact manifold of dimension greater than four. Let $Riem_\epsilon(M)$ denote the space of Riemannian structures (i.e. isometry classes of Riemannian metrics) on M of volume one and the injectivity radius greater than or equal to ϵ. This space can be endowed with the Gromov-Hausdorff metric d_{GH} (cf. [Gro], [Pet], [Pete], [GrLP], [G] for the definition of this metric and some properties of the metric space of Riemannian structures). Our main result is the following theorem:

This work was partially supported by the New York University Research Challenge Fund grant, by NSF grant DMS 9114456 and by the NSERC operating grant OGP0155879.

Theorem 1. *For all sufficiently small positive ϵ $Riem_\epsilon(M)$ can be represented as the union of two non-empty subsets A and B such that for any $\mu_1 \in A$, $\mu_2 \in B$ $d_{GH}(\mu_1, \mu_2) > \epsilon/9$. Furthermore, let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be any increasing function such that for any x $\theta(x) \leq x$ but for some Turing computable function $\alpha : \mathbb{N} \to \mathbb{N}$ $\theta(1/N) \geq \frac{1}{\alpha(N)}$ for any $N \in \mathbb{N}$. (For example, $\theta(x)$ can be equal to x^Λ for any $\Lambda \geq 1$ or to $\frac{1}{\exp(\exp(\ldots(\exp(x^{-\Lambda})\ldots)))}$ (k times) for any positive k and $\Lambda \geq 1$. Below we will call such functions effectively minorizable.) Then there exists $\epsilon_0(\theta, M) > 0$ with the following property: For any positive $\epsilon \leq \epsilon_0(\theta, M)$ there exist $\mu, \nu \in$ $Riem_\epsilon(M)$ such that there is no finite sequence $\mu_0 = \mu, \mu_1, \ldots, \mu_{q-1}, \mu_q = \nu$ of Riemannian structures from $Riem_{\theta(\epsilon)}(M)$ such that for any i $d_{GH}(\mu_i, \mu_{i+1}) \leq \frac{1}{9} \min\{inj(\mu_i), inj(\mu_{i+1})\}$. (In particular, the last statement informally implies that there exist two Riemannian structures in $Riem_\epsilon(M)$ which cannot be connected by a finite sequence of "jumps" of length not exceeding $\theta(\epsilon)/9$ in the set of Riemannian structures on M of volume one and injectivity radius $\geq \theta(\epsilon)$.)*

Remark 1: It follows from the proof of Theorem 1 below that the coefficient 1/9 at ϵ in the statement of Theorem 1 can be replaced by any positive number $a < 1/8$. We did not try to find the best possible value of this constant.

Remark 2: It is not difficult to deduce from the precompactness of $Riem_\epsilon(M)$ (which follows from the Gromov precompactness theorem ([GrLP]) and the Croke inequality ([Cr, Theorem 14]) that Theorem 1 will not be true without the assumption that θ is effectively minorizable.

Remark 3: According to the Church-Turing thesis the class of Turing computable functions coincides with the class of functions computable in an intuitive and informally defined sense. Although this thesis is only an empirical principle, one can state and prove its numerous rigorous versions, where the notion of functions computable in an intuitive sense is replaced by a rigorous notion of computability. In particular, a function is Turing computable if and only if there is a program computing this function written in FORTRAN (or BASIC, or PASCAL, or C, etc.) and using only the integer data type. The same remark can be made about the notion of "algorithm" which will be used below. Formally speaking, we will use the term "algorithm" just as a synonym of the term "Turing computable function". However, the reader will make no mistake thinking of an algorithm as a program for an abstract computer (that is, for a computer with unlimited storage space and able to perform computations with integer numbers of arbitrary length) written in one of the programming languagues mentioned and using only the integer data type.

In section 5 we discuss various generalizations of Theorem 1 and related results. In particular, the proof of Theorem 1 is easily adaptable to prove the disconnectedness of many other interesting subsets of the space of Riemannian structures on M, formed by imposing constraints on curvature, volume, diameter, injectivity radius, etc. Note that such subsets have been studied by many mathematicians (starting from the pioneering work of Cheeger ([C]) and Gromov (cf. [GrLP])). See, for example, [P], [CGr], [Fu], [Gr1], [Gro], [GPW], [GP], [G], [BuGrPe], [AnC], [KrS], [Y], etc.

The only results known to me which establish disconnectedness of subsets of $Riem(M)$ are those of Gromov and Lawson (Theorem 4.47 in [GrLa]) and of M. Kreck and S. Stolz [KrS]. However, the results and the methods of [GrLa] and [KrS] are completely different from ours.

Disconnectedness results similar to Theorem 1 can be applied to prove the existence of infinitely many solutions for some geometric variational problems (for example, see Theorems 5.1, 5.2 in [N3]). In section 5 we give another example of such applications (Theorem 11).

Here is the idea of the proof of Theorem 1. The general case follows from the case $M = S^n$. To prove the theorem in this case we introduce the following invariant of simply connected Riemannian manifolds. Gromov ([Gr2]) defined the filling length $fl(\gamma)$ of a closed curve γ in a Riemannian manifold as the minimal number L such that γ can be contracted to a point by a homotopy passing only through closed curves of length $\leq L$. For a simply connected compact Riemannian manifold (M, μ) we first define $Fl(\mu)$ as the minimal number C such that every closed curve γ in (M, μ) can be contracted to a point by a homotopy passing through closed curves of length not exceeding $C\ length(\gamma)$. Then we define $F_M(\epsilon)$ as $\sup_{\mu\in Riem_\epsilon(M)} Fl(\mu)$. It is not difficult to prove that if μ_1 and μ_2 are sufficiently close Riemannian structures in $Riem_\epsilon(M)$, then $Fl(\mu_1)$ can be majorized in terms of ϵ and $Fl(\mu_2)$ (Lemma 2). Now one can easily see, using the proof of precompactness of $Riem_\epsilon(M)$ (cf. [Pete]), that if μ_1 and μ_2 are in the same connected component of $Riem_\epsilon(M)$, or more generally, μ_1 and μ_2 can be connected by a sequence of "jumps" as in the text of Theorem 1, then $Fl(\mu_1)$ does not exceed $f_0(\epsilon)Fl(\mu_2)$, where $f_0(\epsilon)$ is a certain function of ϵ, which can be explicitly written (Lemma 3). As a corollary, the assumption that Theorem 1 is false for $M = S^n$ implies that the restriction of the function $F_{S^n}(\epsilon)$ on a certain strictly decreasing sequence $\epsilon_i \to 0$ can be majorized by a Turing computable function of $\left[\frac{1}{\epsilon_i}\right]$. (Here and below $[x]$ denotes the integer part of x.) Now Theorem 1 for $M = S^n$, $n \geq 5$, follows from Theorem 4 stating that for these values of n and for any Turing computable function $\phi : \mathbb{N} \to \mathbb{N}$, $F_{S^n}(\epsilon) > \phi([\frac{1}{\epsilon}])$ for all sufficiently small positive ϵ. (Similar results establishing that certain functions naturally arising in geometry cannot be majorized by a recursive function can be found in [N1,3] and [ABBr].) This result follows from the S. Novikov theorem establishing for any fixed $n \geq 5$ the non-existence of an algorithm deciding whether or not a given n-dimensional manifold is homeomorphic to S^n. (A detailed proof of this theorem can be found in the Appendix of [N2].) Roughly speaking, the reason why there is no such algorithm is the algorithmic unsolvability of the triviality problem for finitely presented groups. It is impossible to decide whether or not a given manifold is homeomorphic to S^n because it is impossible to decide whether or not its fundamental group is trivial. But the possibility of computing an upper bound for $F_{S^n}(\epsilon)$ would make possible an explicit construction of an algorithm recognizing whether or not a given homology sphere is simply connected (Lemma 7). This observation implies Theorem 4 and, hence, Theorem 1.

In section 5.F we will describe an idea due to M. Gromov of a different proof of disconnectedness of sets $Riem_\epsilon(M)$ for all sufficiently small positive ϵ.

1. Proof of Theorem 1

Here we will prove Theorem 1 modulo several lemmae which will be proven later in the case when the manifold M is the standard sphere S^n. However, it is not difficult to modify this proof to make it work for an arbitrary compact manifold M of dimension $n \geq 5$ (see the remark at the end of this section).

