Non-recursive functions, knots "with thick ropes" and self-clenching "thick" hyperspheres

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Abstract

We introduce an approach to certain geometric variational problems based on the use of the algorithmic unrecognizability of the n-dimensional sphere for $n \geqq 5$. Sometimes this approach allows one to prove the existence of infinitely many solutions of a considered variational problem. This recursion-theoretic approach is applied in this paper to a class of functionals on the space of $C^{1,1}$-smooth hypersurfaces diffeomorphic to S^n in $\mathbb{R}^{n+1}$, where n is any fixed number $\geqq 5$. The simplest of these functionals κ_v is defined by the formula $\kappa_v(\Sigma^n) = (vol(\Sigma^n))^{\frac{1}{n}}/r(\Sigma^n)$, where $r(\Sigma^n)$ denotes the radius of injectivity of the normal exponential map for $\Sigma^n \subset \mathbb{R}^{n+1}$. We prove the existence of an infinite set of distinct locally minimal values of κ_v on the space of $C^{1,1}$-smooth topological hyperspheres in $\mathbb{R}^{n+1}$ for any $n \geqq 5$.

The functional κ_v naturally arises when one attempts to generalize knot theory in order to deal with embeddings and isotopies of "thick" circles and, more generally, "thick" spheres into Euclidean spaces. We introduce the notion of knot "with thick rope" types. The theory of knot "with thick rope" types turns out to be quite different from the classical knot theory because of the following result: There exists an infinite set of non-trivial knot "with thick rope" types in codimension one for every dimension $\geqq 5$.

0. Introduction

Define for a smooth compact submanifold Σ^n of $\mathbb{R}^{n+k}$ its v-crumpledness $\kappa_v(\Sigma^n)$ as the ratio $(vol\ (\Sigma^n))^{\frac{1}{n}}/r(\Sigma^n)$, where $r(\Sigma^n)$ is the supremum of all r such that any two different normals to Σ^n of length r have no common points. (In other words, $r(\Sigma^n)$ is the maximal radius of an open non-selfintersecting tube around Σ^n in $\mathbb{R}^{n+k}$.)

In this paper we start an investigation of (generalized) knots "with thick ropes". Let $k \subset \mathbb{R}^3$ be a knot. If $x \leqq r(k)$ (i.e. there exists a non-selfintersecting tube of radius x around k), then we can regard k as an axis of a knot with the rope of thickness x. Since rescalings do not change the shape of knots from any intuitive point of view, we consider $length(k)/r(k)$ (i.e. $\kappa_v(k)$) as an invariant of the knot. We say that two knots k_1 and k_2 have the same knot "with thick rope" x-type if there exists an isotopy between k_1 and k_2 passing only through knots k such that $\kappa_v(k) \leqq x$.

These definitions can be straightforwardly generalized for the case of greater dimensions/codimensions.

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One of the main results of this paper is that the theory of (generalized) knot "with thick rope" types differs drastically from the usual theory of multidimensional knots. Namely, for any $n \geqq 5$ there exists an infinite increasing sequence x_i such that 1) $\lim_{i\to+\infty} x_i = +\infty$; 2) For any x_i there exists a non-trivial knot "with thick rope" x_i-type K_i of n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$ diffeomorphic to S^n ; 3) For any two i, j such that $i \neq j$ K_i and K_j are disjoint. (Here we regard K_i as sets of hypersurfaces diffeomorphic to S^n in $\mathbb{R}^{n+1}$). Recall that there are no nontrivial usual knot types of embeddings of S^n in $\mathbb{R}^{n+1}$ for $n \geqq 5$.

We deduce from this result the existence of an infinite set of essentially distinct local minima of κ_v on the space of $C^{1,1}$-smooth hypersurfaces in $\mathbb{R}^{n+1}$ diffeomorphic to S^n. "Essentially distinct" means here that any two of these minimizing hypersurfaces cannot be transformed one into another by a composition of a homothetic transformation and an isometric transformation of $\mathbb{R}^{n+1}$. The same result holds for a wide class of functionals similar to κ_v.

The proof of these theorems uses the author's results on the non-recursivness of some functions in real algebraic geometry [26] based on S.P. Novikov's Theorem stating the algorithmic unrecognizability of the n-dimensional sphere in $\mathbb{R}^{n+1}$ for any $n \geqq 5$ ([40] , ch.10). The present paper also contains the detailed exposition of the results of [26] .

In [15], p. 212 Gromov noted that if M is a compact Riemannian manifold such that its fundamental group has unsolvable word problem then M has infinitely many *contractible* closed geodesics. This result of Gromov seems to be the only application of recursion theory to calculus of variations prior to the present work. However, it is interesting to note that the theory of computations was applied also in a paper of Mnev [24] (see also [38]) to prove the non-connectedness of certain spaces of combinatorial configurations. But the technique of the proof there is completely different from the ours.

The results of the paper are contained in the author's Ph.D. thesis ([29]) done at the Weizmann Institute of Science.

1. Basic definitions and results

Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a non-singular compact algebraic hypersurface of degree d. Let Σ^n be isotopic to the standard hypersphere $S^n \subset \mathbb{R}^{n+1}$. It is well-known that it is not always possible to connect Σ^n and S^n by an isotopy passing only through non-singular algebraic hypersurfaces of degree not higher than d. But one can show that it is always possible to connect Σ^n and S^n by an isotopy passing through algebraic hypersurfaces of some degree A which depends on n and d alone (Theorem 3.2.1(a)). Consider the smallest of such degrees as a function $A_n(d)$ of d. What can be said about these functions?

We proved in [25] that for each $n \geqq 5$ the function $A_n(d)$ cannot be majorized by any recursive function. We state this result here as Theorem 3.2.1(b). (But $A_1(d)$ and $A_2(d)$ are recursive. We presented an explicit upper bound for $A_1(d)$ in [27]). This result can be generalized for higher codimensions. (It is natural to consider regular complete intersections of compact hypersurfaces as a multicodimensional analog of compact hypersurfaces. Numerous facts about such regular complete intersections can be found in [6] and [12].)

The proof of nonrecursiveness of $A_n(d)$ is based on S.P. Novikov's Theorem [40], ch. 10, stating that there is no algorithm which verifies for a given hypersurface $M^n \subset \mathbb{R}^{n+1} (n \geqq 5)$ whether or not this hypersurface is diffeomorphic to S^n. The idea is to show that the recursiveness of $A_n(d)$ implies the possibility to construct an algorithm recognizing hypersurfaces diffeomorphic to S^n, thus obtaining the contradiction with S.P. Novikov's theorem.

Let $M_n(\mathbb{R}^{n+k})$ be the space $\bigsqcup Emb(M^n, \mathbb{R}^{n+k})/Diff(M^n)$, where the disjoint union is taken over all compact C^2-smooth manifolds M^n embeddable in $\mathbb{R}^{n+k}$ and $Emb(M^n, \mathbb{R}^{n+k})$ is the space of C^2-embeddings endowed by C^1-topology. Let us call an isotopy $F : [0,1] \to M_n(\mathbb{R}^{n+k})$ *D-rigid isotopy* for some $D \in \mathbb{N}$ if for any $t \in [0,1]$ $F(t)$ is a regular complete intersection of compact non-singular algebraic hypersurfaces of degree $\leqq D$. In the codimension one case this means that for any $t \in [0,1]$ $F(t)$ is the compact zero set of a polynomial p of degree $\leqq D$ such that grad p does not vanish at any point of $F(t)$. (This definition is a slightly modified and generalized version of V. Rokhlin's definition of rigid isotopy ([32]). Our definition differs from his in the respect that we permit singularities in complex not real part of the zero set of polynomials during the isotopy and also permit the degrees of the polynomials vanishing at $F(t)$, $t \in (0,1)$, to be greater than the degrees of the polynomials vanishing at $F(0)$ and $F(1)$. Some information about rigid isotopy classes of plane algebraic curves can be found in [39]). Note that we do not require in this definition any continuity of the variation of polynomials vanishing at $F(t)$. (Only the complete intersection regarded as an element of $M_n(\mathbb{R}^{n+k})$ should vary continuously). Moreover, in the case of codimension > 1 the intersecting hypersurfaces can also vary discontinuously. The purpose of this slackness of the definition is to make the relation "to be D-rigidly isotopic" transitive. Nevertheless, there exists an algorithm checking for any two compact non-singular algebraic hypersurfaces M_1^n, $M_2^n \subset \mathbb{R}^{n+1}$ whether or not M_1^n and M_2^n are D-rigidly isotopic (Theorem 3.1.2). (Actually the result is true not only for hypersurfaces but for regular complete intersections of compact non-singular algebraic hypersurfaces, too.) This result and the possibility to approximate any isotopy by a D-rigid isotopy allow us to obtain the necessary contradiction between S.P. Novikov's Theorem and the assumption about recursiveness of functions $A_n(d)$.

An informal explanation of S.P. Novikov's theorem can be found in [4]; a formal proof is given in [40],ch. 10. A more detailed exposition of some ideas used in the proof of Novikov's theorem and, also, several related results can be found in the paper of Boone, Haken and Poénaru [7]. (Added in August, 1994. A detailed proof of Novikov's theorem and, in particular, of its version used in the present paper can be found in [30].)

In our proof of the existence of the functions $A_n(d)$ and of the nonrecursiveness of the functions $A_n(d)$ for $n \geqq 5$ we often use the following well-known definitions and results from semialgebraic geometry (cf. [9] or [5]):

— A *semialgebraic set* is any subset $X \subseteq \mathbb{R}^n$ with the *presentation* of the form $\bigcup_{i=1}^k \bigcap_{j=1}^h \{x \in \mathbb{R}^n | P_{ij}(x) \ ?_{ij} 0\}$, where P_{ij} are polynomials and $?_{ij}$ denotes one of the signs $<$, $>$ or $=$. A *semialgebraic map* is a map between two semialgebraic sets such that its graph is a semialgebraic set.

— (Tarski-Seidenberg Theorem). Any subset $X \subseteq \mathbb{R}^n$ which can be represented by a first order formula in the language of the real number field theory is a semialgebraic set. There exists an algorithm which constructs a presentation of the semialgebraic set starting from a given formula. "First order formula in the language of the real number field theory" means here a formula obtained starting from polynomial sign conditions $P_i \ ?_i \ 0$, where $?_i$ means $<$, $>$, or $=$, and using a finite constant number of conjunctions, disjunctions, negations, and universal or existential quantifications of the variables. (Of course, all the variables in such a formula should be real variables).

— Any semialgebraic set has a finite number of connected components.

— (Generalized Thom's Lemma). Any connected component of a semialgebraic set is a semialgebraic set. Given a presentation of a semialgebraic set it is possible to construct (effectively) a list of presentations of all its connected components. As a corollary, there exists an algorithm checking whether or not two given points belong to the same connected component of a given semialgebraic set.

The main point of this paper is the observation that one can use the nonrecursiveness of the functions A_n for $n \geqq 5$ to prove the purely geometric results sketched in the introduction, which will be described below.

We introduce geometric characteristics of compact submanifolds of $\mathbb{R}^{n+k}$ which we call d-crumpledness and v-crumpledness. Let $M^n \subset \mathbb{R}^{n+k}$ be a compact submanifold and $C(M^n)$ be the cut locus of $\mathbb{R}^{n+k}$ with respect to M^n. (Recall the definition of the cut locus. For any point $x \in M^n$ and any unit normal $n(x)$ at x to M^n we consider the supremum $t_{x,n(x)}$ of all $t > 0$ such that $dist\ (x + tn(x), M^n) = t$. The set $\bigcup_{x,n(x)}\{x + t_{x,n(x)}n(x)\}$ is called the cut locus of $\mathbb{R}^{n+k}$ with respect to M^n. In the case when M^n is a hypersurface we shall call the cut locus of $\mathbb{R}^{n+k}$ with respect to M *the central set of* M^n.) Numerous results about the central sets of smooth compact hypersurfaces can be found in [18], [19], [20], [21], [34], [37]. In particular, it is very easy to show that $C(M^n)$ contains the *skeleton* $S(M^n)$ of M^n, which is defined as follows: $S(M^n)$ is the set of all points $x \in \mathbb{R}^{n+1}$ such that there exist distinct points $y_1, y_2 \in M^n$ such that $dist(x, y_1) = dist(x, y_2) = dist(x, M^n)$. For C^2-smooth hypersurfaces $C(M^n)$ coincides with the closure of $S(M^n)$ ([37], [21]). (However, we should warn the reader that the definitions in these papers are somewhat different from our definitions. Namely, they regard a compact hypersurface M^n as the boundary of a bounded domain U in $\mathbb{R}^{n+1}$. This domain is the union of the bounded connected components of $\mathbb{R}^{n+1} \setminus M^n$. The central set defined there is the intersection of $C(M^n)$ (in our terms) with U. Similarly, the skeleton defined there is the part of the skeleton defined here situated in the union of the bounded connected components of $\mathbb{R}^{n+1} \setminus M^n$. Nevertheless many results stated and proven in [37] and [21] are immediately true for central sets/skeletons defined as above.) For $C^{1,1}$-smooth hypersurfaces $C(M^n)$ may differ from $\overline{S(M^n)}$, but the inclusion $C(M^n) \subset \overline{S(M^n)}$ still holds. (By the definition, a $C^{1,1}$-smooth hypersurface is a C^1-smooth hypersurface such that the corresponding Gauss map is Lipschitz. Recall that the Gauss map maps any point of a hypersurface to the (outer) unit normal vector to the hypersurface at this point.) Denote dist $(M^n, C(M^n))$ by $r(M^n)$. If M^n is C^2-smooth, or even $C^{1,1}$-smooth then $r(M^n) > 0$. Let $M^n \subset \mathbb{R}^{n+1}$ be a C^2-smooth compact hypersurface and $x \in M^n$ be a point. The definition of $t_{x,n(x)}$ easily implies that $t_{x,n(x)}$ does not exceed the distance from x to the closest focal point of M^n on the normal to M^n at x. But it is well-known (cf. [35]) that this distance is equal to $1/\max_{i\in\{1,\ldots n\}}|k_i(x)|$, where k_i are principal curvatures of M^n at x. Thus, $1/r(M^n) \geqq \max_{i\in\{1,\ldots,n\};x\in M^n}|k_i(x)|$. This inequality will be frequently used in this work. An analogue of this inequality holds for $C^{1,1}$-smooth hypersurfaces. However, in the $C^{1,1}$-smooth case the maximum at the right hand side should be taken over the set of all $x \in M^n$ such that the Gaussian map is differentiable at x (and, hence, the principal curvatures at x exist). (By virtue of the well-known Rademacher theorem the Gaussian map is differentiable almost everywhere.)

