THE AREA OF A MINIMAL EMBEDDED 2-SPHERE IN A MANIFOLD DIFFEOMORPHIC TO S^3

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Abstract. It is known that each Riemannian manifold Σ^3 diffeomorphic to S^3 contains an embedded minimal submanifold diffeomorphic to S^2. The goal of this paper is to give an upper bound for the minimal area of an embedded minimal S^2 in terms of the diameter, volume and curvature of Σ^3.

1. Main result.

Our main result is the following theorem:

Theorem 1. *Let Σ^3 be a Riemannian manifold diffeomorphic to S^3. Assume that the absolute value of sectional curvature of Σ^3 does not exceed one, diam$(\Sigma^3) \leq D$ and vol$(\Sigma^3) \geq v > 0$ for some v and D.*

Then there exists a minimal embedded two-dimensional sphere in Σ^3 such that its area does not exceed $\exp\left(\exp\left(C\frac{\exp(32D)}{v^{12}}\right)\right)$, where C is an absolute constant that, in principle, can be estimated.

Remark. The assumption that the sectional curvature of Σ^3 is pinched between -1 and 1 is just a matter of convenience: It immediately follows by rescaling that in the general case the minimal area of a non-trivial minimal embedded sphere is at most $\exp(\exp(C\ \frac{\exp(32\sqrt{K}D)}{K^{18}v^{12}}))$, where K is an upper bound for the sectional curvature of Σ^3.

The starting point of the proof of the Theorem is the well-known existence result proven by F.R. Smith [S] (see also an expository paper [J2] and a new survey [CL]): For any Riemannian manifold diffeomorphic to S^3 there exists an embedded minimal two-dimensional sphere. The work of Smith was his Ph.D. thesis written under the direction of L. Simon. It relies on methods of the work of J. Pitts [P] and on results of Meeks, Simon and Yau [MSY]. Note also that J. Jost proved that each Riemannian manifold diffeomorphic to S^3 contains at least *four* distinct embedded minimal two-dimensional spheres; see [J]. From [S] it can be immediately deduced that the minimal area of the minimal embedded sphere in Σ^3 is majorized by any X such that there exists a diffeomorphism of the standard sphere $S^3 = \{x \in \mathbb{R}^4 | \|x\| = 1\}$ to Σ^3 that maps all two-dimensional spheres $S^2_t = \{x \in \mathbb{R}^4 | \|x\| = 1, x_4 = t\}$, $t \in [-1,1]$ into submanifolds of Σ^3 of area that does not exceed X. This X can obviously be majorized by $4\pi L^2$, where L is such that there exists a diffeomorphism φ from the standard sphere S^3 of radius one to Σ^3 such that $\|D\varphi(x)\| \leq L$ at any point $x \in S^3$. Furthermore, here one can also use a Lipshitz homeomorphism with the Lipschitz constant L instead of the diffeomorphism φ.

Thus, everything boils down to the following problem (of independent interest): Find an explicit upper bound for the minimal Lipschitz constant of a Lipschitz homeomorphism $S^3 \to \Sigma^3$ in terms of the geometry of Σ^3.

In order to answer the question we are going to use a theorem proven by H. Rubinstein (cf. [T]) establishing the existence of an algorithm deciding whether or not a given three-dimensional PL-manifold is PL-homeomorphic to S^3. More precisely, what we need is an effective form of H. Rubinstein's result. Such results were recently proven by S. King ([K]), and independently, by A. Mijatovic ([M]). Either of these results can be applied to our situation, but the result of Mijatovic is especially convenient for our purposes. It will be described in the next section.