Let M be a compact simply connected manifold. Consider the space $Riem_0(M)$ of all Riemannian structures of volume one on M endowed with the Gromov-Hausdorff metric. For any Riemannian metric g on M consider the space ΩM_g of all continuous piecewise smooth maps of S^1 into M endowed with the C^0 metric. Following [Gr2], define for every $\gamma \in \Omega M_g$ *the filling length* $fl(\gamma)$ as follows: Consider the set S of all homotopies $H : [0,1] \to \Omega M_g$ between γ and a constant loop. Let $fl(\gamma)$ be the infimum over this set of the maximal length of loops during the homotopy (i.e. $fl(\gamma) = \inf_{H \in S} \sup_{t\in[0,1]} length(H(t))$). Now define $Fl : Riem_0(M) \to \mathbb{R}^+$ by the formula $Fl([g]) = \sup_{\gamma \in \Omega M_g} fl(\gamma)/length(\gamma)$, where $[g]$ denotes the isometry class of a metric g on M. Note that according to [GrLP] (Scolie 2.27), $Fl(g) < \infty$ for any $g \in Riem_0(M)$. We will need the following property of Fl:

LEMMA 2. *Let* $h_1, h_2 \in Riem_0(M)$, $\epsilon = \min\{inj(h_1), inj(h_2)\}$, *and* $d_{GH}(h_1, h_2) < \epsilon/8$. *Then*

$$Fl(h_2) \le 128 \left(1 + \frac{\epsilon}{\epsilon - 8d_{GH}(h_1, h_2)}\right)^3 Fl(h_1)\ .$$

This lemma will be proved in section 2. Note, that Lemma 2 implies that if $d_{GH}(h_1,\ h_2) \le \min\{inj(h_1), inj(h_2)\}/8.5$, then $Fl(h_2) \le 10^6 Fl(h_1)$. Lemma 2 has the following corollary:

LEMMA 3. *Let $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ be any increasing function such that $\theta(\epsilon) \le \epsilon$. Let $h_1,\ h_2 \in Riem_\epsilon(M)$ be two Riemannian structures such that there exist Riemannian structures $g_1, \dots, g_q \in Riem_{\theta(\epsilon)}(M)$ with the following properties:*

(a) *$g_1 = h_1$, $g_q = h_2$;*

(b) *For any $j \in \{1, \dots, q-1\}$ $d_{GH}(g_j, g_{j+1}) \le \min\{inj(g_j), inj(g_{j+1})\}/9$.*

Then

$$Fl(h_2) \le Fl(h_1) 10^{6N([1/\theta(\epsilon)])}\ ,$$

where $N(m)$, $(m \in \mathbb{N})$, is a Turing computable function which can be written down explicitly. (In fact, one can take for $N(m)$ any function which majorizes the number of points in a minimal $1/(20(m+1))$-net in $Riem_{\frac{1}{m+1}}(M)$.)

Proof. The idea of the proof is to replace the sequence $g_1, \dots, g_q$ by its subsequence which has the same properties (a), (b) but contains not more than $N([1/\theta(\epsilon)])$ elements, where a function N satisfies the requirements of Lemma 3. Then the obvious inductive application of Lemma 2 will complete the proof.

To construct the required subsequence we use the following algorithm. Let $k = 1$. We find the maximal j such that $d_{GH}(g_k, g_j) \le \min\{inj(g_k), inj(g_j)\}/9$. Then, if $j > k+1$, we delete terms $g_{k+1}, \dots, g_{j-1}$ from the sequence. Afterwards, if $j < q$, we let $k = j$ and repeat the same procedure, etc. The algorithm stops, when the next value of j turns out to be equal to q. This algorithm yields a subsequence of the original sequence satisfying properties (a), (b) in the text of Lemma 3 and the following additional property: For any $\mu \in Riem_{\theta(\epsilon)}(M)$ there are not more than two elements of the constructed subsequence in the $\theta(\epsilon)/20$-ball centered at μ. Hence, the number of elements in the subsequence does not exceed the number of elements in any $\theta(\epsilon)/20$-net in $Riem_{\theta(\epsilon)}(M)$ multiplied by two. But the Croke inequality ([Cr, Proposition 14]) and the standard volume comparison argument used in the proof of the Gromov precompactness theorem (cf. [Pete]) imply that for some constant $c(n) > 0$ and for any $\delta > 0$ there exists a $\delta/20$-net in $Riem_\delta(M)$ with less than $f_0(\delta) = \exp(c(n)/\delta^{2n+1})$ elements. Hence the number of elements in the constructed subsequence does not exceed $2f_0(\theta(\epsilon))$. This fact implies Lemma 3 for $N(m) = \left[2f_0\left(\frac{1}{m+1}\right)\right] + 1$. □

It is clear that in the particular case, when M is the standard sphere, Theorem 1 immediately follows from Lemma 3 and the following theorem. Define $F_M(\epsilon)$ as $\sup_{h\in Riem_\epsilon(M)} Fl(h)$.

Theorem 4. *Let S^n be the standard sphere of dimension $n \geq 5$. For any Turing computable function $\phi : \mathbb{N} \to \mathbb{N}$, $F_{S^n}(\epsilon) > \phi([1/\epsilon])$ for all sufficiently small positive ϵ.*

Indeed, if Theorem 1 is false for $M = S^n$, then there exists an infinite decreasing sequence ϵ_i, converging to zero, and an effectively minorizable function θ such that for any $h \in Riem_{\epsilon_i}(S^n)$ there exists a finite sequence $h_1, \ldots, h_q = h$ such that h_1 is the canonical metric of volume one on S^n, the distance between h_j and h_{j+1} does not exceed $\min\{inj(h_j),\ inj(h_{j+1})\}/9$, and $h_j \in Riem_{\theta(\epsilon_i)}(S^n)$ for any j. Hence, Lemma 3 applied to h_1 and to h implies that $Fl(h)$ is bounded from above by a Turing computable function of $[1/\epsilon_i]$. But this contradicts Theorem 4.

REMARK: Lemma 2 and the precompactness of $Riem_\epsilon(M)$ (cf. [GrLP], [Pete]) imply that for any $\epsilon > 0$, $F_M(\epsilon) < \infty$.

The proof of Theorem 4 is based on the following theorem and two lemmae:

Theorem 5. *For any $n \geq 5$ there exists an algorithm producing for every Turing machine T and its input w a vector of coefficients of a polynomial $p_T \in Q[x_1, \ldots, x_{n+1}]$ such that:*

(i) *The zero set $Z(p_T)$ of p_T is a non-empty compact hypersurface in $\mathbb{R}^{n+1}$; $grad\ p_T$ does not vanish at any point of $Z(p_T)$.*

(ii) *$Z(p_T)$ is diffeomorphic to S^n if and only if T eventually halts when it starts to work with w.*

(iii) *$Z(p_T)$ is a homology sphere.*

Here and below the term "algorithm" means a Turing computable function. In this case this function has two arguments: a Turing machine (represented, for example, by its number in a certain Gödel numbering) and its input. This theorem is a version of S. Novikov's theorem on the algorithmic unsolvability of the problem of recognition of the sphere S^n for $n \geq 5$ ([VKuF, ch. 10]). Its detailed proof can be found in the Appendix of [N2]. (The proof of this theorem can be briefly explained as follows. Several proofs of the unsolvability of the word problem for finitely presented groups (cf. [R]) provide an algorithm constructing for a given Turing machine T and its input w a finite presentation of a group $G(T)$ and a word $\omega(w)$ such that T eventually halts, when it starts to work with w, if and only if $\omega(w) = e$ in $G(T)$. Then one can apply a version of Rabin's "witness" construction described on pp. 13-14 of [M] to construct a finite presentation of a perfect group G_ω such that G_ω is trivial if and only if $\omega(w) = e$ in $G(T)$. Then, following S. Novikov's idea one writes down a finite presentation of the universal central extension $\bar{G}_\omega$ of G_ω. One can show using results of the M. Kervaire paper [K] that $\bar{G}_\omega$ is the fundamental group of a n-dimensional homology sphere, which can be realized as a non-singular algebraic hypersurface in $\mathbb{R}^{n+1}$. Also, $\bar{G}_\omega$ is trivial if and only if G_ω is trivial. It remains to note, that one of the corollaries of the h-cobordism theorem ([Mi, ch. 9, Theorem A]) implies that for $n \geq 5$ a smooth hypersurface in $\mathbb{R}^{n+1}$, which is a homology sphere, is diffeomorphic to S^n if and only if its fundamental group is trivial.)

LEMMA 6. *Let $\{n_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integer numbers. There is no Turing machine T with the following properties:*

(a) *T terminates its work with any input in a finite time;*
(b) *If the input of T is equal to one of the numbers n_i, then the output of T is the list of all Turing machines with $\leq n_i$ states, which eventually halt when they start to work with the blank tape;*
(c) *If the input l of T is not one of the numbers n_i, then the output of T is the list of some (maybe none) Turing machines with $\leq l$ states which eventually halt when they start to work with the blank tape.*

This lemma is slightly stronger than the theorem about the algorithmic unsolvability of the halting problem for Turing machines. It will be proven in section 3. The proof is a modification of one of the classical proofs of the algorithmic unsolvability of the halting problem for Turing machines using the Rado "busy beaver" function (cf. [E] or [BooJ]). Note also, that in Lemma 6 the set $\{n_i\}_{i=1}^{\infty}$ is *not* assumed to be recursive or recursively enumerable.