Define d-crumpledness $\kappa_d : M_n(\mathbb{R}^{n+k}) \to \mathbb{R}^+$ by the formula $\kappa_d(M^n) = diam_{\mathbb{R}^{n+k}}(M^n)/r(M^n)$ and v-crumpledness $\kappa_v(M^n)$ as $vol_n^{\frac{1}{n}}(M^n)/r(M^n)$. These dimensionless characteristics are related. We show for hypersurfaces that $\kappa_v(M^n)$ can be majorized in terms of $\kappa_d(M^n)$ and, vice versa, $\kappa_d(M^n)$ can be majorized in terms of $\kappa_v(M^n)$ (Lemma 2.1.4). The similar results can be proven for higher codimensions. Several useful properties of κ_d and κ_v are proven in subsection 2.1.

The functional κ_v has the following "physical" sense. Let $k \subset \mathbb{R}^3$ be a knot. One can consider it as an axis of a physical knot tied with a rope of a finite constant thickness. The maximal possible thickness of the rope is equal to $2r(k)$. If one is interested in isotopies of a physical knot with a flexible but incompressible rope (i.e. if the thickness and the length are supposed to be constant during isotopies) then it is natural to study path components of the sets of embeddings k of S^1 into $\mathbb{R}^3$ such that $r(k) \geqq C_1$ and length $(k) \leqq C_2$ where C_1 and C_2 are arbitrary constants. Note that homothetic transformations of $\mathbb{R}^{n+1}$ induce diffeomorphisms with simple metric properties of these sets of embeddings with the same value of the ratio C_2/C_1. Thus, it is sufficient to study path components of sets of embeddings k of S^1 in $\mathbb{R}^3$ such that $\kappa_v(k) = \frac{length(k)}{r(k)} \leqq C(= \frac{C_2}{C_1})$. This idea can be extended to higher dimensional/codimensional embeddings of S^n (and to embeddings of other compact manifolds). In particular, we can introduce for any x the notion of x-type of a generalized knot $S^n \subset \mathbb{R}^{n+k}$ "with thick rope". Let two embeddings $k_1 : S^n \to \mathbb{R}^{n+k}$, $k_2 : S^n \to \mathbb{R}^{n+k}$ be given. If there exists an isotopy $F : [0,1] \to M_n(\mathbb{R}^{n+k})$ such that $F(0) = k_1(S^n)$, $F(1) = k_2(S^n)$ and for any $t \in [0,1]$ $\kappa_v(F(t)) \leqq x$, then we shall say that k_1 and k_2 are of the same knot "with thick rope" x-type. The relation "to be of the same knot "with thick rope" x-type" is an equivalence relation. The equivalence classes which we shall call knot "with thick rope" x-types are just path components of spaces $\kappa_v^{-1}([0,x])$. One of the main results of this paper is that the theory of generalized knots "with thick rope" differs drastically from the classical theory of generalized knots. For example, it is known that for any $n \geqq 5$ there are no nontrivial n-dimensional knots in $\mathbb{R}^{n+1}$ ([22], Ch. 9, Proposition D). (This result is a corollary of S. Smale's h-cobordism theorem). However, we show that for any $n \geqq 5$ there exists an infinite set of different non-trivial x_i-types of generalized n-dimensional knots "with thick rope" of codimension one (Theorem 4.1). Here $i \in \mathbb{N}$ and $\lim_{\to +\infty} x_i = +\infty$. The notion of the knot "with thick rope" can be generalized in order to permit a certain increase of the v-crumpledness during isotopies. (The exact definition is given in section 4.1). An analogue of Theorem 4.1 for such knot types also takes place (Theorem 4.2). These results follow from Theorem 3.2.1, mentioned at the beginning of this section, stating the non-recursiveness of the functions $A_n(d)$, $(n \geqq 5)$ via the following result of an independent interest:

(a) For any x there exist y such that any smooth hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ of v-crumpledness $\leqq x$ and isotopic to the standard sphere S^n can be connected with S^n by an isotopy passing only through hypersurfaces of v-crumpledness $\leqq y$.

(b) Consider the infimum of all such y as a function of x. Denote it by $C_n(x)$. For any $n \geqq 5$ the restriction of $C_n(x)$ on integer numbers cannot be majorized by any recursive function of x.

This result is stated as Theorem 3.2.3. The following observation is crucial to deduce part (b) from the non-recursiveness of the functions $A_n(d)$, $n \geqq 5$: Let a compact smooth hypersurface $M^n \subset \mathbb{R}^{n+1}$ be given. Suppose that we want to approximate it by a non-singular algebraic hypersurface. It turns out that the degree of this algebraic hypersurface and its v-crumpledness (or d-crumpledness) can be estimated in terms of $\kappa_v(M^n)$ (or $\kappa_d(M^n)$) and the closeness of approximation. Of course, to make the last statement rigorous we should define how we measure the closeness of an approximation of a hypersurface. We prefer to use the following modification of the definition given by Wallace in [41]. Let M_1^n, $M_2^n \subset \mathbb{R}^{n+k}$ be smooth compact submanifolds and $\varepsilon < r(M_2^n)$ be a given positive number. We shall say that M_1^n *ε-approximates* M_2^n if

(1) For any point $x \in M_2^n$ there exists exactly one point $x^*(x) \in M_1^n$ such that (a) $x^*(x)$ is on a normal to M_2^n at x; (b) $|x^*(x) - x| \leqq \varepsilon$;

(2) The map $x \longrightarrow x^*(x)$ of M_2^n into M_1^n is a homeomorphism.

(3) It is possible to choose the orthonormal bases $\{e_i\}_{i=1}^k$ of the normal linear space of M_2^n at x and $\{e_i^*\}_{i=1}^k$ of the normal linear space of M_1^n at $x^*(x)$ in such a way that for any $i \in \{1, \dots, k\}$,

$$(e_i, e_i^*) \geqq 1 - \frac{\varepsilon}{diam(M_2^n)}.$$

In this paper we shall use this definition in the case of codimension $k = 1$. In this case property (3) in the text of the definition above has the following more simple form: (3) Let $n(x)$ be the outer unit normal to M_2^n at x and $n(x^*(x))$ be the outer unit normal to M_1^n at $x^*(x)$. Then

$$(n(x),\ n(x^*(x))) \geqq 1 - \frac{\varepsilon}{diam\ (M_2^n)} \quad .$$

Note that if M_1^n and M_2^n are C^2-smooth, then the property (3) in the definition above easily implies that the map $x \longrightarrow x^*(x)$ is a C^1-diffeomorphism.

An obvious drawback of this definition is its asymmetry and non-transitivity. Nevertheless, one can prove a weak analogue of these properties (Proposition 2.2.3) sufficient for our goals. One of the advantages of this definition is its "good behaviour" under homothetic transformations of $\mathbb{R}^{n+1}$. Indeed if M_1^n ε-approximates M_2^n, then after a transformation $x \to \alpha x$ $(\alpha > 0)$ of $\mathbb{R}^{n+1}$ the image of M_1^n will $\alpha\varepsilon$-approximate the image of M_2^n. Another important property of this notion is its compatibility with the topology of $M_n(\mathbb{R}^{n+1})$ (see Lemma 2.2.4 for the exact statement of this property). Some other properties of ε-approximation are proven in section 2.2.

In section 5 we refine the above mentioned result stating the existence of an infinite set of different x-types of codimension one knots "with thick ropes". We prove the existence of an infinite set of essentially distinct local minima of v-crumpledness on the space of $C^{1,1}$-smooth compact hypersurfaces in $\mathbb{R}^{n+1}$, diffeomorphic to S^n, for any fixed $n \geqq 5$ (Theorem 5.2.1). By "essentially distinct" we mean that any of these minimizing hypersurfaces cannot be transformed into another one by a composition of a homothetic transformation and a isometric transformation of $\mathbb{R}^{n+1}$. These minimizing hypersurfaces together with their maximal open non-selfintersecting tubular neighborhoods in $\mathbb{R}^{n+1}$ can be regarded as self-clenching "thick" hypersurfaces. To deduce this result from the infiniteness of the set of codimension one knot "with thick ropes" x-types we prove in Section 5 the lower semicontinuity of κ_v on the space of $C^{1,1}$-smooth compact hypersurfaces $M_n^{(1,1)}(\mathbb{R}^{n+1})$. This space is rigorously defined as $\coprod_{M^n} Emb^{(1,1)}(M^n, \mathbb{R}^{n+1})/Diff(M^n)$, where $Emb^{(1,1)}(M^n, \mathbb{R}^{n+1})$ is the space of C^1-smooth embeddings of M^n into $\mathbb{R}^{n+1}$ with $C^{1,1}$-smooth image endowed by C^1-topology, $Diff(M^n)$ is the group of C^1-smooth diffeomorphisms of M^n, and the disjoint union is taken over all compact C^1-smooth manifolds admitting a C^1-smooth embedding in $\mathbb{R}^{n+1}$ with $C^{1,1}$-smooth image. Obviously, $M_n(\mathbb{R}^{n+1}) \subset M_n^{(1,1)}(\mathbb{R}^{n+1})$. Lemma 2.2.4 shows how one can express convergence in the topology of $M_n^{(1,1)}(\mathbb{R}^{n+1})$ in terms of ε-approximation. Note that the convergence in the topology $M_n^{(1,1)}(\mathbb{R}^{n+1})$ implies the convergence of hypersurfaces in the topology defined using the Hausdorff distance between subsets of an Euclidean space. The opposite is not true. However, we shall demonstrate that all local minima of κ_v in $M_n^{(1,1)}(\mathbb{R}^{n+1})$ will be also local minima if the space of all $C^{1,1}$-smooth hypersurfaces is equipped with the topology defined using the Hausdorff distance. It follows from the proof of the existence of an infinite set of essentially distinct local minima of v-crumpledness on the space of $C^{1,1}$-smooth topological hyperspheres that the same result holds for a wide class of similar functionals (including κ_d and numerous functions of κ_d and κ_v, as, for example, $\ln \kappa_v + \exp(\kappa_d)$) (Theorem 5.2.2).

2. Crumpledness and algebraic approximation

In sections 2.1 and 2.2 we prove several technical results describing some properties of v- and d-crumpledness and of the ε-approximation of hypersurfaces. All statements and proofs (with the exception of the proof of Lemma 2.2.4) are given for the case of C^2-smooth hypersurfaces, although we shall need in section 5 also the extensions of some of these results for the case of $C^{1,1}$-smooth hypersurfaces. These results with the exception of Lemma 2.2.4 can be so straightforwardly generalized for the $C^{1,1}$-smooth case that we prefer just to point out in remarks after the proofs which modifications should be introduced into the statements and proofs of these results to extend them for the $C^{1,1}$-smooth case. (However, we do not need the $C^{1,1}$-smooth version of Proposition 2.2.3, and therefore do not state it. Proposition 2.2.4 will be proven immediately for the more general $C^{1,1}$-smooth case.)

The section 2.3 contains two quantitative results about algebraic approximation of compact hypersurfaces and isotopies of hypersurfaces.

2.1. Geometric properties of d-crumpledness and v-crumpledness

2.1.1. Estimates for d-crumpledness

Here we prove the inequality $\kappa_d(M^n) \geqq 2$ for compact C^2-hypersurfaces in IR^{n+1}. (Note that $\kappa_d(S^n) = 2$.) We also present an upper bound for $\kappa_d(M^n)$ in the case when M^n is the (non-singular) zero set of a smooth function $f : \mathrm{IR}^{n+1} \to \mathrm{IR}$ in terms of analytic characteristics of f.

Lemma 2.1.1. Let M^n be a compact C^2 hypersurface in IR^{n+1}. Then $\kappa_d(M^n) \geqq 2$.

Proof. Let y be a point of M^n and $N(y)$ denote the normal line to M^n at y. It is easy to show that for the points of $N(y)$ sufficiently close to y y is the closest point of M^n. The compactness of M^n implies that there exists a point $z \in N(y) \bigcap M^n$ such that $z \neq y$. Obviously, z itself and not y is the closest to z point of M^n. Thus, there exists a point $x_0 \in (y,\ z]$ such that $x_0 \in C(M^n)$. The point x_0 can be chosen in such a way that for any point a from the interval $(y,\ x_0)$

y is the closest to a point of M^n. This constraint implies that $|x_0 - y| \leqq |x_0 - z|$. Hence, $|x_0 - y| \leqq |y - z|/2$. Now we can see that

$$r(M^n) = \ dist(C(M^n),\ M^n) \ \leqq |x_0 - y| \leqq |y - z|/2 \leqq diam_{\mathbb{R}^{n+1}}(M^n)/2.$$

□

Remark. It is clear from the proof that Lemma 2.1.1 holds for compact $C^{1,1}$-smooth hypersurfaces in $\mathbb{R}^{n+1}$.