2. Bistellar transormations of simplicial 3-manifolds.

In order to describe the result of Mijatovic mentioned at the end of the last section we need the notion of *bistellar transformations* (a.k.a. *Pachner moves*). Bistellar transformations are operations on simplicial complexes that transform an n-dimensional simplicial complex K into a PL-homeomorphic simplicial complex. A bistellar operation is performed as follows: Take any subcomplex S of the original complex K that is simplicially isomorphic to the union of i closed n-dimensional simplices in the boundary of the $(n+1)$-dimensional simplex σ^{n+1}, where $i \in \{1, \dots, n+1\}$. Remove S from K. Fill the resulting hole by attaching the closure of $\partial\sigma^{n+1} \setminus S$. (Observe, that S and $\partial\sigma^{n+1} \setminus S$ have the same boundary.) In dimension 3 there are 4 types of bistellar transformations (depending on the value of i). One of them is equivalent to subdividing a three-dimensional simplex into four three dimensional simplices by adding a new vertex in the center, $(i = 1)$. The second operation is the inverse of the first, $(i = 4)$. The third operation consists in replacing of two three-dimensional simplices having a comon two-dimensional face by three three-dimensional simplices such that each two of them have a common 2-face, and each of these 3 new simplices has exactly one common 2-face with each of 2 simplices that are being replaced, ($i = 2$). The fourth operation is the inverse of the third, ($i = 3$). (We refer the reader to Fig. 1 in [M] illustrating four types of bistelar operations in dimension three.) According to a theorem proven by Pachner for each pair of PL-homeomorphic finite simplicial complexes there exists a finite sequence of bistellar transformations transforming one into the other ([Pa]). Mijatovic proved that any two smooth triangulations of S^3 with $\leq N$ vertices can be transformed one into the other by at most $6000000N^2\, 2^{50000\, N^2} < 2^{100000\, N^2}$ bistellar transormations.

A conected three-dimensional simplicial complex Q can be endowed with the metric space structure by first defining a metric on each 3-dimensional simplex by identifying it with the equilateral tetrahedron with sides of length one, and then defining the distance between any two points in K as the minimal length of a path that connects this points. For brevity we will call this metric *the canonical metric* on Q. Then the theorem of Mijatovic implies the following proposition:

Proposition 1. *Let K_1, K_2 be simplicial complexes of two smooth triangulations of S^3 with $\leq N$ simplices endowed with the canonical metric. Then K_1 and K_2 are bi-Lipschitz homeomorphic by a bi-Lipschitz homeomorphism Ψ such that both Ψ and Ψ^{-1} satisfy the Lipschitz inequality with the Lipschitz constant $\leq$* $\exp(\exp(100000N^2))$.

Proof. Indeed, it is sufficient to show that if two complexes K_1 and K_2 can be transformed one into the other by exactly *one* bistellar transformation, then K_1 and K_2 endowed with the canonical metric will be homeomorphic by a homeomorphism ϕ such that ϕ and ϕ^{-1} satisfy Lipschitz inequality with the Lipschitz constant $< 10^{100}$. Note that after the canonical metrization the four types of bistellar transormations in dimension three amount to replacing one specific metric three-dimensional disc $S_1 \subset K_1$ (that consist of i equilateral tetrahedra, $i = 1, 2, 3, 4$)

in the complex by the other metric simplicial disc $S_2 \subset K_2$ (that consist of $5-i$ equilateral tetrahedra). For each of these 4 types of operations we can fix a bi-Lipschitz homeomorphism between the corresponding metric discs S_1 and S_2 such that its restriction on the boundary is the identity map. It is easy to ensure that these four Lipschitz homeomorphisms and their inverses satisfy the Lipschitz inequality with constant 10^{100}. (Here and below we do not make any attempt to optimize the constants we use.) The desired homeomorphism ϕ from K_1 to K_2 will map S_1 into S_2 using such a homeomorphism and will be constant outside of S_1. QED.

In order to relate this result about triangulations of S^3 with the situation that is of interest to us we need to triangulate the manifold into controllably small 3-dimensional simplices.