LEMMA 7. *For any $n \geq 5$ there exists an algorithm with the following properties. Its input consists of a vector of coefficients of a polynomial p of $n+1$ variables with rational coefficients and of a positive integer number N. Assume that a polynomial $p \in Q[x_1, \dots, x_{n+1}]$ satisfies the following conditions:*

(i) *The zero set $Z(p)$ of p is a non-empty compact smooth hypersurface of $\mathbb{R}^{n+1}$ and grad p does not vanish anywhere on $Z(p)$;*
(ii) *$Z(p)$ is a homology sphere.*

Then the output of the algorithm will be either 1 or 0. The output can be equal to 1 only in the case, if $\pi_1(Z(p))$ is trivial. If, in addition, $N \geq F_{S^n}(inj(Z(p))/\operatorname{vol}(Z(p))^{\frac{1}{n}})$, then the output of the algorithm will be equal to 1 if and only if $\pi_1(Z(p))$ is trivial (and otherwise it will be equal to 0).

(Here $Z(p)$ is regarded as a Riemannian manifold with the metric, induced by the inclusion into $\mathbb{R}^{n+1}$.)

This lemma will be proved in section 4. Now we are going to deduce Theorem 4 from Theorem 5 and Lemmae 6 and 7.

Proof of Theorem 4. The idea is to assume that Theorem 4 is false and to use this assumption to construct an algorithm, the non-existence of which is stated in Lemma 6. So, assume that Theorem 4 is false. That is, there exists a strictly decreasing sequence ϵ_i, tending to 0, such that $F_{S^n}(\epsilon_i)$ does not exceed the value of a Turing computable function ϕ at $[1/\epsilon_i]$. Without loss of generality we can assume that ϕ is increasing. We are going to prove the existence of an infinite strictly increasing sequence n_i and the algorithm, specified in the text of Lemma 6 above. Let us define n_i as the maximal integer such that the value of $inj(Z(p_T))/\operatorname{vol}^{\frac{1}{n}}(Z(p_T))$ for any Turing machine T with $\leq n_i$ states and for the input $w =$ "the blank tape" is not less than ϵ_i. (Here p_T is the polynomial the existence of which is stated in Theorem 5. Also, we regard here and below the hypersurfaces $Z(p_T)$ as Riemannian manifolds with metric induced by the embedding in the ambient Euclidean space $\mathbb{R}^{n+1}$.) To see that such n_i exist first note that the statement of Theorem 5 easily implies that when T runs over the set of all Turing machines, the set of fundamental groups $\pi_1(Z(p_T))$ is infinite. On the other hand according to [GPW] for any fixed $n, \epsilon > 0$, the set of homotopy types of compact n-dimensional Riemannian manifolds of volume one and injectivity radius $\geq \epsilon$ is finite.

Now consider the following algorithm. For each m we consider all Turing machines T with $\leq m$ states. For every of these machines and its zero input the algorithm constructs the polynomial p_T the existence of which is stated in Theorem 5. Then the algorithm computes an integer upper bound for $\operatorname{vol}(Z(p_T))^{\frac{1}{n}}/inj(Z(p_T))$. (Such a bound can be found using the Cheeger inequality ([Co], [Pet]) to find a lower bound for $inj(Z(p_T))$ in terms of $\operatorname{diam}(Z(p_T)), \operatorname{vol}(Z(p_T))$ and an upper bound for sectional curvature of $Z(p_T)$. One can compute the required lower and upper bounds for $\operatorname{vol}(Z(p_T))$, and the upper bounds for $\operatorname{diam}(Z(p_T))$, and the sectional curvature first constructing a semialgebraic atlas of $Z(p_T)$ (for example, as described in the proof of Theorem 1 in [N2]) such that the metric coefficients of the Riemannian metric induced by the embedding of $Z(p)$ into $\mathbb{R}^{n+1}$ will be given in the local coordinates by semialgebraic functions. See [Co] or [BoCoRo] concerning computations with semialgebraic functions.) Then we find the maximum $I(m)$ of these upper bounds over the set $\Sigma(m)$ of all machines T with $\leq m$ states. Now, by our choice of n_i, $I(n_i + 1) \geq 1/\epsilon_i$.

Since we assumed that ϕ is increasing

$$\phi(I(n_i+1)) \geq \phi([1/\epsilon_i]) \geq F_{S^n}(\epsilon_i) \geq F_{S^n}\left(\inf_{T\in\Sigma(n_i)} \frac{inj(Z(p_T))}{\mathrm{vol}(Z(p_T))^{\frac{1}{n}}}\right) .$$

Also, note that $\phi(I(m))$ is a Turing computable function of m. Now we can complete the description of the algorithm as follows: For every Turing machine T with $\leq m$ states we compute $N = \phi(I(m+1))$ and then use this value of N in the data for the algorithm of Lemma 7, which will determine whether or not $Z(p_T)$ is simply connected. Note, that if $m = n_i$ then the answer provided by this algorithm will be correct. If $m \neq n_i$ and the algorithm will decide that $Z(p_T)$ is simply connected then $Z(p_T)$ is, indeed, simply connected. Now it remains to note that by virtue of Theorem 5 $Z(p_T)$ will be simply connected if and only if T will eventually halt, when it starts to work with the blank tape. Thus, we constructed the required algorithm and obtained the desired contradiction with Lemma 6. □

To prove Theorem 1 in the case, when M is not necessarily diffeomorphic to S^n one can construct a representative from a non-trivial connected component of $Riem_\epsilon(M)$ taking an appropriately defined "Riemannian connected sum" of a "simple" Riemannian metric μ_0 on M and a representative μ from a non-trivial connected component of $Riem_{2\epsilon}(S^n)$ such that $Fl(\mu)$ is very large. It is easy to see that $fl(\gamma)$ (for a loop $\gamma \in \Omega S^n_\mu$) cannot become significantly smaller as a result of taking an appropriately defined "Riemannian connected sum" with (M, μ_0). And an argument similar to the proofs of Lemmae 2, 3 shows that the resulting Riemannian structure on M and the Riemannian structure corresponding to μ_0 are in different connected components of $Riem_\epsilon(M)$.

2. Proof of Lemma 2

Proof of Lemma 2. For any two metrics $g_1, g_2 \in Riem_\epsilon(M)$ such that $d_{GH}(g_1, g_2) < \epsilon/8$ we can define the map $F_{g_2,g_1} : \Omega M_{g_2} \rightarrow \Omega M_{g_1}$ as follows: Let γ be a closed parametrized loop in (M, g_2). Choose points $t_1 = 0,\ t_1, \ldots, t_k < 1,\ t_{k+1} = 1$ in a manner such that the for every $j \in \{1, \ldots, k\}$ the length of the segment of γ between $\gamma(t_j)$ and $\gamma(t_{j+1})$ is equal to $length(\gamma)/(4[length(\gamma)/(\epsilon/2 - 4d_{GH}(g_1, g_2))] + 4)$. Let, for any $j \in \{0, \ldots, k\}$, $F_{g_2,g_1}(t_j)$ be (one of) the closest point(s) of (M, g_1) to $\gamma(t_j) \in (M, g_2)$ in a metric on the union of (M, g_2) and (M, g_1) (regarded as two different but not necessarily disjoint Riemannian manifolds) realizing the Gromov-Hausdorff distance between (M, g_1) and (M, g_2). Connect, for every $j \in \{1, \ldots, k\}$, $F_{g_2,g_1}(\gamma)(t_j)$ and $F_{g_2,g_1}(\gamma)(t_{j+1})$ by the shortest geodesic in (M, g_1). Note that the distance between $F_{g_2,g_1}(\gamma)(t_j)$ and $F_{g_2,g_1}(\gamma)(t_{j+1})$ in (M, g_1) is strictly less than ϵ ($\leq inj(g_1)$), and hence the shortest geodesic is unique. (Here and below inj denotes the injectivity radius.) Combining these geodesics together we obtain a piecewise smooth parametrized closed curve $F_{g_2,g_1}(\gamma)$. Observe that using the triangle inequality one can easily show that:

(a) For any $\gamma \in \Omega M_{g_2}$

$$\begin{aligned}
&\max_{t\in[0,1]} \mathrm{dist}_{g_2}\left(\gamma(t), F_{g_1,g_2}(F_{g_2,g_1}(\gamma))(t)\right)\\
&\qquad \leq 4d_{GH}(g_1,g_2) + \frac{length(\gamma)}{4([length(\gamma)/(\epsilon/2-4d_{GH}(g_1,g_2))]+1)}\\
&\qquad\qquad + \frac{length(F_{g_2,g_1}(\gamma))}{4([length(F_{g_2,g_1}(\gamma))/(\epsilon/2-4d_{GH}(g_1,g_2))]+1)}\\
&\qquad < \epsilon/2 - \left(\epsilon/4 - 2d_{GH}(g_1,g_2)\right) .
\end{aligned}$$

(b) Let γ_1, γ_2 be two parametrized closed curves on (M, g_2). Then

$$\begin{aligned}
&\max_{t\in[0,1]} \mathrm{dist}_{g_1}\left(F_{g_2,g_1}(\gamma_1)(t), F_{g_2,g_1}(\gamma_2)(t)\right)\\
&\quad \leq 4d_{GH}(g_1,g_2) + \max_{i\in\{1,2\}} \frac{length(\gamma_i)}{4([length(\gamma_i)/(\epsilon/2-4d_{GH}(g_1,g_2))]+1)}\\
&\qquad + \max_{t\in[0,\ 1]} \mathrm{dist}_{g_2}(\gamma_1(t),\gamma_2(t))\\
&\quad < \max_{t\in[0,1]} \mathrm{dist}_{g_2}\left(\gamma_1(t),\gamma_2(t)\right) + \epsilon/2 - \frac{\epsilon/2-4d_{GH}(g_1,g_2)}{2} .
\end{aligned}$$

(c) The length in g_1 metric of the curve $F_{g_2,g_1}(\gamma)$ does not exceed $length_{g_2}(\gamma) + 8d_{GH}(g_1,g_2)([length_{g_2}(\gamma)/(\epsilon/2-4d_{GH}(g_1,g_2))]+1)$.

In our proof of the proposition we will also need the following statement:

(d) Let (M, g) be a compact Riemannian manifold such that $inj(M, g) \geq \epsilon$ and $\gamma_1, \gamma_2 \in \Omega M_g$ be two closed curves such that $\delta = \max_{t\in[0,1]} \mathrm{dist}_g(\gamma_1(t), \gamma_2(t)) < \epsilon/2$. Then there exists a homotopy between γ_1 and γ_2 which passes only through curves of length not exceeding $\max\{length(\gamma_1), length(\gamma_2)\}(1 + 8\delta/(\epsilon/2-\delta)) + 8\delta$.

Indeed, this homotopy can be constructed as follows. Consider points $t_1 = 0, \ldots, t_k < 1$, $t_{k+1} = 1 \in S^1 = [0,1]/\{0,1\}$ such that for any $j \in \{1,\ldots,k\}$ the length of the segment of γ_1 between $\gamma_1(t_j)$ and $\gamma_1(t_{j+1})$ is equal to $length(\gamma_1)/(2([length(\gamma_1)/(\epsilon/2-\delta)]+1))$. Define a curve γ_3 which for every $j \in \{1,\ldots,k\}$ goes along γ_1 between $\gamma_1(t_j)$ and $\gamma_1(t_{j+1})$, then goes along the shortest geodesic in (M, g) from $\gamma_1(t_{j+1})$ to $\gamma_2(t_{j+1})$ and then returns to $\gamma_1(t_{j+1})$ along the same geodesic. The parametrization of

γ_3 is not important. It is clear that γ_1 and γ_3 can be connected by a homotopy passing only through curves of length not exceeding the length of γ_3. Now note that for every j the segment σ_j of γ_3 passing from $\gamma_2(t_j)$ to $\gamma_1(t_j)$, then to $\gamma_1(t_{j+1})$ and then to $\gamma_2(t_{j+1})$ can be connected with the shortest geodesic τ_j connecting $\gamma_2(t_j)$ and $\gamma_2(t_{j+1})$ by a homotopy fixing the endpoints $\gamma_2(t_j)$ and $\gamma_2(t_{j+1})$ of both these curves and passing only through curves of length not exceeding $length(\sigma_j)$. (Indeed, the assumptions made in the formulation of the statement (d) and the triangle inequality imply that $\sigma_j \bigcup \tau_j$ lies in a geodesic ball centered at $\gamma_2(t_j)$ of radius less than the injectivity radius of (M, g).) Thus, γ_3 can be connected with the curve γ_4, defined as $\bigcup_{j=1}^{k} \tau_j$ (with any parametrization), by a homotopy passing only through curves of length not exceeding $length(\gamma_3)$. Similarly, γ_4 and γ_2 can be connected by a homotopy passing only through curves of length not exceeding $length(\gamma_2)$. Combining the homotopies between γ_1 and γ_3, γ_3 and γ_4, γ_4 and γ_2 we get a homotopy between γ_1 and γ_2 which passes only through curves of length not exceeding $\max\{length(\gamma_3), length(\gamma_2)\}$. But

$$length(\gamma_3) \le (k+1)\big(4\delta + length(\gamma_1)/(k+1)\big) \ ,$$

which implies the statement (d).

Now we are ready to prove Lemma 2. Note that any closed curve γ in (M, h_2) contained in a ball of radius $< \epsilon$ in h_2 can be contracted via closed curves of length not exceeding the length of γ. Thus, it is sufficient to majorize $fl(\gamma)/length(\gamma)$ only for closed curves of length $> \epsilon$ in (M, h_2). The statement (c) (above) implies that the length of $F_{h_2,h_1}(\gamma)$ does not exceed $L = length(\gamma)\big(2 + \frac{2\epsilon}{\epsilon - 8d_{GH}(h_1,h_2)}\big) > length(\gamma)$. (We used here also the inequality $\epsilon > 8d_{GH}(h_1, h_2)$.) $F_{h_2,h_1}(\gamma)$ can be contracted in (M, h_1) by a homotopy $H : [0,1] \to \Omega M_{h_1}$, $H(0) = F_{h_2,h_1}(\gamma)$ passing through curves of the length not exceeding $Fl(h_1)L$ to a constant loop. Choose $s_1 = 0, s_2, \ldots, s_r, s_{r+1} = 1$ such that for any $j \in \{1, \ldots, r\}$, $t \in [0,1]$ the distance in (M, h_1) between $H(s_j)(t)$ and $H(s_{j+1})(t)$ does not exceed $\epsilon/8 - d_{GH}(h_1, h_2)$. Now the statement (c) implies that all curves $F_{h_1,h_2}(H(s_j))$ have length not exceeding $L_1 = Fl(h_1)length(\gamma)\big(2 + 2\frac{\epsilon}{\epsilon - 8d_{GH}(h_1,h_2)}\big)^2$. Note, that $F_{h_1,h_2}(H(1))$ is a constant loop. Further, the statement (b) implies that for any $j \in \{1, \ldots, r\}$, $t \in [0,1]$

$$\mathrm{dist}_{h_2}\big(F_{h_1,h_2}(H(s_j))(t), F_{h_1,h_2}(H(s_{j+1}))(t)\big) < \epsilon/2 - \big(\epsilon/8 - d_{GH}(h_1, h_2)\big) \ .$$

Also, the statement (a) implies that for any $t \in [0,1]$

$$\mathrm{dist}_{h_2}\big(\gamma(t), F_{h_2,h_1}(F_{h_1,h_2}(\gamma))(t)\big) < \epsilon/2 - \big(\epsilon/4 - 2d_{GH}(h_1, h_2)\big) \ .$$

Now the statement (d) implies the existence of homotopies connecting γ and $F_{h_2,h_1}(F_{h_1,h_2}(\gamma)) = F_{h_2,h_1}(H(s_1))$, $F_{h_2,h_1}(H(s_1))$ and $F_{h_2,h_1}(H(s_2)),\ldots,$ $F_{h_2,h_1}(H(s_r))$ and $F_{h_2,h_1}(H(1))$ by homotopies passing through curves of length not exceeding $\left(2+\frac{32\epsilon}{\epsilon-8d_{GH}(h_1,h_2)}\right)4\left(1+\frac{\epsilon}{\epsilon-8d_{GH}(h_1,h_2)}\right)^2 Fl(h_1)length(\gamma)$. Combining these homotopies we obtain a homotopy between γ and a constant loop which passes only through loops of length (in the h_2-metric) not exceeding $128(1+\epsilon/(\epsilon-8d_{GH}(h_1,h_2)))^3 Fl(h_1)length(\gamma)$. □

3. Proof of Lemma 6

In the literature there are many versions of the notion of Turing machines. It makes virtually no difference for our purposes which of these versions to use. Still, for the sake of clarity let us agree that all Turing machines used in this paper are "classical" one-tape Turing machines, and the alphabet contains just two symbols: 1 and "Blank". Also, let us agree that a machine must start its work scanning the leftmost symbol of the input and, if it halts, it must halt scanning the leftmost symbol of the output.