Lemma 2.1.2. Let M^n be a compact C^2-submanifold of $\mathbb{R}^{n+1}$. Then

$$r(M^n) \geqq \inf_{\substack{x,y \in M^n \\ x \neq y}} \ \frac{|x - y|}{|n(x) - n(y)|} \ .$$

Proof. The skeleton $S(M^n)$ is the set of points $s \in \mathbb{R}^{n+1}$ such that there exist points $x_s, y_s \in S^n$ such that $x_s \neq y_s$ and $x_s + dist(s, M^n)n(x_s) = y_s + dist(s, M^n)n(y_s) = s$, where $n(x_s)$ and $n(y_s)$ are unit normal vectors to M^n at x_s and y_s, respectively. It is well-known that $C(M^n) = \overline{S(M^n)}$ (cf. [37], [21], Theorem 1). (As it was noted, the definitions of $S(M^n)$ and $C(M^n)$ given in [37], [21] slightly differ from the ours. Actually it was proven in these works that the intersection of $C(M^n)$ with the union U of all bounded connected components of $\mathbb{R}^{n+1} \setminus M^n$ coinsides with $\overline{S(M^n)} \bigcap U$ (in our terms). However, the proofs there can be straighforwardly and insignificantly modified to yield the equality $C(M^n) = \overline{S(M^n)}$.) The inclusion $C(M^n) \subset \overline{S(M^n)}$ implies that for any $\varepsilon > 0$ there exists $s \in S(M^n)$ such that $dist(s, M^n) \leqq r(M^n) + \varepsilon$. Thus, $|x_s - y_s| = dist(s, M^n)|n(x_s) - n(y_s)| \leqq (r(M^n) + \varepsilon)|n(x_s) - n(y_s)|$. Hence, $r(M^n) \geqq \frac{|x_s - y_s|}{|n(x_s) - n(y_s)|} - \varepsilon$. Since ε is an arbitrary positive number, this inequality implies the lemma. □

Remark. If M^n is a $C^{1,1}$-smooth compact hypersurface then $C(M^n)$ does not coincide in general with $\overline{S(M^n)}$. However, in the proof of Lemma 2.1.2 we used only the inclusion $C(M^n) \subset \overline{S(M^n)}$. This inclusion (Lemma 2 in [37]) still holds in the $C^{1,1}$-smooth case, and the proof given in [37] is still correct. Thus Lemma 2.1.2 holds in the $C^{1,1}$-smooth case too.

Lemma 2.1.3. Let M^n be the compact zero set of some C^1 map $f : \mathbb{R}^{N+1} \to \mathbb{R}$ such that $\inf_{x \in M^n} |grad\ f| > 0$ for any $x, y \in Conv\ (M^n)$ and

$$|grad\ f(x) - grad\ f(y)| \leqq L|x - y|$$

for some constant L. Here $Conv\ (M^n)$ denotes the convex hull of M^n in $\mathbb{R}^{n+1}$. Then

$$\kappa_d(M^n) \leqq \frac{2L\ diam(M^n)}{\inf_{x \in M^n} |grad\ f(x)|} \ .$$

Proof. Note that for any $x, y \in M^n$ $n(x) = \frac{grad\ f(x)}{|grad\ f(x)|}$ and $n(y) = \frac{grad\ f(y)}{|grad\ f(y)|}$. It follows from Lemma 2.1.2 and the remark after its proof that it is sufficient to prove that for any $x, y \in M^n$ $\frac{2L|x-y|}{\inf_{x\in M^n}|grad\ f(x)|} \geqq |n(x) - n(y)|$.

Since $|n(x) - n(y)| \leqq 2$ this inequality holds for $|x - y| \geqq \frac{\inf_{x\in M^n}|grad\ f(x)|}{L}$. If $|x - y| < \frac{\inf_{x\in M^n}|grad\ f(x)|}{L}$ then for any $z \in [x, y]$ there exists $\frac{grad\ f}{|grad\ f|}(z)$ (i.e. $|grad\ f(z)| \neq 0$) and a simple computation shows that

$$|\frac{grad\ f}{|grad\ f|}(x) - \frac{grad\ f}{|grad\ f|}(y)| \leqq |x - y|\frac{2L}{\inf_{x\in M^n}|grad\ f(x)|} .$$

□

Notation. Let a hypersurface M^n and a function f satisfy the assumptions of Lemma 2.1.3. We shall denote the ratio $(L\ diam(M^n))/\inf_{x\in M^n}|grad\ f|$ by $\kappa(f)$. According to Lemma 2.1.3 $\kappa_d(M^n) \leqq 2\kappa(f)$.

2.1.2. d-crumpledness and v-crumpledness

In this section we show how to deduce an upper bound for d-crumpledness of a C^2-smooth submanifold M^n of the Euclidean space in terms of the v-crumpledness of M^n and, vice versa, how to majorize $\kappa_v(M^n)$ if $\kappa_d(M^n)$ is known. The reason why we need these estimates is the following one: To work with d-crumpledness is more convenient from the technical point of view (see the proof of Theorem 3.2.2(a) below), and we are going to deduce some results about v-crumpledness from the (easier) analogous results about d-crumpledness.

Lemma 2.1.4. For any C^2-smooth compact connected hypersurface $M^n \subset \mathbb{R}^{n+1}$

(a)

$$\kappa_v(M^n) < C_1(n)(\kappa_d(M^n))^{1+\frac{1}{n}}, \tag{2.1.1}$$

(b)

$$\kappa_d(M^n) < C_2(n)\kappa_v^n(M^n), \tag{2.1.2}$$

where $C_1(n)$, $C_2(n)$ depend only on n but not on M^n.

Proof. (a) Both sides of the inequality (2.1.1) are invariant under homothetic transformations (more rigorously, under transformations of $M_n(\mathbb{R}^{n+1})$ induced by homothetic transformations of $\mathbb{R}^{n+1}$). Thus, we can assume without loss of generality that $r(M^n) = 1$. Now we want to prove that the volume of the radius one neighbourhood $N_1(M^n)$ of M^n in $\mathbb{R}^{n+1}$ satisfies the inequality

$$vol_{n+1}(N_1(M^n)) \geqq vol_n(M^n) \cdot C_3(n) \tag{2.1.3}$$

where $C_3(n)$ depends only on n. For any $\rho < 1$ consider ρ-neighbourhood $N_\rho(M^n)$ of M^n in $\mathbb{R}^{n+1}$.The classical result of H. Weyl implies that $vol(N_\rho(M^n))$ is a polynomial of degree $n+1$ in ρ.

Let $vol_{n+1}(N_\rho(M^n)) = \sum_{i=0}^{n+1} a_i\rho^i$. It is easy to show that $a_0 = 0$ and $a_1 = 2\ vol_n(M^n)$. Note that $q(\rho) \equiv \frac{\partial\ vol_{n+1}(N_\rho(M^n))}{\partial \rho}$ is non-negative for any $\rho \in [0,1)$ because $vol_{n+1}(N_\rho(M^n))$ grows with ρ. Thus,

$$(2.1.4) \qquad vol_{n+1}(N_1(M^n)) = \|q\|_{L^1[0,1]} = 2vol_n(M^n) \cdot \|q_0\|_{L^1[0,1]},$$

where $q_0(\rho) = \frac{1}{2\ vol_n(M^n)} q(\rho)$.

Note that $q_0(\rho)$ is a non-negative polynomial on [0,1] and the norm of the vector of coefficients of q_0 is greater or equal to one. Let $U_{\geqq 1}(d)$ denote the set of polynomials of degree d on [0,1] such that the norm of the vector of coefficients of p is greater or equal to one, and $U_1(d) \subset U_{\geqq 1}(d)$ be formed by polynomials $p \in U_{\geqq 1}(d)$ such that the norm of the vector of coefficients of p is equal to one. Then $q_0 \in U_{\geqq 1}(n)$. Hence,

$$\|q_0\|_{L^1[0,1]} \geqq \inf_{p \in U_{\geqq 1}(n)} \|p\|_{L^1[0,1]} = \inf_{p \in U_1(n)} \|p\|_{L^1[0,1]} > 0 \ .$$

(The last inequality follows from the compactness of $U_1(n)$).
Denote $2\inf_{p\in U_1(n)} \|p\|_{L^1[0,1]}$ by $C_3(n)$. Now (2.1.4) implies (2.1.3).

To deduce (2.1.1) from (2.1.3) note that $N_1(M^n)$ is contained in a ball of radius $(diam\ (M^n) + 2)$ in $\mathbb{R}^{n+k}$. Thus,

$$(2.1.5) \qquad vol_{n+1}(N_1(M^n)) \leqq (\ diam\ (M^n) + 2)^{n+1} \cdot v_{n+1} \quad ,$$

where v_{n+1} is the volume of the unit $(n+1)$-dimensional ball. Now using (2.1.3) and Lemma 2.1.1 we see that

$$\frac{v_{n+1}}{C_3(n)}\ (2\ diam\ (M^n))^{n+1} \geqq vol_n(M^n)\ ,$$

which yields (2.1.1). Part (a) of the lemma is proven.

(b) Since both sides of the inequality (2.1.2) are invariant under homothetic transformations, it is sufficient to prove the inequality in the case of $r(M^n) = 1$. We consider the neighbourhood $N_{1/2}(M^n)$ of radius $\frac{1}{2}$ of M^n. This time our first aim will be to prove an upper bound for $vol_{n+1}(N_{1/2}(M^n))$ of the form $C_4(n)\ vol_n(M^n)$, where $C_4(n)$ depends only on n. Let for any non-negative $\rho < 1$ M_ρ denote the set $\{x \in \mathbb{R}^{n+1} |$ the distance from x to M^n is equal to $\rho\}$. By the definition of $r(M^n)$ for any $x \in M_\rho$ there exists a single point $m(x) \in M^n$ such that $dist(x, M^n) = dist(x, m(x))$. M_ρ is a smooth manifold. It is easy to see that

$$(2.1.6) \qquad vol_{n+1}(N_{1/2}(M^n)) = \int_0^{1/2} vol_n(M_\rho) d\rho\ .$$

Consider the map $m : M_\rho \to M^n$. This map is differentiable. Let $n(x) = \frac{1}{\rho}(x - m(x))$. Then $n(x)$ is a unit normal to M^n at $m(x)$. It is easy to show that the determinant of Dm satisfies the inequality

$$det(Dm(x)) \geqq (1 + \rho||L||)^{-n} \ ,$$

where $L : T_{m(x)}M^n \longrightarrow T_{m(x)}M^n$ denotes the second fundamental form of M^n at x. It is a standard argument in differential geometry to show that

$$||L|| \leqq \max_{x\in M^n; i\in\{1,\dots,n\}} |k_i(x)| \leqq \frac{1}{r(M^n)}$$

(and $\frac{1}{r(M^n)} = 1$ by our assumption). (The proof of the left hand side inequality can be found in $[T]$.) Thus, for $\rho \leqq \frac{1}{2}$

$$det(Dm(x)) \geqq \left(\frac{2}{3}\right)^n > 0 \ .$$

$$\text{(2.1.7)} \qquad \left(\frac{2}{3}\right)^n vol_n(M_\rho) \leqq \int_{M_\rho} det(Dm(x))dx = 2\int_{M^n} dy = 2\ vol_n(M^n) \ .$$

Here "dx" denotes the volume element of M_ρ at some point $x \in M_\rho$ and "dy" denotes the volume element of M^n at $y \in M^n$. Using (2.1.6) we obtain the necessary upper bound for $vol_{n+1}(N_{1/2}(M^n))$:

$$\text{(2.1.8)} \qquad vol_{n+1}(N_{1/2}(M^n)) \leqq C_4(n) vol_n(M^n)$$

for some $C_4(n)$ depending only on n.But there exists a set of at least $[diam(M^n)]$ pairwise disjoint open balls in $\mathbb{R}^{n+1}$ of radius $\frac{1}{4}$ centered at some points of M^n. Since all such balls are subsets of $N_{1/2}(M^n)$,

$$vol_{n+1}(N_{1/2}(M^n)) \geqq [diam(M^n)] \left(\frac{1}{4}\right)^{n+1} v_{n+1} \ .$$

Lemma 2.1.1 implies that $diam(M^n) \geqq 2r(M^n) = 2$. Hence $[diam\ (M^n)] > \frac{2}{3}\ diam(M^n)$. Thus,

$$\text{(2.1.9)} \qquad vol_{n+1}(N_{1/2}(M^n)) \geqq \frac{2}{3} \left(\frac{1}{4}\right)^{n+1} v_{n+1} diam(M^n)$$

But (2.1.8) and (2.1.9) immediately imply (2.1.2). □

Remark 1. The lemma holds also for $C^{1,1}$-smooth compact hypersurfaces. The proof is essentially the same, but the following two modifications should be made. In the proof of part a) the statement that $vol(N_\rho(M^n))$ is a polynomial of degree $n+1$ of ρ is correct, but we cannot refer to the Weyl's tube formula. The proof of this statement for $C^{1,1}$-smooth hypersurfaces follows, for example, from a result of Federer [10], Theorem 5.6. In the proof of part b) the map m will not be differentiable but only Lipschitz. The well-known Rademacher's theorem implies that this map is differentiable almost everywhere. Thus, the equalities in the right hand side of (2.1.7) are still valid. Estimating $det(Dm(x))$ in smooth points of $m(x)$ exactly as it was done in the proof of the lemma we see that the inequality in the left hand side of (2.1.7) holds also. The remaining part of the proof of Lemma 2.1.4 in the $C^{1,1}$-smooth case does not differ from the proof of Lemma 2.1.4 in the C^2-smooth case.