3. Triangulations of Riemannian manifolds.

More precisely, we need the following proposition:

Proposition 2. *Let M^3 be a closed Riemannian manifold such that the absolute value of its sectional curvature does not exceed one, its diameter, $diam(M^3) \leq D$, and its volume, $vol(M^3) \geq v > 0$. Then there exists a simplicial complex K with at most $N = C\exp(16D)/v^6$ simplices triangulating M^3 and a homeomorphism f from K endowed with the canonical metric to M^3 that satisfies the Lipschitz condition with constant one. Here C is an absolute constant that, in principle, can be explicitly evaluated.*

Remark. This proposition can be generalized for manifolds of any dimension n for $N = C(n)\exp(2(n^2-1)D)/v^{2n}$.

Sketch of the proof.

A. Some well-known facts from Riemannian Geometry. One starts from the existence of the strong convexity radius $C(M^n) > 0$. This is, by definition, the largest number r such that all metric balls of radius $\leq r$ in M^3 are strongly convex. That is, for any two points x, y in a metric ball of radius $\leq C(M^n)$ there exists a unique minimizing geodesic in M^n between x and y, this geodesic is contained in the metric ball, and no other (not necessarily minimizing) geodesic between x and y is contained in the metric ball. Observe that the strong convexity radius

$$C(M^n) \geq \min\left\{\frac{\pi}{2\sqrt{K}}, \frac{\mathrm{inj}(M^n)}{2}\right\},$$

where $\mathrm{inj}(M^n)$ denotes the injectivity radius of M^n, and K denotes a positive upper bound for the absolute value of sectional curvature (cf. [Ch], Theorem 7.9). Now we can use the well-known Cheeger inequality

$$inj(M^n) \geq \min\left\{\frac{\pi}{\sqrt{K}}, \frac{n-1}{2c_{n-2}}\ \frac{v}{S_{-K}^{n-1}(D)}\right\},$$

where $c_{n-2} = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ and $S_{-K}(D) = \frac{\sinh(\sqrt{K}D)}{\sqrt{K}}$, ([C], see also [Ch], Theorem 7.6). This inequality implies that if $|sec(M^n)| \leq 1$ then $\mathrm{inj}(M^3) \geq \min\left\{\pi, \frac{2v}{\pi(\exp(2D)+\exp(-2D)-2)}\right\}$. Combining these two inequalities together, and observing that the Bishop volume comparison inequality implies that $v/\exp(2D) \leq \pi/2$, we see that, if $K \leq 1$, then $C(M^3) \geq \frac{v}{\pi\ \exp(2D)}$. Choose $\epsilon = \frac{v}{4000\exp(2D)}$ $(< 10^{-3})$. The standard comparison theorem in Riemannian geometry (cf. Theorem 7.3 in [Ch] or Corollary 2.4 in ch. 6 of [Pe]) implies that the norm of the difference of the differential of the exponential map defined on a metric ball of radius 2ϵ in M^3 and the identity map does not exceed 10^{-6}. Similarly, we have an upper bound for the Hessian of the distance function from a point $p \in M^3$ on the metric ball of radius 2ϵ centered at p: The norm of the Hessian at x is bounded by $1.01/dist(x,p)$.

B. Proofs of the triangulability of smooth manifolds. In order to prove the proposition we can use either Cairns's proof of the triangulability of smooth manifolds ([Ca]) or Whitney's proof ([Wh]). Both these proofs imply Proposition 2. However, the manifold is assumed to be embedded in a Euclidean space in both of these proofs. Both proofs easily imply the lower bound for the size of the simplices of the triangulation of the form $r/const$, where r is the injectivity radius of the normal exponential map of the embedding. So first we need to construct an embedding $g: M^3 \longrightarrow \mathbb{R}^M$ for some M and to estimate from below the injectivity radius of the normal exponential map for this embeding.

It is known that the injectivity radius of the normal exponental map of a manifold K^n embedded into an Euclidean space is equal to $\min\{\frac{1}{\sup|k(K^n)|}, \frac{1}{2}R(K^n)\}$ (see e.g. [D], Proposition 12), where $\sup|k(K^n)|$ is the maximal absolute prinicipal curvature, and $R(K^n)$ denotes the minimal length of a straight line segment in the ambient Euclidean space such that its endpoints are on K^n, and it is perpendicular to K^n at each of its endpoints.