The "busy beaver" function $b:\mathbb{N}\to\mathbb{N}$, introduced by T. Rado, is defined as follows. Let $T(n)$ be the set of all Turing machines with not more than n states which eventually halt when they start to work with the blank tape. Obviously, $T(n)$ is a finite set. For every $\tau\in T(n)$ define its *productivity* $u(\tau)$ as follows. If τ halts after the computation that begins with the blank tape scanning the leftmost an unbroken string of 1's its productivity is said to be the length of this string. Otherwise, it is zero. If τ is a Turing machine with not more than n states but $\tau\notin T(n)$, then $u(\tau)$, by definition, is zero. Define $b(n)$ as $\max_{\tau\in T(n)} u(\tau)$. Obviously, $b(n)$ is an increasing function. Rado proved that b cannot be majorized by any Turing computable function (cf. [E] or [BooJ]). Also, it is easy to see that there exists an algorithm computing the productivity of any given Turing machine τ which halts after the computation that begins with the blank tape. The existence of this algorithm and the mentioned result of T. Rado immediately imply the algorithmic unsolvability of the halting problem for Turing machines (cf. [BooJ]). To prove Lemma 6 we need the following version of Rado's theorem which is a very slight modification of Theorem 2.2 in [E]. For the sake of completeness we give a detailed proof of this (known) version of Rado's theorem:

Lemma 8. *For any Turing computable function α there exists N_0 such that for any $n\geq N_0$*

$$b(n)>\alpha(n)\ .$$

Proof. We are going to prove Lemma 8 by contradiction. Assume that the lemma is false. Without any loss of generality we can assume that there exists an *increasing* Turing computable function α and a strictly increasing infinite sequence of positive integers $\{n_i\}$ such that $\alpha(n_i) \geq b(n_i)$. (Indeed, if $\alpha(n)$ is not increasing we can replace it by $\alpha^*(n) = \max_{1\leq k\leq n} \alpha(k)$.) Also, without any loss of generality we can assume that $\alpha(0) = 0$. Let t_i denote the minimal t such that $b(t) \geq n_i$. Hence, $b(t_i - 1) < n_i$. Note that $\alpha(b(n))$ does not exceed $b(n+const_\alpha)$, where $const_\alpha$ denotes the minimal number of states of a Turing machine, computing α. Indeed, for any Turing machine $\tau \in T(n)$ we can consider the following Turing machine τ^* with $n + const_\alpha$ states: First, τ^* makes the same computations as τ, starting with the blank tape. When τ halts, τ^* works as α on the string of 1's to the right of the head of τ (representing $u(\tau)$, if $u(\tau) \neq 0$). Hence,

$$b(t_i + const_\alpha) \geq \alpha\big(b(t_i)\big) \geq \alpha(n_i) \geq b(n_i) \ ,$$

whence

$$t_i + const_\alpha \geq n_i \ ,$$

and

$$t_i - 1 \geq n_i - const_\alpha - 1 \ .$$

But $b(t_i - 1) < n_i$. Hence

$$n_i > b(n_i - const_\alpha - 1) \ .$$

But for any n and for an appropriate constant c

$$b(n + c) \geq 2n \ .$$

(Indeed, we can consider only Turing machines which use first n states to write n 1's, then use c states to double this number of 1's. According to [BooJ] one can take $c = 11$ here.) Thus, for all sufficiently large i and for some $C(\alpha)$ not depending on i

$$n_i > 2n_i - C(\alpha) \ ,$$

which yields the desired contradiction. □

Now we will prove Lemma 6 by contradiction. We are going to assume that Lemma 6 is false, and to deduce the existence of a Turing computable function coinciding with the "busy beaver" function $b(n)$ on the same infinite sequence $\{n_i\}_{i=1}^{\infty}$ for which there exist a Turing machine T solving the halting with the zero input problem for all Turing machines with $\leq n_i$ states. The Turing machine S computing the required function can be constructed as follows. We apply T. Then for every Turing machine τ on the list produced by T, S computes the productivity $u(\tau)$ of τ. Then S finds the maximal of the obtained productivities. If $n = n_i$, then this number will be equal to $b(n)$. □

4. Proof of Lemma 7

We are going to describe the following algorithm: Its input data are a non-singular compact algebraic hypersurface $Z(p) \subset \mathbb{R}^{n+1}$, presented as the zero set of a polynomial p with rational coefficients given by the vector of its coefficients, and an integer number N. (This number N supposedly is an upper bound for $F_{S^n}(inj(Z(p))/\operatorname{vol}^{\frac{1}{n}}(Z(p)))$.) It is given, that $Z(p)$ is a homology sphere (and, thus, according to Theorem A in ch. 9 of [Mi] is diffeomorphic to S^n if and only if it is simply connected). The algorithm must find out whether or not the fundamental group of $Z(p)$ is trivial. If N is, indeed, an upper bound for $F_{S^n}(inj(Z(p))/\operatorname{vol}^{\frac{1}{n}}(Z(p)))$, then the answer provided by the algorithm must be correct. If N is not an upper bound for $F_{S^n}(inj(Z(p))/\operatorname{vol}^{\frac{1}{n}}(Z(p)))$, and the algorithm decides that $\pi_1(Z(p))$ is trivial, then the answer must be correct, but if it decides that $\pi_1(Z(p))$ is non-trivial, then this answer is allowed to be incorrect.

On the first stage the algorithm finds a triangulation of $Z(p)$ (cf. [Co], [BoCoRo] for an algorithm triangulating any semialgebraic set. But, of course, our case of a compact non-singular algebraic hypersurface is much simpler than the general case. So, alternatively, we can use the procedure for triangulation described in [W].) Then the algorithm writes down a finite presentation for the fundamental group of the considered hypersurface. In particular, it obtains a finite list of generators $\gamma_1, \ldots, \gamma_l$ of $\pi_1(Z(p))$. Then we want to check which of these loops γ_i can be contracted to a constant curve by a homotopy passing only via closed curves of length not exceeding $N\ length(\gamma_i)$. If all these loops γ_i can be contracted to a constant curve by homotopies passing through closed curves of length not exceeding $N\ length(\gamma_i)$, then $Z(p)$ is simply connected. On the other hand, if $Z(p)$ is simply connected, then it is diffeomorphic to S^n. Now the definition of F_{S^n} and the scale invariance of the ratio $inj(Z(p))/\operatorname{vol}^{\frac{1}{n}}(Z(p))$ imply that every closed curve γ on $Z(p)$ can be contracted to a constant curve by a homotopy passing only through closed curves of length $\leq length(\gamma)F_{S^n}(inj(Z(p))/\operatorname{vol}^{\frac{1}{n}}(Z(p)))$. Therefore, it is not difficult to see that an algorithm which will declare $\pi_1(Z(p))$ trivial if all loops γ_i can be contracted to a constant curve by homotopies passing only through closed curves of length not exceeding $Nlength(\gamma_i)$ and only if $\pi_1(Z(p))$ is, indeed, trivial has all the required properties. Note also, that we can use in this algorithm any number $L_i \geq Nlength(\gamma_i)$ instead of $Nlength(\gamma_i)$.

To check whether or not a given loop γ on $Z(p)$ can be contracted to a constant curve by a homotopy passing only through closed curves of length not exceeding $Nlength(\gamma)$ we are going first to construct a finite $inj(Z(p))/100$-net NET in a neighborhood of the subspace $\Omega_0 \subset \Omega(Z(p))$

formed by all continuous maps λ of S^1 into $Z(p)$ of length not exceeding $10\ length(\gamma)N$ and such that $|d\lambda(t)/dt| = 1$ identically. Such a net can be presented as a finite set of finite objects and effectively constructed as follows. We can look for this net among piecewise geodesic closed curves, representable by finite sequences of points $x_1, \dots, x_{l-1}, x_l \in Z(p)$ with algebraic coordinates, and such that (i) $x_l = x_1$; (ii) the distance between any two consecutive points x_j and x_{j+1} in the inner metric of $Z(p)$ does not exceed $inj(Z(p))/200$. To any such set of points we assign the closed curve $\gamma(x_1, \dots, x_l) \in \Omega_0$ which starts and ends at $x_1 = x_l$ and passes consecutively through all points x_j via the shortest geodesics in $Z(p)$ connecting them. All points x_j will be chosen from a $inj(Z(p))^2/(10^7 N\ length(\gamma))$-net ν_p in $Z(p)$ made of points with algebraic coordinates. The number l of points from ν_p used to construct a closed curve from NET should not exceed $10N\ length(\gamma)/\min_{x,y\in\nu_p;x\neq y} \mathrm{dist}_{\mathbb{R}^{n+1}}(x,y)$.