Remark 2. Let M^n be a C^2-smooth compact connected submanifold of $\mathbb{R}^{n+k}$. Then the proof of Lemma 2.1.4 (a) can be straightforwardly generalized to prove that for an appropriate constant $C_1(n,k)$

$$\kappa_v(M^n) < C_1(n,k)\kappa_d(M^n)^{1+\frac{k}{n}}.$$

It seems highly plausible that the proof of Lemma 2.1.4(b) can also be generalized in order to prove that $\kappa_d(M^n) < C_2(n,k)\kappa_v^n(M^n)$. (In (2.1.7) it will be necessary to use the coarea formula (cf. [8], Theorem 13.4.2, instead of the integration by parts).

2.2. ε-approximation

Here we prove some properties of ε-approximation. The relation "A ε-approximates B" is neither symmetric nor transitive. However, there are some weak analogs of symmetry and transitivity of this relation (Proposition 2.2.3). Proposition 2.2.4 states that the topology of the space of submanifolds $M_n(\mathbb{R}^{n+1})$ coincides with a topology defined using the notion of ε-approximation.

We start from the proposition establishing a possibility to connect two compact hypersurfaces which are zero sets of two functions f_1 and f_2 by an isotopy passing through zero sets of functions $tf_1+(1-t)f_2$, $(t\in[0,1])$, under some conditions on f_1 and f_2. Under the same conditions this proposition provides an upper bound for the d-crumpledness of the zero set of the function $tf_1+(1-t)f_2$ for any $t\in(0,1)$.

Proposition 2.2.1. Let $f_1,\ f_2:\mathbb{R}^{n+1}\to\mathbb{R}$ be C^2-functions such that 1) f_1 and f_2 have non-empty zero sets $Z(f_1)$ and $Z(f_2)$ and $grad\ f_i\neq 0$ on $Z(f_i)$ $(i\in\{1,2\})$; 2) f_1 and f_2 have the same sign at all points x such that $|x|$ is large enough; 3) $Z(f_i)$ is a compact hypersurface contained in a ball B $(i\in\{1,2\})$; 4) $Z(f_2)$ δ-approximates $Z(f_1)$, where

$$\delta<\delta_0\overset{def}{=}\frac{1}{3}\min_{i\in\{1,2\}}\frac{\min_{x\in Z(f_i)}|grad\ f_i(x)|}{\max_{x\in B}||Hess\ f_i(x)||}\quad , \tag{2.2.1}$$

Then

A) For any $t \in (0,1)$ the zero set $Z((1-t)f_1+tf_2)$ of the function $(1-t)f_1+tf_2$ is a smooth hypersurface. Moreover, $\kappa((1-t)f_1+tf_2) \leqq \frac{2\ diam(B)}{\delta_0}$ (recall that $\kappa(f) = \max_{x\in Conv(Z(f))} \|Hessf\|\ diam(Z(f))/\min_{x\in Z(f)} |grad(f)|$);

B) For any $t \in [0,1]$

$$\kappa_d(Z((1-t)f_1+tf_2)) \leqq \frac{4}{\delta_0}\, diam\ B \quad .$$

Proof. We shall assume for simplicity that $Z(f_1)$ is connected. The general case follows easily from this particular case. Consider an open δ-neighborhood U of $Z(f_1)$. By the virtue of Lemma 2.1.3 $r(Z(f_1)) > \delta$. Therefore, $Z(f_1)$ is a deformation retract of U. Hence Jordan's theorem implies that $\mathbb{R}^{n+1}\setminus U$ contains exactly two connected components. Denote the unbounded connected component by C_1 and the bounded one by C_2. Note that the assumption 4) of the lemma implies that $Z(f_2)\subset U$. The assumption 2) implies that f_1 and f_2 are of the same sign in C_1. Without any loss of generality we can assume that f_1 and f_2 are positive on C_1. The assumption 1) implies that f_1 is negative on C_2.

Fix an arbitrary point $x\in Z(f_1)$. Denote the unit outer normal to $Z(f_1)$ by $n(x)$ and the interval of the normal line to $Z(f_1)$ at x $[x-\delta\ n(x),\ x+\delta\ n(x)]$ by γ. Note that γ connects C_1 and C_2. The assumption 4) of the lemma implies that f_2 vanishes in exactly one point of γ. Denote this point by y. By the virtue of assumption 1) f_2 changes its sign at y. Hence f_2 is negative on C_1. Now for any $t\in(0,1)$ we are able to conclude that the function $(1-t)f_1+tf_2$ is positive everywhere on C_1 and negative on C_2. Thus, $Z((1-t)f_1+tf_2)\subset U\subset B$. Moreover, we are able to conclude that $(1-t)f_1+tf_2$ has at least one zero in the interval γ. Now we are going to prove a uniform positive lower bound for the directional derivative of $(1-t)f_1+tf_2$ in the direction $n(x)$ on γ. The existence of such bound will immediately imply that $(1-t)f_1+tf_2$ vanishes in exactly one point x_t of γ and that $Z((1-t)f_1+tf_2)$ is smooth in the neighborhood of x_t. Since x is an arbitrary point of $Z(f_1)$ this will imply that $Z((1-t)f_1+tf_2)$ is a smooth hypersurface.

Let x_0 be an arbitrary point of γ. It follows from the formula (2.2.1) that

$$|(grad\ f_1(x_0), n(x))| \geqq \frac{2}{3}|grad\ f_1(x_0)| \ .$$

It follows from the definition of ε-approximation and from Lemma 2.1.1 that $|(grad\ f_2(y), n(x))| \geqq 1/2\ |grad\ f_2(y)|$. Now we can write that

$$\begin{aligned}&|(grad\ f_2(x_0), n(x))| = |(grad\ f_2(y), n(x)) -\\ &-(grad\ f_2(y) - grad\ f_2(x_0), n(x))| \geqq \frac{1}{2}|grad\ f_2(y)| -\\ &-\frac{1}{3}\ |grad\ f_2(y)| = \frac{1}{6}|grad\ f_2(y)| \quad .\end{aligned}$$

Note that because of the assumption and 2) $(grad\ f_1(x_0), n(x))$ and $(grad\ f_2(x_0), n(x))$ are of the same sign. Thus,

$$(2.2.2)\qquad \begin{aligned} &|t\ grad\ f_2(x_0) + (1-t) grad\ f_1(x_0)| \geqq \\ \geqq\ &|(t\ grad\ f_2(x_0) + (1-t) grad\ f_1(x_0), n(x))| \geqq \\ \geqq\ &\frac{1}{6}\,(t \min_{x\in Z(f_2)} |grad\ f_2(x)| + (1-t) \min_{x\in Z(f_1)} |grad\ f_1(x)|) \quad . \end{aligned}$$

But $((1-t)grad\ f_1(x_0) + tgrad\ f_2(x_0), n(x))$ is the value of the directional derivative of $(1-t)f_1 + tf_2$ in the direction of $n(x)$ at x_0. Since x_0 is an arbitrary point of γ and x is arbitrary point of $Z(f_1)$ we can conclude from here, as it was mentioned before, that $Z((1-t)f_1 + f_2)$ is a smooth hypersurface. Note that (2.2.2) implies, in addition, the required upper bound for $\kappa(Z((1-t)f_1 + tf_2))$. This completes the proof of part A) of the lemma.

Part B) follows immediately from A) and Lemma 2.1.3. □

Remark. Let $f : \mathrm{IR}^{n+1} \to \mathrm{IR}$ be a $C^{1,1}$-smooth function and $B \subset \mathrm{IR}^{n+1}$ be a closed ball. Denote the minimal constant C such that for any $x, y \in B$

$$|grad\ f(x) - grad\ f(y)| \leqq\ C|x-y|$$

by $C(f, B)$. Note that the Proposition 2.2.1 holds also for $C^{1,1}$ functions f_1, f_2 if instead of the inequality (2.2.1) one requires that

$$(2.2.1')\qquad \delta < \delta_0 \overset{def}{=} \frac{1}{3} \min_{i\in\{1,2\}} \frac{\min_{x\in Z(f_i)} |grad\ f_i(x)|}{C(f_i, B)} \quad .$$

The proof goes on exactly as the proof of original Proposition 2.2.1.

Fix once and for all an arbitrary C^∞ function $\psi_0 : \mathrm{IR} \to \mathrm{IR}$ which has the following properties:

$$1)\qquad \psi_0(x) = \begin{cases} -1 & \text{if}\quad x \in (-\infty, -1]\ , \\ x & \text{if}\quad x \in [-\frac{1}{2},\ \frac{1}{2}]\ , \\ 1 & \text{if}\quad x \in [1, +\infty)\ \ , \end{cases}$$

2) ψ_0 is strictly increasing in the interval [-1,1].

Let M^n be a compact $C^{1,1}$-smooth hypersurface and $\sigma : \mathrm{IR}^{n+1} \setminus M^n \to \mathrm{IR}$ denote the function which is identically equal to one in the unbounded connected component of M^n and to minus one in the union of bounded connected components of $\mathbb{R}^{n+1} \setminus M^n$. Let for any $r > 0$ $N_r(M^n)$ denote the open neighbourhood of M^n in $\mathbb{R}^{n+1}$ of radius r. We define *the characteristic function* $\chi_{M^n} : \mathbb{R}^{n+1} \to \mathbb{R}$ of M^n by the formula

$$\chi_{M^n}(x) = \begin{cases} \sigma(x), & \text{if } x \in \mathbb{R}^{n+1} \setminus N_{r(M^n)/2}(M^n) \\ \psi_0\left(\sigma(x)\frac{dist(x,M^n)}{r(M^n)/2}\right), & \text{if } x \in N_{r(M^n)/2}(M^n) \setminus M^n \\ 0\ , & \text{if } x \in M^n\ . \end{cases}$$

Now let M^n be C^2-smooth. It follows from the construction of χ_{M^n} that it has the following properties:

1) χ_{M^n} is a C^2 function,
2) The zero set of χ_{M^n} coincides with M^n ,
3) If $x \in N_{r(M^n)/4}(M^n)$, then $|grad\ \chi_{M^n}(x)| = 2/r(M^n)$. For any x $|grad\ \chi_{M^n}(x)| \leqq const_2/r(M^n)$.
4) For any $x \in \mathbb{R}^{n+1}$

$$\|Hess\ \chi_{M^n}(x)\| \leqq \frac{const_3}{r^2(M^n)}\ ,$$

where $const_2$, $const_3$ are some absolute constants (depending only on the choice of the function ψ_0).

Properties 1), 2) are obvious from the construction of χ_{M^n}. In order to prove property 3) let us differentiate χ_{M^n}: For any $x \in N_{r(M^n)}(M^n)$

$$(*) \qquad grad\ \chi_{M^n}(x) = (2/r(M^n))\psi_0'(\sigma(x)\frac{dist(x,M^n)}{r(M^n)/2})n(\pi(x)),$$

where $\pi(x)$ is the closest to x point of M^n, and $n(\pi(x))$ denotes the outer unit normal to M^n at $\pi(x)$. (And for any x outside $r(M^n)/2$-neighborhood of M^n $grad\ \chi_{M^n} = 0$.) Since $|\psi_0'|$ is uniformly bounded in $\mathbb{R}^{n+1}$, this formula for $grad \chi_{M^n}$ immediately implies property 3). In order to derive property 4) we shall differentiate once more. Choose in a neighborhood of a point $x \in N_{r(M^n)/2}(M^n)$ an orthonormal coordinate system $x_1, \ldots, x_{n+1}$ such that x_1-axis is directed along the outer normal to M^n at $\pi(x)$. Denote by P the subspace spanned by $x_2, \ldots, x_{n+1}$. Observe that

$$D_P[dist(x,M^n)\sigma(x)] = D_P dist(x,M^n) = 0$$

for any x (even if $x \in M^n$). Note that

$$D_{x_1}\pi(x) = 0.$$

Hence

$$D_P grad\ \chi_{M^n}(x) = \frac{2}{r(M^n)}\psi_0'(\sigma(x)\frac{dist(x,M^n)}{r(M^n)/2})L(\pi(x))D_P\pi(x),$$

where $L(\pi(x))$ denotes the second fundamental form of M^n at point $\pi(x)$. Since $\pi(x) + dist(x, M^n)n(\pi(x)) = x$

$$D_P\pi(x) = (E + dist(x, M^n)L(\pi(x)))^{-1}.$$

But $\|L(\pi(x))\| = \max_i |k_i(\pi(x))| \leqq \frac{1}{r(M^n)}$. Here, as before, $k_i(\pi(x))$ are principal curvatures of M^n at $\pi(x)$. Therefore $\|D_P\pi\| \leqq (1 - \frac{r(M^n)}{2}\frac{1}{r(M^n)})^{-1} = 2$. Since $|\psi_0'|$ is uniformly bounded, we see that for an appropriate constant $const_{3_a}$

$$\|D_P grad\chi_{M^n}\| \leqq \frac{const_{3_a}}{r^2(M^n)}.$$

On the other hand

$$D_{x_1} grad\chi_{M^n}(x) = ((2/r(M^n))^2\psi_0''(\sigma(x)\frac{dist(x,M^n)}{r(M^n)/2})n(\pi(x)).$$

Since $|\psi_0''|$ is uniformly bounded,

$$\|D_{x_1} grad\chi_{M^n}\| \leqq \frac{const_{3_b}}{r^2(M^n)}$$

for an appropriate constant $const_{3_b}$. This completes the proof of property 4).