Therefore, it will be necessary to find an upper bound U for the norm of the second fundamental form of the embedded manifold (or, equivalently, for $\sup|k|$). After that we will be immediately able to find a lower bound, namely $\frac{\pi}{U}$, for the distance in the inner metric of the embedded manifold between endpoints of any straight line segment in the ambient Euclidean space such that this segment is perpendicular to the manifold at each of its endpoints. In order to estimate $R(g(M^3))$ from below we will also need to find an upper bound for the ratio of the distances between points of the embedded manifold in the inner and the outer metrics. Such an upper bound can be found as the product of upper bounds U_1 for $\|Dg\|$ and U_2 for the Lipschitz constant g^{-1} $g^{-1}: g(M^3) \longrightarrow M^3$, where $g(M^3)$ is endowed with the outer metric. Thus, $R(g(M^3)) \geq \frac{\pi}{UU_1U_2}$.

We construct g and estimate Lipschitz constants for g and g^{-1} in section 3.C below. In section 3.E we will explain how to find an upper bound for the absolute principal curvatures of $g(M^3)$.

C. Construction of an embedding of M^3 into an Euclidean space and estimates for the Lipschitz constants. In order to construct the embedding we first choose an $\frac{\epsilon}{1000}$-net $\{p_i\}$ in M^3 such that the metrics balls of radius $\frac{\epsilon}{2000}$ centered at p_i are pairwise disjoint. Now choose a collection of non-negative functions $\{\theta_i\}$ corresponding to all elements the covering of M^3 by the metric annuli of the outer radius ϵ and inner radius $\epsilon/10000$ with centers at p_i with the following properties:

1) θ_i is equal to zero outside of the corresponding annulus;

2) θ_i is equal to one on the concentric annulus with inner radius equal to $\epsilon/500$ and outer radius equal to 0.9ϵ;

3) $|\theta_i'| < 10000/\epsilon$ and $|\theta_i''| < 10^{10}/\epsilon^2$.

Let L be equal to the number of these balls. We construct an embedding g of M^3 into $\mathbb{R}^L$ using functions $\theta_i(x)\ dist_{M^3}(x, p_i)$ as the coordinate functions. It is easy to see that for any $x \in M^n$ less than 10^{12} coordinate functions of the embedding are not zero. This observation follows from the Bishop-Gromov comparison theorem. Indeed, coordinate functions that are not zero correspond to points p_i such that $d(x,p_i)<\epsilon$. Since all metric balls of radius $\epsilon/2000$ centered at p_i are pairwise disjoint, this number can be majorised by the ratio of the volume of the ball of radius ϵ in the 3-dimensional hyperbolic space to the volume of the ball of radius $\frac{\epsilon}{2000}$ (where $\epsilon<10^{-3}$, as we agreed in section A of this proof).

Now it is obvious that the embedding satisfies the Lipschitz condition with constant $10^{20}/\epsilon$.

Note that it is easy to prove that the embedding can *decrease* distances by at most the factor of 1000. (In other words, the homeomorphism of $g(M^3)$ into M^3 inverse to the embedding satisfies the Lipschitz condition with constant 1000. Here we consider $g(M^3)$ with the outer metric induced by the embedding into the Euclidean space.)

The idea of the proof of the last assertion can be briefly explained as follows: Let $x,y\in M^3$ be two close points (e.g. assume that $dist(x,y)<\epsilon/10$). Consider the geodesic passing through x and y and consider a point z on this geodesic such that $dist(z,x)=\epsilon/10; dist(z,y)=\epsilon/10+dist(x,y)$. Let p_z be any of the points of the net $\{p_i\}$ closest to z. Then it is easy to demonstrate that $dist(p_z,y)-dist(p_z,x)>dist(x,y)/10$ using the already used estimate for the differential of the exponential map. (See Theorem 7.3 in [Ch] or Corollary 2.4 in ch. 6 of [Pe]. At the considered distances the norm of the difference of the differential of the exponential map and the identity map does not exceed 10^{-6}).