Note that, as a corollary from the Tarski-Seidenberg theorem (cf. [Co], [BoCoRo]), every open set in a non-singular real algebraic hypersurface, defined over the field of algebraic numbers, contains a point with algebraic coordinates. This observation implies the existence of a net ν_p with the required properties in $Z(p)$. It is not difficult to develop a formal algorithm, constructing such a net. For example, one can proceed as follows: The injectivity radius $r(Z(p))$ of the normal exponential map for $Z(p)$ is a semialgebraic function of coefficients of p and, hence, can be evaluated to any accuracy. (Moreover, since coefficients of p are algebraic, $r(Z(p))$ can be computed *exactly*, but we do not need this.) In particular, one can effectively compute $r(Z(p))/2$ to an accuracy $r(Z(p))/20$. The points of ν_p will correspond to points from a sufficiently dense net ν_p^* in the $r(Z(p))/2$-neighborhood $N_{r(Z(p))/2}(Z(p))$ of $Z(p)$ formed by points with rational coordinates. Every point $y^* \in \nu_p^*$ codes the closest to y^* point y of $Z(p)$ (which is unique). The Tarski-Seidenberg theorem implies that the coordinates of y are algebraic and can be effectively found if the coordinates of y^* are known. To complete the construction of ν_p we need only to decide how dense the net ν_p^* must be. In order to do that we need to be able to compute for any given p an upper bound for the ratio $r = \sup_{x,y\in Z(p);\mathrm{dist}_{\mathbb{R}^{n+1}}(x,y)\leq r(Z(p))} \frac{\mathrm{dist}_{Z(p)}(x,y)}{\mathrm{dist}_{\mathbb{R}^{n+1}}(x,y)}$ which is *not* a semialgebraic function of the coefficients of p. If $\mathrm{dist}_{\mathbb{R}^{n+1}}(x,y) \leq r(Z(p))$, then we can consider the direct line segment $[x,y]$ in $\mathbb{R}^{n+1}$ connecting x and y. Consider the map $\mu : [x,y] \to Z(p)$ assigning to every point $z \in [x,y]$ the closest to z point of $Z(p)$. It is not difficult to see, that $\|D\mu(z)\| \leq 2$. Since μ maps $[x,y]$ onto a curve in $Z(p)$ connecting x and y, we see that the ratio $r \leq 2$. Thus, ν_p^* may be chosen $\min\{inj(Z(p))^2/(10^8 N\ length(\gamma)), r(Z(p))/2\}$-dense. Further, in the construction described above one can use instead of the actual value of $inj(Z(p))$ a computed positive lower bound for $inj(Z(p))$ obtained, for example, using Cheeger's inequality, as explained in the proof of Theorem 4.

After the net NET is constructed, we form a graph GR such that its set of vertices is in a one-to-one correspondence with elements of NET. Two vertices are connected by an edge if the corresponding paths are sufficiently close in the Hausdorff metric. The loop γ is declared null-homotopic if a sufficiently close to γ closed curve $\gamma_0 \in NET$ belongs to the same (connected) component of GR as a vertex corresponding to a constant path from NET (that is, to a path corresponding to a finite sequence of points $x_1, \ldots x_l$ such that $x_1 = x_2 = \ldots = x_l$).

The statement (d) in Section 2 implies that, indeed, the above-described algorithm will declare γ null-homotopic if γ can be contracted to a point by a homotopy passing only via closed curves of length not exceeding $length(\gamma)N$.

It is not difficult to see that this algorithm can be implemented by a computer program (say, in FORTRAN or PASCAL) using only the integer data type and, hence, by virtue of a rigorous and provable version of Church's thesis can be implemented by a Turing machine (cf. [BooJ, ch. 6]).

5. Other Disconnectedness Results and Applications to Variational Problems on the Space of Riemannian Structures

In this section we discuss some disconnectedness results similar to Theorem 1, which can be proven using the technique of this paper.

A. There exists an integer number k such that Theorem 1 is true for any *four-dimensional* compact manifold M representable as the connected sum of an arbitrary compact four-dimensional manifold M and k copies of $S^2 \times S^2$. This result can be proved using the same ideas as in the proof of Theorem 1, but instead of S. Novikov's theorem used here one must use Markov's theorem (cf. [BHP]). This theorem establishes that for a certian k there is no algorithm deciding whether or not a given compact four dimensional manifold X is diffeomorphic to the connected sum of k copies of $S^2 \times S^2$. It seems plausible that Theorem 1 is valid for any compact four-dimensional manifold M.

What happens if $dim(M) = 2$ or 3? It is not difficult to see that if either $dim(M) = 2$ or $M = S^3$, then the conclusion of Theorem 1 is false. Namely, there exists an effectively minorizable function θ_M such that for any positive ϵ any two Riemannian structures $\mu, \nu \in Riem_\epsilon(M)$ can be connected by a finite sequence of "jumps" of length $\leq \theta_M(\epsilon)/9$ in $Riem_{\theta_M(\epsilon)}(M)$. (Indeed, notice that the algorithmic recognizability of these manifolds implies that for any such manifold there exists an algorithm which constructs for any given rational $\epsilon > 0$ a finite $\epsilon/9$-net in $Riem_\epsilon(M)$. (A proof of the algorithmic recognizability of S^3 can be found in [T].) Any two elements μ_i, μ_j of this net can be connected by a path continuous in C^2-topology in the space of Riemannian structures of volume one on M. Hence, one can effectively find a positive rational $\epsilon_{ij} \leq \epsilon$ and a finite sequence of "jumps" of length $\leq \frac{1}{9}\epsilon_{ij}$ in $Riem_{\epsilon_{ij}}(M)$ between μ_i and μ_j using a trial-and-error algorithm. Now consider $\min_{i,j} \epsilon_{ij}$ as a function of ϵ. The computability of this function implies the statement above.) However, I do not know whether or not sets $Riem_\epsilon(S^2)$ (or $Riem_\epsilon(M)$ for any other compact manifold M of dimension two or three) are connected for all sufficiently small positive ϵ.

B. The method of the proof of Theorem 1 can be used virtually without any modifications to prove the disconnectedness of many other subsets of the space of Riemannian structures on a fixed compact manifold of dimension ≥ 5 formed by imposing constraints on the curvatures, volume, diameter, injectivity or convexity or contractibility radius, etc. For example, define $\psi(n, K, v, D)$ by the formula

$$\psi(n, K, v, D) = \frac{\pi}{K^{1/2}} \min\left\{1, \frac{2vK^{n/2}}{\mathrm{vol}(S^n)\sinh(DK^{1/2})}\right\},$$

where $\mathrm{vol}(S^n)$ denotes the volume of the standard n-dimensional sphere of the unit radius. The well-known Cheeger inequality ([C]) provides the following lower bound for the injectivity radius of a Riemannian structure μ on a compact manifold: $inj(\mu) \geq \psi(n, |K(\mu)|, \mathrm{vol}(\mu), \mathrm{diam}(\mu))$, where $|K(\mu)|$ denotes the supremum of the absolute value of sectional curvature of μ.

Theorem 9. *Let M^n be a compact manifold of dimension $n \geq 5$. For any positive x, y, z denote by $Riem_{x,y,z}(M^n)$ the set of Riemannian structures μ on M^n such that $|K(\mu)| \leq x$, $\mathrm{vol}(\mu) \geq \frac{1}{y}$, $\mathrm{diam}(\mu) \leq z$. Then for all sufficiently large x, y and z $Riem_{x,y,z}(M^n)$ will be disconnected. Moreover, for any increasing function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that its restriction on $\mathbb{N} \subset \mathbb{R}^+$ can be majorized by a Turing computable function there exist two Riemannian structures μ_1, μ_2 on M^n with the following property. Let $K = \max_{i\in\{1,2\}} |K(\mu_i)|$, $D = \max_{i\in\{1,2\}} \mathrm{diam}(\mu_i)$, $v = \min_{i\in\{1,2\}} \mathrm{vol}(\mu_i)$. Then, there is no finite sequence of Riemannian structures on M^n, $\nu_1, \nu_2, \ldots, \nu_p$ with the following properties:*

(a) *$\nu_1 = \mu_1$, $\nu_p = \mu_2$;*
(b) *For any j $\nu_j \in Riem_{\theta(K),1/\theta(1/v),\theta(D)}(M^n)$;*

(c) *For any* j

$$d_{GH}(\nu_j, \nu_{j+1}) \le \psi\big(n, \theta(K), 1/\theta(1/v), \theta(D)\big)/9 \ .$$

The function $\theta(x)$ in this theorem can be equal, for example, to x^λ for any positive λ or to, say, $\exp(\exp(\exp(x)))$. The proof of this theorem is quite similar to the proof of Theorem 1.