Remark. For a $C^{1,1}$-smooth hypersurface M^n the characteristic function χ_{M^n} is only $C^{1,1}$-smooth but properties 2) and 3) hold with exactly the same proof. In the $C^{1,1}$ case property 4) holds in the following form: For any $x, y \in \mathbb{R}^{n+1}$

$$(**) \qquad |grad\ \chi_{M^n}(x) - grad\chi_{M^n}(y)| \leqq \frac{const_3}{r^2(M^n)}|x-y|\ .$$

This inequality can be easily proven using Rademacher's theorem and the argument used to prove property 4) for the C^2-smooth case. Indeed, it is sufficient to prove this inequality only for $x, y \in N_{2r(M^n)/3}(M^n)$. Let us start from the formula (*) above for the gradient of χ_{M^n}. Since by our assumption $n(x)$ is Lipschitzian and since by Theorem 4.8(8) from [10] π is Lipschitzian in $N_{2r(M^n)/3}(M^n)$, $grad\chi_{M^n}$ will be Lipschitzian, too. Moreover, Theorem 4.8(8) from [10] implies that for any $x, y \in N_{2r(M^n)/3}(M^n)$

$$|\pi(x) - \pi(y)| \leqq \frac{r(M^n)}{r(M^n) - \frac{2}{3}r(M^n)}|x-y| = 3|x-y|.$$

By the virtue of Rademacher's theorem π will be differentiable almost everywhere in $N_{2r(M^n)/3}(M^n)$. Exactly as in the C^2-smooth case one can show that $|Dn(z)| = \|L(z)\| \leqq \frac{1}{r(M^n)}$ for any $z \in M^n$ such that n is differentiable at z. By the virtue of Rademacher's theorem $n(z)$ will be differentiable almost everywhere on M^n, i.e. the set $\Omega_1 \subset M^n$ of points z such that n is not differentiable in z has measure zero. (We mean here the Lebesque measure on M^n, of course). It is not difficult to show that $\pi^{-1}(\Omega_1)$ will have measure zero. (Here we again consider the Lebesque measure on $\mathbb{R}^{n+1}$.) Thus, for almost all $x \in N_{2r(M^n)/3}(M^n)$ π will be differentiable at x and n will be differentiable at $\pi(x)$. For all these x $|D[n(\pi(x))]| \leqq \frac{3}{r(M^n)}$. Hence, for any $y, z \in N_{2r(M^n)/3}(M^n)$

$$|n(\pi(y)) - n(\pi(z))| \leqq \frac{3}{r(M^n)}|y - z|.$$

This inequality and (*) easily imply (**).

Lemma 2.2.2. There exists a constant $C > 0$ with the following property. Suppose $M_1^n,\ M_2^n \subset \mathbb{R}^{n+1}$ are compact connected C^2-hypersurfaces such that M_2^n ε-approximates M_1^n for some $\varepsilon < C \cdot \min\{r(M_1^n), r(M_2^n)\}$. Then there exists an isotopy $F : [0,1] \to M_n(\mathbb{R}^{n+1})$ between M_1^n and M_2^n such that for any $t \in [0,1]$

$$\kappa_v(F(t)) \leqq \theta(\max\{\kappa_v(M_1^n), \kappa_v(M_2^n)\}) \tag{2.2.3}$$

where $\theta(x) \overset{def}{=} (C_5(n)x^n + C_6(n))^{1+\frac{1}{n}}$ for some positive constants $C_5(n), C_6(n)$ depending only on n.

Proof. As it was noted above, there exist C^2 functions $\chi_{M_i^n}$, $(i \in \{1,2\})$, (the characteristic functions of M_i^n) vanishing only on M_i^n such that $|grad\ \chi_{M_i^n}(x)| = 2/r(M_i^n)$ for all $x \in M_i^n$ and $\|Hess\ \chi_{M_i^n}(x)\| \leqq \frac{const}{r^2(M_i^n)}$ for some universal constant $const > 1$ and for any $x \in \mathbb{R}^{n+1}$.

Applying Proposition 2.2.1 one can see that if

$$\varepsilon < \frac{1}{3\ const}\ \min\{r(M_1^n), r(M_2^n)\} \tag{2.2.4}$$

then $F : [0,1] \to M_n(\mathbb{R}^{n+1})$ defined as $F(t) = Z((1-t)\chi_{M_1^n} + t\chi_{M_2^n})$ is an isotopy and $\kappa_d(Z((1-t)\chi_{M_1^n} + t\chi_{M_2^n})) \leqq 12\ const\ \frac{diam\ M_i^n + 2\varepsilon}{min\{r(M_1^n), r(M_2^n)\}}$, $(i \in \{1,2\})$. Hence, (2.2.4) implies that

$$\kappa_d(Z((1-t)\chi_{M_1^n} + t\chi_{M_2^n})) \leqq 12\ const\ \max_{i\in\{1,2\}} \kappa_d(M_i^n) + 8\ . \tag{2.2.5}$$

But by Lemma 2.1.4

$$\kappa_d(M_i^n) < C_2(n)\kappa_v^n(M_i^n)$$

and

$$\kappa_v(Z((1-t)\chi_{M_1^n}+t\chi_{M_2^n}))<C_1(n)(\kappa_d(Z((1-t)\chi_{M_1^n}+t\chi_{M_2^n})))^{1+\frac{1}{n}}\ .$$

These formulae together with (2.2.5) imply that F is the necessary isotopy. □

Remark. Lemma 2.2.2 can be stated in the $C^{1,1}$ case exactly as in the C^2-smooth case. There is only one minor modification in the proof of this lemma: the characteristic functions $\chi_{M_i^n}$ vanishing on M_i^n will be only $C^{1,1}$-smooth and not C^2-smooth. The inequality $|grad\ \chi_{M_i^n}(x)-grad\ \chi_{M_i^n}(y)|\leqq\frac{const}{r^2(M_i^n)}|x-y|$ will hold instead of the inequality $\|Hess\ \chi_{M_i^n}\|\leqq\frac{const}{r^2(M_i^n)}$. However, one can apply the $C^{1,1}$ version of Proposition 2.2.1 and to carry out the proof exactly as it was done in the C^2-smooth case.

The next proposition states weakened versions of symmetry and transitivity properties of the relation "to be ε-approximating".

Proposition 2.2.3. Let n be fixed.

(a) There exist constants $\alpha, A>0$ such that for any two compact C^2-smooth hypersurfaces $M_1^n,\ M_2^n\subset \mathbb{R}^{n+1}$ such that M_1^n ε-approximates M_2^n, where $\varepsilon<\alpha\ \min\{r(M_1^n), r(M_2^n)\}$, M_2^n $(\varepsilon\kappa_d(M_2^n)A)$- approximates M_1^n .

(b) There exist constants $\beta, B>0$ such that for any three compact C^2-smooth hypersurfaces $M_1^n,\ M_2^n,\ M_3^n\subset\mathbb{R}^{n+1}$ and a positive $\varepsilon<\beta\ \min_{i\in\{1,2,3\}} r(M_i^n)$ the following statement is true. If M_1^n ε-approximates M_2^n and M_2^n ε-approximates M_3^n then M_1^n $(\varepsilon\kappa_d(M_1^n)B)$ - approximates M_3^n.

Proof. Properties 3) and 4) of the characteristic functions $\chi_{M_i^n}:\mathbb{R}^{n+1}\to\mathbb{R}$ of M_i^n imply that $|n(x)-n(y)|\leqq\frac{const}{r(M_i^n)}|x-y|$ for any $x,y\in M_i^n$ $(i\in\{1,2\})$ and an universal constant $const$. Here $n(x),\ n(y)$ denote unit normal vectors to M_i^n at x and y. Now the proposition follows from a quantitative version of the implicit function theorem (cf. [28], Proposition 3.2). We omit the calculations. □

The last lemma in this section describes the topology of the hypersurface space $M_n(\mathbb{R}^{n+1})$ in terms of ε-approximation. Everywhere below we shall use only the description of the convergence in this space provided by the statement (B) of the lemma and not the initial definition given above.

Lemma 2.2.4. Let $\{M_i^n\}_{i=1}^\infty$ be a sequence of compact C^2-smooth hypersurfaces in $\mathbb{R}^{n+1}$ and M_∞^n be also a compact smooth hypersurface in $\mathbb{R}^{n+1}$. The following two statements are equivalent:

(A) The sequence $\{M_i^n\}_{i=1}^\infty$ converges in $M_n(\mathbb{R}^{n+1})$ to a hypersurface M_∞^n. (Recall, that $M_n(\mathbb{R}^{n+1})$ is $\bigsqcup_{M^n} Emb(M^n,\mathbb{R}^{n+1})/Diff(M^n)$, where $Emb(M^n,\mathbb{R}^{n+1})$ and $Diff(M^n)$ are equipped with C^1-topology and the disjoint union is taken over all compact smooth manifolds embeddable in $\mathbb{R}^{n+1}$).

(B) For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $k \geqq N$ M^n_k ε-approximates M^n_∞.

This lemma immediately follows from its generalization for $C^{1,1}$-smooth hypersurfaces:

Lemma 2.2.4 for $C^{1,1}$ case. For any sequence $\{M^n_i\}_{i=1}^\infty \subset M_n^{(1,1)}(\mathbb{R}^{n+1})$ and a hypersurface $M^n_\infty \in M_n^{(1,1)}(\mathbb{R}^{n+1})$ the following two statements are equivalent:

(A) The sequence $\{M^n_i\}_{i=1}^\infty$ converges in $M_n^{(1,1)}(\mathbb{R}^{n+1})$ to the hypersurface M^n_∞.

(B) For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $i > N$, M^n_i ε-approximates M^n_∞.

Proof. To prove the implication $(B) \Longrightarrow (A)$ we shall construct for all sufficiently large i C^1-smooth diffeomorphisms $g_i : M^n_\infty \longrightarrow M^n_i$ such that $incl_i \circ g_i$ converges in C^1 topology to $incl_\infty$. (Here $incl_i$ and $incl_\infty$ denote identical maps of M^n_i and M^n_∞ into $\mathbb{R}^{n+1}$.) In the C^2-smooth case we can take for g_i the map $x \longrightarrow x^*(x) \in M^n_i$ from the definition of ε-approximation given at the end of Section 1. But if the hypersurfaces are assumed to be only $C^{1,1}$-smooth, this map will not be C^1-smooth in general. However we can proceed as follows.

Although the unit outer normal field $n(x)$ on M^n_∞ is not C^1-smooth, a result of Whitney [42], ch. IV, Theorem 10A, implies that it can be arbitrarily closely approximated by a C^1-smooth vector field on M^n_∞ such that the corresponding direct lines fill an open neighborhood U of M^n_∞ in $\mathbb{R}^{n+1}$ in a one-to-one way. Consider a family of these approximating vector fields $n_\varepsilon(x)$, parametrized by a positive index ε, uniformly converging to the outer unit normal field $n(x)$ on M^n_∞ when $\varepsilon \longrightarrow 0$. It is easy to prove that for sufficiently large i we can find a sufficiently small $\varepsilon(i)$ such that the direct line passing through x in the direction $n_{\varepsilon(i)}(x)$ has a unique point of intersection $m_i(x)$ with M^n_i in some open neighborhood $U^*(i) \subset U$ of M^n_∞. Let $g_i(x){=}m_i(x)$. It is easy to see that if $\varepsilon(i)$ are small enough then $\{g_i\}$ will be the necessary sequence of C^1-smooth diffeomorphisms.

To prove the implication $(A) \Longrightarrow (B)$ fix some positive $\varepsilon < r(M^n_\infty)$ and note that (A) implies the existence of a sequence of C^1-smooth maps $g_i : M^n_\infty \to M^n_i$ such that $\|incl_i \circ g_i - incl_\infty\|_{C^1} \to 0$, where $incl_i$, $incl_\infty$ denote identical inclusions of M^n_i, M^n_∞ into $\mathbb{R}^{n+1}$. For all sufficiently large i, g_i is a C^1 diffeomorphism. We are going now to prove (B) assuming the existence of such a sequence $\{g_i\}$.