Thus, we found upper bounds for $\|Dg\|$ and for the Lipschitz constant of g^{-1} mentioned at the end of the section 3.B above. It remains to majorize the absolute value of principal curvatures of $g(M^3)$.

D. Local coordinates on the embedded manifold. Now let $x\in M^3$. We are going to find three points p_{i_1},p_{i_2},p_{i_3} of the net such that the three distance functions $d_{p_{i_j}}$, ($j=1,2,3$), on M^3 form a local coordinate system of the $\epsilon/1000$-neighborhood of x: Consider the exponential map at x in M^3. Consider the images of vectors $(\epsilon/10)e_1$, $(\epsilon/10)e_2$, $(\epsilon/10)e_3$, where e_1,e_2,e_3 is the standard basis. Take points p_x,p_y,p_z of the net that are the closest to these three points. It is not difficult to show that the projection of $\epsilon/1000$-neighborhood of the image of x in $\mathbb{R}^L$ in the embedded M^3 to the three-dimensional subspace of $\mathbb{R}^L$ corresponding to p_x,p_y,p_z is a bi-Lipschitz homeomorphism and to estimate the Lipschitz constants of this homeomorphism and its inverse. (Both of these

Lipschitz constants will be very close to 1 .)

E. Estimates for the norm of the second fundamental form of the embedding. Now it is easy to majorize the norm of the second fundamental form of the embedded copy of M^3 in $\mathbb{R}^L$ by $\frac{const}{\epsilon^2}$ for an appropriate constant $const$. (Here one needs the explicit upper bounds for the Hessian of the distance function mentioned above.) Then one is able to minorize the injectivity radius of the normal exponential map of the embedding of M^3 into $\mathbb{R}^L$ by $\epsilon^2/const$ for an approptiate constant $const$, as it was explained at the end of section 3.B. At this point the assertion about the size, and, thus, the number of the simplices follows directly from [Ca] (or [Wh]).

E. The Lipschitz constant of a homeomorphism between M^3 and the triangulating simplicial complex endowed with the canonical metric. It is easy to find a homeomorphism f mapping the simplicial complex K triangulating M^3 , obtained as described above and endowed with the canonical metric, to M^3 that satisfies the Lipschitz condition with constant one (or even much less than one). It is sufficient to define the required homeomorphism on every three-dimensional simplex of the triangulating simplicial complex. Now the key observations are:
1) Simplices of the triangulation of M^3 are very small;
2) At distances $\leq 2\epsilon$ the exponential map is almost an isometry;
3) The inverse image of the exponential map defined at any point of any simplex of the triangulation of M^3 (regarded as a subset of M^3) maps this simplex into "almost" straight simplices in the tangent space $\mathbb{R}^3$.
(The last assertion requires some analysis of the construction of these simplices in [Ca] (or [Wh]).) We omit the rest of the calculations. This completes the demonstration of Proposition 2. QED.

4. Proof of Theorem 1.

Now the required Lipschitz homeomorphism from the standard sphere S^3 of radius one to Σ^3 can be obtained as the composition of:
1. A Lipschitz homeomorphism from S^3 to the boundary of the standard four-dimensional simplex $\partial\sigma^4$ endowed by the canonical metric.
2. A Lipschitz homeomorphism from $\partial\sigma^4$ to the simplicial complex K triangulating Σ^3 as in Proposition 2 and endowed by the standard metric. This homeomor-

phism is the composite of homeomorphisms corresponding to individual bistellar transformations. The number of bistellar transformations is bounded by Mijatovic. Therefore, as we observed above (Proposition 1), the Lipschitz constant of this composite homeomorphism is less than $\exp(\exp(10^5\ N^2))$, where N is the number of simplices in K .

3. The Lipschitz homeomorphism f mapping the simplicial complex K triangulating $M^3 = \Sigma^3$ and endowed with the canonical metric to M^3 as in Proposition 2.

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