C. Similarly, one can prove the disconnectedness of subsets of other moduli spaces arising in real differential geometry and associated with a fixed compact manifold of dimension ≥ 5. For example, the following theorem is a slightly strengthened version of Theorem 4.1 in [N3]. It can be proven similarly to the proof of Theorem 4.1 in [N3] but using Lemma 6 of the present paper instead of the algorithmic unsolvability of the halting problem for Turing machines used in [N3]:

Theorem 10. *Denote for any compact smooth hypersurface* $\Sigma^n \subset \mathbb{R}^{n+1}$ *the injectivity radius of the normal exponential map of* Σ^n *by* $r(\Sigma^n)$. *(Informally,* $r(\Sigma^n)$ *is the maximal radius of an open non-selfintersecting tubular neighborhood around* Σ^n *in* $\mathbb{R}^{n+1}$.*) Define* $\kappa(\Sigma^n)$ *as* $\mathrm{vol}^{\frac{1}{n}}(\Sigma^n)/r(\Sigma^n)$. *Then for any* $n \ge 5$ *for all sufficiently large* x *the set of all hypersurfaces* $\Sigma^n \subset \mathbb{R}^{n+1}$, *diffeomorphic to* S^n, *and satisfying the inequality* $\kappa(\Sigma^n) \le x$ *is disconnected.*

D. The disconnectedness results like Theorems 1,9 and 10 can be applied to prove the existence of infinitely many solutions for some geometric variational problems. For example, consider the functional κ, introduced in the text of Theorem 10, on the (connected) space of $C^{1,1}$-smooth hypersurfaces in $\mathbb{R}^{n+1}$, diffeomorphic to S^n. According to Theorem 5.1 in [N3] for any $n \ge 5$ the set of values of the functional κ at its local minima is infinite and unbounded.

The technique of the present paper and the Gromov-Cheeger compactness theorem easily imply the following result. Let M be a compact manifold and $Al_1(M)$ denote the space of C^1-smooth Alexandrov spaces of curvature $-1 \le K \le 1$, C^1-diffeomorphic to M (see [BuGrPe], [Ni] for the definition of Alexandrov spaces). ($Al_1(M)$ is endowed with the Gromov-Hausdorff metric.) Denote by $Al_1^{smooth}(M)$ the closure in $Al_1(M)$ of its subspace formed by all C^∞-smooth Riemannian structures. The Gromov-Cheeger compactness theorem implies that all elements of $Al_1^{smooth}(M)$ are $C^{1,\alpha}$-smooth Riemannian structures on M for any $\alpha \in (0,1)$. A result of I. Nikolaev ([Ni]) is that $Al_1^{smooth}(M)$ coincides with $Al_1(M)$. (See also [Pl] for a related result.)

Theorem 11. *Let* M^n *be a compact smooth manifold of dimension* $n \ge 5$ *satisfying at least one of the following three conditions:*

(i) *The Euler characteristic of* M^n *is not equal to zero;*

(ii) *n is divisible by four and at least one of the Pontryagin numbers of M^n is not equal to zero;*

(iii) *The simplicial volume of M^n is not equal to zero.*

Then the functional diam : $Al_1^{smooth}(M^n) \to \mathbb{R}^+$ *has infinitely many local minima.*

REMARK 1: This result can be strengthened in many ways. For example, one can prove that the set of values of diam at its local minima is unbounded. The details will appear in a sequel of this paper.

REMARK 2: The simplicial volume of a compact manifold M^n is defined as the infimum of $\Sigma_i|a_i|$ over the set of all singular chains with real coefficients $\Sigma_i a_i\sigma_i$, ($a_i \in \mathbb{R}$), representing the fundamental class of M^n in $H_n(M^n, \mathbb{R})$ (cf. [Gr1]). It is known that if M^n admits a Riemannian metric of negative sectional curvature then the simplicial volume of M^n is not equal to zero ([Gr1, Corollary (b) in section 0.4]). Thus, such manifolds satisfy the condition (iii) of Theorem 11.

Sketch of the proof. The Gauss-Bonnet formula, the Chern-Weyl theorem and Corollary (A) in section 0.5 of [Gr1] imply that if M^n satisfies any of the conditions (i)-(iii) then for any Riemannian metric on M^n such that $|K| \leq 1$ the volume of M^n is not less than a certain positive constant $const(M^n)$ depending only on M^n (cf. [Gr1, p. 6 and 12]). Therefore the Cheeger inequality mentioned implies that for Riemannian metrics on such manifolds satisfying the condition $|K| \leq 1$ the injectivity radius can be minorized by a (positive) computable function of diameter. (More precisely, there exists a positive integer constant $Const(M^n)$ such that $inj^{-1} \leq Const(M^n)3^{[\mathrm{diam}]}$.) Hence the Bishop inequality and Lemma 3 imply that there exists a Turing computable function ρ such that for any two Riemannian structures $\mu_1, \mu_2 \in Al_1^{smooth}(M^n)$ which belong to the same connected component of the sublevel set $\mathrm{diam}^{-1}([0, D])$ for some D, $Fl(\mu_1) \leq Fl(\mu_2)\rho([D])$. Now, as in the proof of Theorem 1, the obvious appropriate version of Theorem 4 (which can be proven exactly as Theorem 4) implies that for all sufficiently large L the set $\mathrm{diam}^{-1}([0, L]) \subset Al_1^{smooth}(M^n)$ is disconnected. Fix one such value of L. Denote this value by L_0. Clearly a local minimum of diam in a connected component of $\mathrm{diam}^{-1}([0, L_0])$ is also a local minimum of diam on $Al_1^{smooth}(M^n)$. Thus, diam will have at least one non-trivial local minimum. If $\mathrm{diam}^{-1}([0, L_0])$ has infinitely many connected components, then diam has infinitely many local minima. Otherwise, it is clear that there exists $L_1 > L_0$ such that $\mathrm{diam}^{-1}([0, L_0])$ is comprised in one connected component of $\mathrm{diam}^{-1}([0, L_1])$. Now either there exists $L_2 \geq L_1$ such that $\mathrm{diam}^{-1}([0, L_2])$ has infinitely many connected components, and hence diam has infinitely many local minima, or we can construct an infinite increasing sequence L_j such that that for any j $\mathrm{diam}^{-1}([0,L_j])$ is disconnected and $\mathrm{diam}^{-1}([0,L_j])$ is comprised in a certain connected component of $\mathrm{diam}^{-1}([0,L_{j+1}])$, which we will call trivial. The local minima of diam in all non-trivial connected components of all sets $\mathrm{diam}^{-1}([0,L_j])$ are also distinct local minima of diam on the whole space $Al_1^{smooth}(M^n)$. $\square$

E. Let $\theta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an effectively minorizable function (for example, $\theta(x) = x$) and δ be a positive number. We will say that two Riemannian structures $\mu_1, \mu_2 \in Riem_\epsilon(M)$ are (θ,δ)-equivalent if there exists a finite sequence $h_1 = \mu_1,\ h_2, \dots, h_{l-1}, h_l = \mu_2$ of Riemannian structures from $Riem_{\theta(\epsilon)}(M)$ such that for any j $d_{GH}(h_1,h_2) \leq \delta$. Denote by $m_{\theta,\delta,\epsilon}(M)$ the number of equivalence classes of Riemannian structures from $Riem_\epsilon(M)$ with respect to this relation. Theorem 1 can be reformulated as the statement that for any compact manifold M of dimension at least five, for any Turing computable function θ for all sufficiently small $\epsilon > 0$

$$m_{\theta,\theta(\epsilon)/9,\epsilon}(M) > 1 \ .$$

In a sequel of this paper we are going to use the technique of [N4] based on the notion of time-bounded Kolmogorov complexity to prove exponentially growing with $\frac{1}{\epsilon^n}$ lower bounds for $m_{\theta,\theta(\epsilon)/9,\epsilon}(M)$ and, as a corollary, for the number of connected components of $Riem_\epsilon(M)$. (As before, M is an arbitrary compact manifold of dimension $n \geq 5$.) One can also derive similar quantitative versions of results described in sections 5.A-5.D above.