First, we prove that for any sufficiently large i and any $x_0 \in M^n_\infty$ there exists a point $x_i \in M^n_i$ on the normal line to M^n_∞ at x_0 such that $|x_i - x_0| \leqq \varepsilon$. Let T_{x_0} be a ball of a small radius ρ_{x_0} around x_0 in the tangent hyperplane H_{x_0} to M^n_∞ at x_0. If ρ_{x_0} is sufficiently small, then the map $\pi : T_{x_0} \to M^n_\infty$ mapping any point of T_{x_0} to its closest in M^n_∞ is a correctly defined continuous map. Moreover, for any $\delta >$

0 one can choose ρ_{x_0} sufficiently small to ensure that $|\pi(x)-\pi(y)| \leqq (1+\delta)|x-y|$ for any $x, y \in T_{x_0}$. (This statement immediately follows from Theorem 4.8(8) in [10] but it is not difficult to verify it straightforwardly). Choose ρ_{x_0} small enough to ensure that $|\pi(x)-\pi(y)| \leqq \frac{4}{3}|x-y|$ and to ensure, in addition, that for any $x, y \in M^n_\infty$ $|(x-\pi(x))-(y-\pi(y))| \leqq \frac{1}{3}|x-y|$. (It is easy to check that this is possible, too.) Denote the orthogonal projection onto H_{x_0} by P_{x_0}. Note that if for some $x_{(i)} \in T_{x_0}$ $P_{x_0}(g_i(\pi(x_{(i)}))) = x_0$ then $g_i(\pi(x_{(i)}))$ is on the normal to M^n_∞ at x_0. Thus, we should prove that $x_0 \in P_{x_0}(g_i(\pi(T_{x_0})))$ for all sufficiently large i. Note that for any $x \in T_{x_0}$ $P_{x_0}(g_i(\pi(x))) = x + \psi(x)$ where $\psi(x) \overset{df}{=} P_{x_0}(g_i(\pi(x)))-x = P_{x_0}(g_i(\pi(x))-x) = P_{x_0}(g_i(\pi(x))-\pi(x))+P_{x_0}(\pi(x)-x)$. But for sufficiently large i $|(g_i(\pi(y))-\pi(y))-(g_i(\pi(x))-\pi(x))| \leqq \frac{1}{4}\,|\pi(y)-\pi(x)| \leqq \frac{1}{3}\,|x-y|$. (Here, of course, we used the fact that the sequence of maps $\{incl_i \circ g_i\}$ of M_n^∞ into $\mathbb{R}^{n+1}$ converges in C^1-topology to the identical inclusion.) Since by the choice of ρ_{x_0} $|(\pi(x)-x)-(\pi(y)-y)| \leqq \frac{1}{3}|x-y|$, then for any $x, y \in M^n_\infty$ $|\psi(x)-\psi(y)| \leqq \frac{2}{3}|x-y|$. Let i be sufficiently large to ensure that $|g_i(x_0)-x_0| \leqq \frac{\rho_{x_0}}{6}$. Hence, $|\psi(x_0)| = |P_{x_0}(g_i(\pi(x_0)))-x_0| = |P_{x_0}(g_i(x_0)-x_0)| \leqq \frac{\rho_{x_0}}{6}$. Thus, the quantitative Lipschitz inverse function theorem of Hirsch and Pugh ([16], 1.3-1.7) (see also Abraham, Marsden and Ratiu [1], 2.5K(iii)) implies that $x_0 \in P_{x_0}(g_i(\pi(T_{x_0})))$, and, thus, the existence of a point $x_i = g_i(\pi(x_{(i)}))$ of M^n_i on the normal to M^n_∞ at x_0. (The used Lipschitz inverse function theorem was proven in [16] using the contraction mapping principle very similarly to the standard modern proof of the inverse function theorem). Taking ρ_{x_0} small enough one can ensure that for sufficiently large i $|x_i - x_0| \leqq \varepsilon$.

Note that $|x_0-\pi(x_{(i)})| = |\pi(x_0)-\pi(x_{(i)})| \leqq \frac{4}{3}\rho_{x_0}$. For large i the unit normal of M^n_i at $g_i(\pi(x_{(i)}))$ will be arbitrarily close to the unit normal to M^n_∞ at $\pi(x_{(i)})$. Now the $C^{1,1}$-smoothness of M^n_∞ implies that by an appropriate initial choice of ρ_{x_0} we can ensure that for all sufficiently large i the unit normal to M^n_i at x_i will be as close as necessary to the unit normal to M^n_∞ at x_0.

We want to prove now by contradiction that for sufficiently large i there is no other point of M^n_i ε-close to x_0 on the normal to M^n_∞ at x_0. Let $I \subset \mathbb{N}$ be an infinite set of indices i such that for any $i \in I$ there exist two distinct points $x_1(i)$ and $x_2(i)$ of M^n_∞ such that the points $g_i(x_1(i))$, $g_i(x_2(i))$ are in $B_\varepsilon(x_0) \cap$ the normal line to M^n_∞ at x_0. Then, denoting by $n(x_0)$ the unit normal to M^n_∞ at x_0, we have

$$|(g_i(x_1(i)) - g_i(x_2(i)), n(x_0))| = |g_i(x_1(i)) - g_i(x_2(i))| \quad .$$

When $i \to +\infty$,

$$|(g_i(x_1(i)) - g_i(x_2(i)), n(x_0))| = |x_1(i) - x_2(i)|(1 + o(1)) \ .$$

Hence

$$|(x_1(i) - x_2(i), n(x_0))| = |x_1(i) - x_2(i)|(1 + o(1)) \quad .$$

Since $\varepsilon < r(M^n_\infty)$, $\lim_{i\in I; i\to+\infty} x_1(i) = \lim_{i\in I; i\to+\infty} x_2(i) = x_0$. Indeed, otherwise a subsequence of one of the sequences $\{x_1(i)\}_{i\in I}$, $\{x_2(i)\}_{i\in I}$ converges to a point $x^0 \in M^n_\infty$ such that $x^0 \neq x_0$. The normal to M^n_∞ at x_0 will pass through x^0 and $|x_0 - x^0| \leqq \varepsilon$. Since by our assumption $\varepsilon < r(M^n_\infty)$ we obtain the necessary contradiction with the assumption that there exists a limit point x^0 different from x_0.

Now the smoothness of M^n_∞ implies that when $i \to +\infty$ (remaining in I)

$$|(x_1(i) - x_2(i), n(x_0))| = |x_1(i) - x_2(i)| \cdot o(1) \quad .$$

Thus

$$|x_1(i) - x_2(i)| \cdot o(1) = |x_1(i) - x_2(i)|(1 + o(1)) \ .$$

We got a contradiction with the assumption that $x_1(i) \neq x_2(i)$. Hence, the uniqueness for sufficiently large i is proven.

For sufficiently large i the continuity of the map $x_0 \to x_i$ mapping a point x_0 of M^n_∞ to the closest point of intersection of the normal to M^n_∞ at x_0 with M^n_i can be proven straightforwardly using the already proven uniqueness of the point of intersection in the open ε-neighborhood of M^n_∞. This uniqueness implies also that the map $x_0 \longrightarrow x_i$ is surjective since the inverse map $x_i \longrightarrow x_0$ mapping a point x_i of M^n_i to the closest point of M^n_∞ is everywhere defined. By the virtue of Theorem 4.8(4) of [10] the map $x_i \to x_0$ is continuous. □

Remark. This lemma is true for sequences of compact $C^{1,1}$-smooth submanifolds of Euclidean spaces of arbitrary codimension (and not only of codimension one). The proof is very similar to the proof of the codimension one case given above.

2.3. Algebraic approximation of hypersurfaces and of isotopies

Let $M^n_1, M^n_2 \subset \mathbb{R}^{n+1}$ be compact non-singular algebraic hypersurfaces in $\mathbb{R}^{n+1}$. We show in this section that if M^n_1 and M^n_2 are isotopic then they are D-rigidly isotopic for some large enough D. Moreover, one can effectively majorize this D in terms of degrees of M^n_1 and M^n_2 and of the maximal crumpledness of hypersurfaces during an isotopy between M^n_1 and M^n_2 (Proposition 2.3.2). We start from an effective version of the well-known result (cf. [33] or [17]) stating the existence of an algebraic hypersurface arbitrary closely approximating a given smooth compact hypersurface.

Proposition 2.3.1. For any n there exist recursive functions $\nu : \mathbb{N} \to \mathbb{N}$; $\mu : \mathbb{N} \to \mathbb{N}$ with the following property. Let $M^n \subset \mathbb{R}^{n+1}$ be a compact C^2-smooth hypersurface. For any positive $\varepsilon < r(M^n)$ there exist $d \in \mathbb{N}$ and a smooth compact algebraic hypersurface A^n ε-approximating M^n such that A^n is the zero set of a polynomial p of degree d and

$$d \leqq \nu\left([\frac{diam\ M^n}{\varepsilon}]\right) \ ,$$

$$\kappa(p) \equiv \frac{\max_{x\in Conv(A^n)} ||Hess(p)||}{\min_{x\in A^n} |grad\ p|}\ diam(A^n) \leqq \mu([\kappa_d(M^n)])\ .$$

Remark. Since by Lemma 2.1.3 $\kappa_d(A^n) \leqq 2\kappa(p)$ one has also effective upper bounds for $\kappa_d(A^n)$ and for $\kappa_v(A^n)$. Note that these bounds do not depend on ε!

Proof. Note that making the transformation $x \longrightarrow x/diam(M^n)$ of $\mathbb{R}^{n+1}$ we can reduce the task of proving the proposition to proving its particular case, when $diam(M^n) = 1$. So, assume that $diam(M^n) = 1$.

To prove the proposition we follow the well-known proof of the fact that any compact smooth hypersurface can be ε-approximated by an algebraic hypersurface (cf. [33], [17]). This proof uses the Weierstrass theorem which states that it is possible to approximate any continuous function on a compact subset of Euclidean space by polynomials. We shall use a constructive version of this theorem (for example, Jackson's theorem) here and, consequently, we shall obtain some effective estimates of d and of $\max_{x\in Conv(M^n)} ||Hess\ p||/\min_{x\in M^n} |grad\ p|$ for some approximating polynomial p.

In the first stage of the proof we approximate the characteristic function χ_{M^n} by a polynomial p_1 in a ball B of radius 4 such that the concentric ball B_2 of radius 2 contains $N_{r(M^n)}(M^n)$. (The definition of χ_{M^n} was given in Section 2.2.) Let $\varepsilon_0 = \min\{\frac{r(M^n)}{4}, \varepsilon\}$. We want to find recursive functions ν_1, μ_1 and a polynomial p_1 such that

1) $Z(p_1) \cap B$ is a smooth manifold ε_0-approximating M^n.
2) $deg\ p_1 \leqq \nu_1\left(\left[\frac{diam(M^n)}{\varepsilon_0}\right]\right)$;
3) $\kappa(p_1) = \frac{\max_{x\in Conv(Z(p_1))} ||Hess\ p_1(x)||}{\min_{x\in Z(p_1)} |grad\ p_1(x)|}\ diam(Z(p_1)) \leqq \mu_1([\kappa_d(M^n)])$

Let a positive $\delta \leqq 1/3$ be such that $||p - \chi_{M^n}||_{C^1(B)} \leqq 2\delta$ implies that $Z(p) \cap B$ ε_0-approximates M^n. Obviously δ can be effectively estimated in terms of ε_0.

To make the estimate in 3) independent of ε we are going to approximate χ_{M^n} by a polynomial in $C^2(B)$ norm. Thus, to obtain an upper bound for the degree and for the absolute values of all coefficients of the approximating polynomial it is necessary to have an upper bound for the absolute values of third order partial derivatives of the approximated function. But actually we intend to approximate M^n by an algebraic hypersurface only in C^1-topology. This suggests the idea to approximate χ_{M^n} by a polynomial in $C^2(B)$ norm in two stages. First, for any $\lambda > 0$ we consider a function $\chi_\lambda(x) = \left(\frac{\lambda}{\pi}\right)^{\frac{n+1}{2}} \int_{\mathbb{R}^{n+1}} \chi_{M^n}(y)\exp(-\lambda(x-y, x-y))dy$. By a standard argument one can show that it is possible to choose effectively λ_0 large enough to ensure that 1) $||\chi_{\lambda_0} - \chi_{M^n}||_{C^1(B)} < \delta/2$; 2) For any $x \in B$ $||Hess\ \chi_{\lambda_0}|| \leqq \frac{const_3}{r^2(M^n)}$, where $const_3$ is the constant defined above in the text of Property 4) of χ_{M^n} ; 3) For any $x \in B$ $||D^3\chi_{\lambda_0}(x)||$ can be majorized by a increasing recursive function of $\left[\frac{1}{\delta}\right]$ and of $\frac{1}{r(M^n)}$. In the second stage we approximate χ_{λ_0} by a polynomial p_1 in B such that $||\chi_{\lambda_0} - p_1||_{C^2(B)} \leqq \delta/2$.

Jackson's theorem (cf. [36]) provides an explicit upper bound for the degree of a polynomial with required properties. Since the approximating polynomials are constructed in the proof of Jackson's theorem as convolutions with an explicit integral kernel, the proof can be used to derive an effective upper bound $U(p_1)$ for the sum of absolute values of the coefficients of p_1.

Now the connected components of the zero set of p_1 which are inside B approximate M^n as we wish. However p_1 can vanish outside B, too. In order to get rid of the connected components of the zero set of p_1 outside B we would like to replace p_1 by a polynomial p such that 1) $\|p_1 - p\|_{C^2(B_2)} < \delta$, 2) $p(x) > 0$ for $x \notin B$. The classical trick here is to add to p_1 a polynomial of the form $\frac{(x-b,x-b)}{c}^m$, where b denotes the center of B and c,m are some appropriate constants. Recall that by our assumption $diam\ (M^n) = 1$, the radius of B is equal to four, and the radius of B_2 is equal to two. Now note that for a sufficiently large integer m, which can be effectively estimated, $\left(\frac{(x-b,x-b)}{9}\right)^m > |p_1(x)|$, when $x \notin B$, and $\|\left(\frac{(x-b,x-b)}{9}\right)^m\|_{C^2(B_2)} < \delta$, where b denotes the center of B. Obviously, the polynomial $p_1(x) + \left(\frac{(x-b,x-b)}{9}\right)^m$ is positive in $B \setminus B_2$. Thus, we can take for $p(x)$ the polynomial $p_1(x) + \left(\frac{(x-b,x-b)}{9}\right)^m$. □

Remark. Actually we have proved that $\mu([\kappa_d(M^n)])$ majorizes not only $\kappa(p)$ but even $\frac{\max_{x\in B}\|Hess\ p(x)\|}{\min_{x\in Z(p)}|grad\ p(x)|}\ diam(B)$, where B is a ball containing both M^n and A^n. This remark will enable us to apply below Proposition 2.2.1 for the hypersurfaces M^n and A^n (regarded as zero sets of χ_{M^n} and p, correspondingly).

One can prove that if there exists an isotopy passing through hypersurfaces of d-crumpledness $\leqq x$ between two algebraic hypersurfaces M_1^n, M_2^n then there exists an $\varphi_n([x], D)$- rigid isotopy between M_1^n and M_2^n, where φ_n is a increasing recursive function. We prove here a weaker result. It is a corrected version of Proposition 4.1(iii) in [26].

Proposition 2.3.2. For any n there exists a recursive function $\varphi_n : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the following property. Let $M_1^n,\ M_2^n \subset \mathbb{R}^{n+1}$ be compact C^2 smooth algebraic hypersurfaces of degree $\leqq D$. Suppose that M_i^n is the zero set of a polynomial p_i of degree D such that $\kappa(p_i) < +\infty (i \in \{1, 2\})$. Suppose that there exists an isotopy connecting M_1^n and M_2^n passing only through hypersurfaces of d-crumpledness $\leqq x$. Then there exists an $\varphi_n([x], D, [\max_{i\in\{1,2\}} \kappa(p_i)])$-rigid isotopy between M_1^n and M_2^n.

Proof. Let $F : [0, 1] \to M_n(\mathbb{R}^{n+1})$ be an isotopy connecting M_1^n and M_2^n such that for any $t \in [0, 1]$ $\kappa_d(F(t)) \leqq x$. Since we can apply, if necessary, a continuously depending on t family of homothetic transformations of $\mathbb{R}^{n+1}$, we can assume that for any $t \in [0, 1]$ $diam(M_1^n) = diam(M_2^n) = diam\ F(t) = 1$. It follows from Proposition 2.3.1 that for any $t \in [0, 1]$, any positive $\delta < x$ and for some recursive functions $\nu : \mathbb{N} \to \mathbb{N}$ and $\mu : \mathbb{N} \to \mathbb{N}$ we can find an algebraic hypersurface $A_{t,\delta}^n$ δ-approximating $F(t)$ which is the zero set of a polynomial $p_{t,\delta}$

of degree not exceeding $\nu([\frac{1}{\delta}])$ such that *grad* $p_{t,\delta} \neq 0$ on $A^n_{t,\delta}$ and $\kappa(p_{t,\delta}) \leqq \mu([x])$. By the virtue of Lemma 2.1.3 $\kappa_d(A^n_{t,\delta}) \leqq 2\mu([x])$. Note that in general $p_{t,\delta}$ will not depend on t continuously. However, Lemmae 2.2.3, 2.2.4 imply that $A^n_{t_1,\delta}$ will $(2AB^2x^2\mu([x])\delta)$-approximate $A^n_{t_2,\delta}$ if δ and $|t_1-t_2|$ are small enough. Here A, B are constants defined in the text of Lemma 2.2.3. (The Lemma 2.2.3 (b) was applied here twice and Lemma 2.2.3(a) was applied once). Let $\varepsilon = 1/(3\mu([x]))$. Choosing δ to be small enough to ensure that $\lambda \overset{df}{\equiv} 2x^2\mu([x])AB^2\delta < \varepsilon$ we ensure that $A^n_{t_1,\delta}$ will ε-approximate $A^n_{t_2,\delta}$ if $|t_1 - t_2| < \tau$ for some $\tau > 0$. Note that although we have no positive lower bound for τ (see Remark after the proof of the proposition), we have an effective positive lower bound for δ (in terms of x, ε and the constants α, β, A and B defined in the text of Lemma 2.2.3). Hence , we can find an effective upper bound for degrees of polynomials $p_{t,\delta}$. Now let $0 = t_0, t_1, \cdots, t_m = 1$ be an increasing sequence such that for any i $t_{i+1} - t_i < \tau$. We can construct for any i an isotopy between $A^n_{t_i,\delta}$ and $A^n_{t_{i+1},\delta}$ by the following way. Multiplying by -1, if necessary, we can assume that all polynomials $p_{t,\delta}$ assume positive values at infinity. Thus, by the virtue of Proposition 2.2.1, we can connect $A^n_{t_i,\delta} = Z(p_{t_i,\delta})$ and $A^n_{t_{i+1},\delta} = Z(p_{t_{i+1},\delta})$ by a $\nu([\frac{1}{\delta}])$-rigid isotopy for $i \in \{1, \ldots, m-2\}$. If, in addition, we choose $\delta < \frac{1}{3\max_{i\in\{1,2\}}\kappa(p_i)}$, then we can also connect $Z(p_{t_0,\delta}) \equiv M^n_1$ with $Z(p_{t_1,\delta})$ and $Z(p_{t_{m-1},\delta})$ with $Z(p_{t_m,\delta}) \equiv M^n_2$ by $\max\{D, \nu([\frac{1}{\delta}])\}$ - rigid isotopies. This completes the proof of the proposition. $\square$

Remark. The absence of the information about parameter τ in the proof is due to the fact that we did not make any assumptions about geometric properties of the approximated isotopy regarded as a path in $M_n(\mathrm{IR}^{n+1})$. (Informally speaking, this isotopy can have arbitrary "length" and "speed".) But the main purpose of the proposition is just to show that we do not need this information!

3. Rigid isotopies and non-recursive functions.

3.1. Algorithmic properties of D-rigid isotopies.

The main result of this section is Theorem 3.1.2 stating the existence of an algorithm recognizing whether or not two given non-singular compact algebraic hypersurfaces are D-rigidly isotopic.

Let $p_1, \ldots, p_k$ be polynomials of degree $\leqq d$ such that for all $i \in \{1, \ldots, k\}$ the zero set $Z(p_i)$ of p_i is compact and non-empty, *grad* p_i and p_i do not vanish simultaneously at any point, and the Jacobi matrix $J(p_1, \ldots, p_k)$ has the maximal rank at any point of $Z(p_1) \bigcap Z(p_2) \bigcap \ldots \bigcap Z(p_k)$. We shall call such systems of polynomials *regular* systems. Fix some numeration of all $\binom{n+k+d}{d} k$ monomial coefficients in such systems. Consider set $\Phi_{n,k,d}$ of vectors of coefficients of all regular systems. We can regard $\Phi_{n,k,d}$ as a subset of $\mathrm{IR}^{k\binom{n+k+d}{d}}$. We call vectors from $\Phi_{n,k,d}$ *fixations* of corresponding regular complete intersections. Corresponding regular complete intersections of compact algebraic hypersurfaces of degree d are called here for brevity d-*CCIs*. Note that the fact that the zero set of a polynomial $p : \mathrm{IR}^{n+1} \longrightarrow \mathrm{IR}$, where $n > 0$, is a compact smooth hypersurface implies that the degree of p is even. Thus, $\Phi_{n,k,2m+1}$ coincides with $\Phi_{n,k,2m}$. Consider also a subset $\Phi^*_{n,k,d}$ of $\Phi_{n,k,d}$ formed by all vectors of coefficients of regular systems of polynomials p_i with the following property: The sum $p_i^{(d^*)}$ of all terms of (the maximal) degree $d^* = 2[\frac{d}{2}]$ of every polynomial p_i of the system vanishes only at zero. This informally means that all polynomials p_i are of degree which is exactly equal to d^*, and every one of them has no "hidden" singularity at infinity. Generalized Thom's separation lemma implies that the sets $\Phi_{n,k,d}$ and $\Phi^*_{n,k,d}$ are semi-algebraic subsets of the Euclidean space. Hence any of these sets has a finite number of connected components. (Moreover, there exists an algorithm yielding for any given triple n, k, d the list of presentations of all connected components of the sets $\Phi_{n,k,d}$ and $\Phi^*_{n,k,d}$.) Denote the number of connected components of $\Phi_{n,k,d}$ by $C_{n,k,d}$, and the number of connected components of $\Phi^*_{n,k,d}$ by $C^*_{n,k,d}$. There exists a canonical projection $\rho : \Phi_{n,k,d} \longrightarrow M_n(\mathrm{IR}^{n+k})$ which maps a fixation into the zero set of the corresponding system of polynomials determined by the fixation. Lemma 2.2.4 easily implies that the restriction of ρ on $\Phi^*_{n,k,d}$ is continuous. Hence, any continuous path in $\Phi^*_{n,k,d}$ corresponds to a d-rigid isotopy of the zero sets of respective regular systems of polynomials. However, this is not always true for a continuous path in $\Phi_{n,k,d}$. (The reason why it might be not true is the possible existence of the hidden singular point at infinity.) But, it is possible to show that $\Phi_{n,k,d}$

can be represented as a disjoint union of the finite number $C^*_{n,k,d}$ of connected semialgebraic sets $\Phi^{(j)}_{n,k,d}$, such that the following conditions are satisfied:

1) There exists an algorithm giving for any given n, k, d the list of presentations of the semialgebraic sets $\Phi^{(j)}_{n,k,d}$;

2) If $f_1, f_2 \in \Phi^{(j)}_{n,k,d}$ for some j then f_1 and f_2 can be connected by a continuous path $\gamma \subset \Phi^{(j)}_{n,k,d}$ such that $\rho(\gamma)$ is a rigid isotopy between $\rho(f_1)$ and $\rho(f_2)$.

Given any such representation we will call sets $\Phi^{(j)}_{n,k,d}$ *rigidly connected components* of $\Phi_{n,k,d}$.

Here we are going to construct (and then to apply) a specific representation of $\Phi_{n,k,d}$ as disjoint union of rigidly connected components only in the case $k = 1$. The construction in the general case is similar.

The general idea of the construction is to define rigidly connected components as unions of a connected component of $\Phi^*_{n,1,d}$ and an appropriate semialgebraic subset of $\Phi_{n,1,d} \setminus \Phi^*_{n,1,d}$. This idea is based on the observation that for any $f \in \Phi_{n,1,d}$ we can get rid of the possible singularity at infinity only slightly perturbing $\rho(f)$ by merely adding $\varepsilon(x_1^2 + \dots x_{n+1}^2)^{[d/2]}$ to the polynomial p_f corresponding to f, where $|\varepsilon|$ is sufficiently small and the sign of ε coincides with the sign of p_f at infinity. Thus, any of the rigidly connected components will be union of a connected component of $\Phi^*_{n,1,d}$ and a piece of $\Phi_{n,1,d} \setminus \Phi^*_{n,1,d}$ formed by all fixations which go to this connected component when we get rid of the singularity at infinity adding a small multiple of $(x_1^2 + \dots + x_{n+1}^2)^{[d/2]}$. Now let us be more formal.

For any fixation $f \in \Phi_{n,1,d}$ denote by p_f the polynomial such that its vector of coefficients is f. It follows from the definition of $\Phi_{n,1,d}$ that for all $x \in \mathrm{I\!R}^{n+1}$ with sufficiently large norm $p_f(x)$ is of the same positive or negative sign. Denote by σ_f $sign(p_f(x))$ at infinity. Let τ_* denote the vector of coefficients of the polynomial $(x_1^2 + \dots + x_{n+1}^2)^{[d/2]}$. We are going to show that it is possible to explicitly define a semialgebraic function α from $\Phi_{n,1,d}$ into $\mathrm{I\!R}$ assuming positive values, sufficiently small to ensure the following. The semialgebraic maps τ_t from $\Phi_{n,1,d}$ into $\mathrm{I\!R}^{\binom{n+1+d}{d}}$ defined for any $t \in [0, 1]$ as identical maps on $\Phi^*_{n,1,d}$ and by the formula

$$\tau_t(f) = f + t\sigma_f \alpha(f)\tau_*$$

for all $f \in \Phi_{n,1,d} \setminus \Phi^*_{n,1,d}$ satisfy the following conditions:

(a) For any f and $t > 0$, $\tau_t(f) \in \Phi^*_{n,1,d}$.

(b) The map $t \longrightarrow \rho(\tau_t(f))$, $(t \in [0,\ 1])$, is a rigid isotopy between $\rho(f)$ and $\rho(\tau_1(f))$.

(c) There exists an algorithm which for any given n, d produces a presentation of the graph of τ_1.

Assuming that these three conditions are satisfied one can easily see that

$$\Phi_{n,1,d} = \bigcup_{j=1}^{C^*_{n,1,d}} \tau_1^{-1}(\Phi^{*(j)}_{n,1,d}),$$

where $\{\Phi^{*(j)}_{n,1,d}\}_{j=1}^{C^*_{n,1,d}}$ is the set of all connected components of $\Phi^*_{n,1,d}$, is a representation of $\Phi_{n,1,d}$ as a disjoint union of rigidly connected components. (Indeed, condition 1) follows immediately from (c) and the Tarski-Seidenberg Theorem. To demonstrate 2) we can construct the required path γ between f_1 and f_2 as union of three paths, namely, of two paths connecting f_1 with $\tau_1(f_1)$ via $\tau_t(f_1)$ and f_2 with $\tau_1(f_2)$ via $\tau_t(f_2)$, $(t \in [0,1])$, and of any continuous path in $\Phi^{*(j)}_{n,1,d}$, connecting $\tau_1(f_1)$ and $\tau_1(f_2)$.)