F. Prof. M. Gromov informed me that he independently came to the conclusion that the set, denoted in the present paper $Riem_\epsilon(M)$, is disconnected for any compact M of dimension ≥ 5 and for all sufficiently small ϵ. Moreover, he drew my attention to the fact that one can explicitly construct a Riemannian structure μ in a non-trivial connected component of $Riem_\epsilon(M)$ using the following informal idea (which is, in fact, very closely related to the main idea of the proof of Theorem 1 in this paper): For simplicity, let M be the sphere S^n, $n \geq 5$. One looks for a short finite presentation P of the *trivial* group such that: 1) Any demonstration of the fact that P is trivial using an appropriate set of elementary moves is very long; 2) There exists a Riemannian structure μ on S^n of volume one such that the injectivity radius of μ is not very small and with the property that P is an evident finite presentation of the fundamental group of the resulting Riemannian manifold. (That is, one fixes a certain algorithm which produces a finite presentation or a list of finite presentations of the fundamental group of a given Riemannian manifold. The finite presentation P must be on the list obtained as the result of the application of this algorithm to

(S^n, μ).) It is clear, that it is not difficult to formalize this informal idea. (Here I would like to draw the attention of the reader to the following curious fact hampering the construction of such P and μ: Let an algorithm A constructing for a given triangulation T of a PL-manifold M a finite list $A(T)$ of finite presentations of the fundamental group of M be fixed. Then there is *no* Turing computable function β and algorithm B constructing for *any* given finite presentation P_{triv} of the trivial group in time not exceeding $\beta(length(P_{triv}))$ a triangulation $T(P_{triv})$ of the standard n-dimensional sphere such that P_{triv} is on the list $A(T(P_{triv}))$. (This fact easily follows from the observation that the proof of the C. Gordon theorem on the algorithmic unsolvability of the problem "Is the second homology group of a given finitely presented group trivial?" given in [M] yields the stronger statement that finite presentations of the trivial group are algorithmically undistinguishable from finite presentations of groups with non-zero second homology group.) To overcome the difficulty caused by this fact one can use, for example, constructions described in section 2 of [N4].)

Acknowledgments. I would like to thank Prof. M. Gromov for his very helpful comments concerning this work. As it was already mentioned, Prof. Gromov independently noticed that for any sufficiently small ϵ the space $Riem_\epsilon(M)$ is disconnected. I would like to thank the referee of this paper for remarks which helped to improve the exposition.

References

[ABBr] F. Acquistapace, R. Benedetti, F. Broglia, Effectiveness-non-effectiveness in semi-algebraic and PL geometry, Inv. Math. 102:1 (1990), 141-156.

[AnC] M. Anderson, J. Cheeger, C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded from below, J. Diff. Geom. 35 (1992), 265-281.

[BoCoRo] J. Bochnak, M. Coste, M.-F. Roy, Geometrie Algebrique Reelle, Springer, 1987.

[BooJ] G. Boolos, R. Jeffrey, Computability and Logic", Third Edition, Cambridge University Press, 1989.

[BHP] W. Boone, W. Haken, V. Poenaru, On recursively unsolvable problems in topology and their classification, in "Contributions to Mathematical Logic" (H. Arnold Schmidt, K. Schutte, H.-J. Thiele, eds.), North-Holland, 1968.

[BuGrPe] Yu. Burago, M. Gromov, G. Perelman, A.D. Alexandrov spaces with curvature bounded below, Russ. Math. Surv. 47 (1992), 1-58.

[C] J. Cheeger, Finiteness theorems for Riemannian manifolds, Amer. J. Math. 92 (1970), 61-74.

[CGr] J. Cheeger, M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded I, II", J. Diff. Geom. 23 (1986), 309-346 and J. Diff. Geom. 32 (1990), 269-298.

[Co] M. Coste, Ensembles semi-algebriques, in "Geometrie algebrique reelle et formes quadratiques", Journees S.M.F. Universite de Rennes, (J.-L. Colliot-Thelene, M. Coste, L. Mahe, M.-F. Roy, eds.) Springer Lecture Notes in Math. 959 (1982), 109-138.

[Cr] C. Croke, Some isoperimetric inequalities and eigenvalue estimates, Ann. Scient. Ec. Norm. Sup. 13 (1980), 419-435.

[E] H.B. Enderton, Elements of recursion theory, in "Handbook of Mathematical Logic", (J. Barwise, ed.), North-Holland, 1977, 527-566.

[Fu] K. Fukaya, Hausdorff convergence of Riemannian manifolds and its applications, in "Recent topics in Differential and Analytic Geometry" (T. Ochiai, ed.) Kinokuniya, 1990.

[G] R. Greene, Some concepts and methods in Riemannian geometry, in "Differential Geometry: Riemannian Geometry" (R. Greene, S.T. Yau, eds.), Proceedings of AMS Symposia in Pure Math. 54:3 (1993), 1-22.

[Gr1] M. Gromov, Volume and bounded cohomology, Publication IHES 56 (1982), 5-99.

[Gr2] M. Gromov, Asymptotic invariants of infinite groups, in "Geometric Group Theory" vol. 2 (G. Niblo, M. Roller,eds.) Cambridge University Press, 1993.

[GrLP] M. Gromov, J. Lafontaine, P. Pansu, Structures metriques pour les varietes Riemannienes, CEDIC/Fernand Nathan, Paris, 1981.

[GrLa] M. Gromov, H.B. Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Publ. Math. IHES 58 (1983), 295-408.

[Gro] K. Grove, Metric differential geometry, in "Differential Geometry", (V. Hansen, ed.), Springer Lecture Notes in Math. 1263 (1987), 171-227.

[GP] K. Grove, P. Petersen V, Bounding homotopy type by geometry, Ann. Math. 128 (1988), 195-206.

[GPW] K. Grove, P. Petersen V, J. Wu, Geometric finiteness theorems via controlled topology, Inv. Math. 99 (1990), 205-213.

[K] M. Kervaire, Smooth homology spheres and their fundamental groups, Trans. Amer. Math. Soc. 144 (1969), 67-72.

[KrS] M. Kreck, S. Stolz, Non-connected moduli spaces of positive sectional curvature metrics, J. of the Amer. Math. Soc. 6:4 (1993), 825-850.

[M] C.F. Miller, Decision problems for groups - survey and reflections, in "Combinatorial Group Theory" (G. Baumslag, C.F. Miller, eds.), Springer, 1989, 1-59.

[Mi] J. Milnor, Lectures on the h-cobordism Theorem, Princeton Univ. Press, 1965.

[N1] A. Nabutovsky, Non-recursive functions in real algebraic geometry, Bull. Amer. Math. Soc. 20 (1989), 61-65.

[N2] A. Nabutovsky, Einstein structures: existence versus uniqueness, Geometric And Functional Analysis 5 (1995), 76-91.

[N3] A. Nabutovsky, Non-recursive functions, knots 'with thick ropes' and self-clenching 'thick' hypersurfaces, Comm. on Pure and Appl. Math. 48 (1995), 381-428.

[N4] A. Nabutovsky, Geometry of the space of triangulations of a compact manifold, Comm. Math. Phys., to appear.

[Ni] I. Nikolaev, Bounded curvature closure of the set of compact Riemannian manifolds, Bull. AMS 24 (1992), 171-177.

[P] P. Pansu, Effondrement des varietes riemanniennes (d'apres J. Cheeger et M. Gromov), Sem. Bourbaki 1983/1984, Asterisque 121-122 (1985), 63-82.

[Pet] S. Peters, Convergence of Riemannian manifolds, Comp. Math. 62 (1987), 3-16.

[Pete] P. PETERSEN V, Gromov-Hausdorff convergence of metric spaces, in "Differential Geometry: Riemannian Geometry" (R. Greene, S.T. Yau, ed.), Proceedings of AMS Symposia in Pure Math. 54:3 (1993), 489-504.

[Pl] C. PLAUT, Metric curvature, convergence and topological finiteness, Duke Math. J. 66:1 (1992), 43-57.

[R] J.J. ROTMAN, An Introduction to the Theory of Groups, Allyn and Bacon, Boston, 1984.

[T] A. THOMPSON, Thin position and the recognition problem for S^3, Math. Res. Lett. 1 (1994), 613-630.

[VKuF] I. VOLODIN, V. KUZNETZOV, A. FOMENKO, The problem of discriminating algorithmically the standard three-dimensional sphere, Russ. Math. Surveys 29:5 (1974), 71-172.

[W] H. WHITNEY, Geometric Integration Theory, Princeton Univ. Press, Princeton, 1957.

[Y] T. YAMAGUCHI, Homotopy type finiteness theorems for certain precompact families of Riemannian manifolds, Proc. Amer. Math. Soc. 102 (1988), 660-666.

Alexander Nabutovsky
Courant Institute of Math. Sci.
New York University
251 Mercer Street
New York, NY 10012, USA
e-mail: nabutvsk@cims.nyu.edu

and

Current Address:
Department of Mathematics
University of Toronto
Toronto, Ontario
Canada M5S 1A1
e-mail: alex@math.toronto.edu

Submitted: June 1995
Revised Version: October 1995