In order to define a required semialgebraic function α we want to introduce the following notations: Let $B(f)$ be the union of bounded connected components of $\mathbb{R}^{n+1} \setminus \rho(f)$, $D(f)$ be the ball of the minimal possible radius centered at the origin in $\mathbb{R}^{n+1}$ such that its interior comprises the open $r(\rho(f))$-neighborhood of $\rho(f)$, $\delta_*(f)$ denote $\frac{\min_{x\in\rho(f)} |\ grad\ p_f(x)|}{4\max_{x\in D(f)}\|Hess\ p_f(x)\|}$, $B_{\geqq\delta_*(f)}(f)$ denote the subset of $B(f)$ formed by all points $x \in \mathbb{R}^{n+1}$ such that $dist(x, \rho(f)) \geqq \delta_*(f)$. Denote $\min_{x\in B_{\geqq\delta_*(f)}(f)} |p_f(x)|$ by $m(f)$ and define α by the formula:

$$\alpha(f) = \frac{\min\{1,\ m(f),\ \min_{x\in\rho(f)} |\ grad\ p_f(x)|\}}{100d(n+1)^{[d/2]+1}(\max\{1,\ diamD(f)/2\})^{2[d/2]}}.$$

Denote the polynomial $t\sigma_f\alpha(f)(x_1^2 + \ldots + x_{n+1}^2)^{[d/2]}$ by p_{tf}. According to our definition of τ_t for $f \in \Phi_{n,1,d} \setminus \Phi^*_{n,1,d}$, $\tau_t(f)$ is the vector of coefficients of the polynomial $p_f + p_{tf}$. One can easily see that $\|p_{tf}\|_{C^0(D(f))} \leqq \frac{1}{100}\min\{1,\ m(f)\}$ and $\|grad\ p_{tf}\|_{C^0(D(f))} \leqq \frac{1}{50}\inf_{x\in B(f)\setminus B_{\geqq\delta_*(f)}(f)} |grad\ p_f(x)|$. Now note that in the complement to the closure of $B(f)$ p_f and p_{tf} are of the same sign and, hence, their sum does not vanish there. On $B_{\geqq\delta_*(f)}(f)$ p_f and p_{tf} are of the opposite signs but $|p_f| > |p_{tf}|$. Hence their sum does not vanish in $B_{\geqq\delta_*(f)}(f)$. Now it is easy to show that $p_f + p_{tf}$ has exactly one zero on every inner normal to $\rho(f)$ of length $\delta_*(f)$ and that $|grad(p_f + p_{tf})|$ will be uniformly bounded from below by a positive constant on $B(f)\setminus B_{\geqq\delta_*(f)}(f)$. This guarantees that all sets $\rho(\tau_t(f))$ are smooth hypersurfaces lying in $D(f)$. Now the implicit function theorem implies that $t \longrightarrow \rho(\tau_t(f)), t \in [0,1]$, is a rigid isotopy. (The uniform boundedness of the hypersurfaces $\rho(\tau_t(f))$ is crucial here.) The Tarski-Seidenberg theorem easily implies (c). It remains only to show that $\tau_t(f)$ is, indeed, in $\Phi^*_{n,1,d}$ and not just in $\Phi_{n,1,d}$.

To show that for any $f \in \Phi_{n,1,d} \setminus \Phi^*_{n,1,d}$ and $t > 0$ $\tau_t(f)$ will be in $\Phi^*_{n,1,d}$ we consider the sum of all terms of degree $2[d/2]$ of the polynomial p_f. Denote this sum which is either zero or a homogeneous polynomial of degree $2[d/2]$ by p^*_f. This polynomial cannot assume both positive and negative values. (Indeed, in the opposite case the standard argument shows that the set of zeroes of p_f is unbounded, which is impossible.) One can easily see now that $p^*_f + p_{tf}$ does not vanish at any non-zero x. Thus, $\tau_t(f)$ is in $\Phi^*_{n,1,d}$. This completes the construction of a decomposition of $\Phi_{n,1,d}$ into union of disjoint rigidly connected components.

In general, not every D-rigid isotopy between d-CCIs $M_1^n \subset \mathbb{R}^{n+k}$ and $M_2^n \subset \mathbb{R}^{n+k}$ can be obtained as the image under ρ of a continuous path in $\Phi_{n,k,D}$. (Recall that in our definition of rigid isotopy we postulated only that the resulting smooth algebraic variety should vary continuously in $M_n(\mathbb{R}^{n+k})$). However, we can prove that a pair of D-rigidly isotopic d-CCIs can be connected by a D-rigid isotopy which is obtained by a piecewise-continuous variation of components of some fixation of M_1^n, such that the number of pieces does not exceed the value of some recursive function $L(n,k,D)$ (which can be defined explicitly). Here we need this result only for the codimension one case. Therefore, we state it only for $k=1$. However, note that both the statement and the proof of this result can be straightforwardly generalized for the case of arbitrary k.

Theorem 3.1.1. For any D-rigid isotopy $F: [0,1] \longrightarrow M_n(\mathbb{R}^{n+1})$ there exist a D-rigid isotopy $G: [0,1] \longrightarrow M_n(\mathbb{R}^{n+1})$, a number $N_o \leq C^*_{n,1,D}$, a numeration of some connected components of $\Phi_{n,1,D}$ by numbers from 1 to N_o (i.e. a not necessarily one-to-one map of $\{1,\dots,N_o\}$ to the set of connected components of $\Phi_{n,1,D}$), numbers t_i, $i \in \{0,\dots,N_o\}$, such that $0 = t_0 \leq t_1 \leq \dots \leq t_{N_o} = 1$, and a system of intervals $\{\Delta_i\}_{i=1}^{N_o}$ such that,

(1) $[0,1] = \bigcup_{i=1}^{N_o} \Delta_i$

(2) Each $\Delta_i = (t_{i-1}, t_i)$, or $(t_{i-1}, t_i]$, or $[t_{i-1}, t_i)$, or $[t_{i-1}, t_i]$.

(3) For any i there exists a continuous lift of the restriction of the isotopy G on Δ_i to the ith connected component of $\Phi_{n,1,D}$. More precisely, there exists a continuous mapping $\bar{G}_i: \Delta_i \longrightarrow \Phi_{n,1,D}$ such that $\rho \circ \bar{G}_i = G|_{\Delta_i}$ and the image of $\bar{G}_i$ is contained in the i-th connected component of $\Phi_{n,1,D}$.

(4) $G(0) = F(0)$; $G(1) = F(1)$.

Remark. Note that according to Theorem 3.1.1 that the number N_o does not exceed $C^*_{n,1,D}$. The constant $C^*_{n,1,D}$ can easily be majorized by some recursive function $L(n,D)$ (using, for example, the estimates from [23] for Betty numbers of semi-algebraic sets).

Proof. Let an d-rigid isotopy F be given. Consider the constructed above representation of $\Phi_{n,1,D}$ as a disjoint union of rigidly connected components $\tau_1^{-1}(\Phi^{*(j)}_{n,1,D})$. We will numerate some of these rigidly connected components by consecutive numbers in such a manner that any of them is numerated only once. After numerating a rigidly connected component we immediately assign the same number to the connected component of $\Phi_{n,1,D}$ comprising the rigidly connected component. We numerate inductively some of the rigidly connected components of $\Phi_{n,1,D}$ by the consecutive positive integer numbers in the following way. Let S_1 be a rigidly connected component containing a fixation of $F(0)$. For any $i \geqq 1$ let t_i be the supremum of the set of $t \in [0,\ 1]$ such that there exists a fixation of $F(t)$ in S_i. If there exists a fixation of $F(t_i)$ in S_i, then let S_{i+1} be a rigidly connected component of $\Phi_{n,1,D}$ such that there exists a strictly decreasing sequence $\{t_{i_n}\}_{n=1}^{\infty}$ converging to t_i such that for any n $F(t_{i_n})$ has a fixation in S_{i+1}. If there is no fixation of $F(t_i)$ in S_i, then let S_{i+1} be any rigidly connected component of $\Phi_{n,1,d}$ containing a fixation of $F(t_i)$. If for some i $t_i = 1$ then we put $N_o = i$ or $i+1$ depending on whether or not $F(1)$ has a fixation in S_i. If $F(1)$

has no fixations in S_i, then we take for S_{i+1} any rigidly connected component containing a fixation of $F(1)$. Then we stop the process of numeration.

Now our aim is to connect $F(t_{k-1})$ and $F(t_k)$ (for all k) by a rigid isotopy which is the image of a continuous path in S_k (and thus in the k th connected component of $\Phi_{n,1,D}$) under ρ.

First, suppose that both $F(t_{k-1})$ and $F(t_k)$ both have fixations in S_k. In this case we consider a pair of fixations $f_{k-1} \in S_k$ of $F(t_{k-1})$ and $f_k \in S_k$ of $F(t_k)$. They can be connected by a continuous path in S_k such that the image of this path under ρ constitutes the necessary isotopy.

However, if $F(t_{k-1})$ and/or $F(t_k)$ have no fixations in S_k the existence of such isotopy is not a priori guaranteed. In this case we proceed as follows. Suppose, for example, that $F(t_k)$ has no fixations in S_k and $F(t_{k-1})$ has a fixation f_{k-1} in S_k (the case when $F(t_{k-1})$ has no fixations in S_k is treated similarly).

Let $\{t_{k_m}\}$ be a increasing sequence converging to t_k such that $F(t_{k_m})$ has a fixation f_{k_m} in S_k. A multiplication of a fixation by a nonzero scalar does not change the corresponding CCI. Thus, we can assume without loss of generality that norms of fixations from the sequence $\{f_{k_m}\}_{m=1}^{\infty}$ are uniformly bounded. Therefore, there exists a subsequence $\{g_i\}_{i=1}^{\infty}$ of $\{f_{k_m}\}_{m=1}^{\infty}$ converging to a limit point g_0 belonging to the closure of S_k. Since S_k is a rigidly connected component, f_{k-1} and g_1 can be connected by a continuous path in S_k such that the image of this path under ρ provides a segment of the necessary isotopy. It remains to construct the necessary isotopy between $\rho(g_1)$ and $F(t_k)$.

It seems natural to connect subsequently pairs of points g_i and g_{i+1} for all i by continuous paths in S_k in order to construct the necessary isotopy between $\rho(g_1)$ and $F(t_k)$. However, such isotopy may be discontinuous at $t = 1$. To overcome this difficulty we proceed as follows: First, by virtue of our construction of rigidly connected components all points $g_1, g_2, \ldots$, and g_0 are in the closure of a connected component of $\Phi^*_{n,1,D}$, containing $\tau_1(g_1)$. (Here τ_1 is the map defined at the beginning of the section to construct rigidly connected components.) Denote this component by Φ^*. We construct the required isotopy in two stages. On the first stage we connect g_1 and $\tau_1(g_1)$ by the path $\beta(t)$ defined as $\beta(t) = \tau_t(g_1)$. As we have shown before, the image of this path under ρ constitutes a rigid isotopy (between $\rho(g_1)$ and $\rho(\tau_1(g_1))$). On the second stage we are going to connect $\tau_1(g_1)$ and g_0 by a continuous path γ^* such that, roughly speaking, its image under ρ is an isotopy between $\rho(\tau_1(g_1))$ and $F(t_k)$. More precisely, we require that the image of $\gamma^* \setminus g_0$ under ρ should be an isotopy and that $\rho(\gamma^*(t))$ should tend to $F(t_k)$ in $M_n(\mathrm{IR}^{n+1})$, when $\gamma^*(t) \longrightarrow g_0$.

First, note two obvious properties of the notion of ϵ-approximation.

P1. Let $F_* : [0,1] \longrightarrow M_n(\mathrm{IR}^{n+1})$ be a mapping which is continuous in $[0,1)$. The mapping F_* is an isotopy (i.e. $F_*(t)$ is continuous at $t = 1$) if and only if for any $\epsilon > 0$ there exists $t_\epsilon < 1$ such that for any $t \in [t_\epsilon, 1]$ $F_*(t)$ ϵ-approximates $F_*(1)$. (This property follows from Lemma 2.2.4).

P2. Let M be a d-CCI, S^* be a connected component of $\Phi^*_{n,1,D}$, f_* be a point belonging to the closure of S^*, and ϵ be a positive constant. Let

$$(() \quad S^*(\epsilon, M, f_*) = \{f \in S^* | \rho(f)\ \epsilon\text{−approximates } M \textit{ and } |f - f_*| < \epsilon\}.3.1)$$

By the virtue of Tarski-Seidenberg Theorem $S^*(\epsilon, M, f_*)$ is a semi-algebraic set. Hence it has only a finite number of connected components.

Let for any p $\varepsilon_p = 2^{-p}$. Consider sets $S^*_p = \Phi^*(\varepsilon_p,\ F(t_k),\ g_0)$ defined by (3.1). Obviously, $\Phi^* \supset S^*_1 \supset S^*_2 \supset \ldots$. Note, that for any p S^*_p is non-empty. Indeed, property P1 and Lemma 2.2.4 imply that for sufficiently large i and for sufficiently small positive ϵ $\tau_\epsilon(g_i)$ will ε_p-approximate $F(t_k)$, and, thus, will be in S^*_p. (Here τ_ϵ are the maps defined in the process of construction of rigidly connected components of $\Phi_{n,1,d}$.) Since by virtue of P2 S^*_1 has a finite number of connected components, there exists a connected component C^*_1 of S^*_1 such that $C^*_1 \bigcap S^*_p$ is not empty for any $p > 1$. Proceeding by induction, for any j there exists a connected component C^*_{j+1} of S^*_{j+1} such that $C^*_{j+1} \subset C^*_j$ and $C^*_{j+1} \bigcap S^*_p$ is not empty for any $p > j$. Consider the constructed sequence of sets $\Phi^* \supset C^*_1 \supset C^*_2 \supset \ldots$. Let us construct the required path γ^* between $\tau_1(g_1)$ and g_0 as follows: Let $\gamma^*(0) = \tau_1(g_1)$, the restriction γ^* on $[0, 1/2]$ be any continuous path in Φ^* connecting $\tau_1(g_1)$ and a point of C^*_1. Define inductively γ^* on all intervals $[1-\frac{1}{l+1}, 1-\frac{1}{l+2}]$, $(l \in \{1, 2, \ldots\})$, as follows: For any l let the restriction of γ^* on $[1 - \frac{1}{l+1}, 1 - \frac{1}{l+2}]$ be any continuous path in C^*_l connecting the already built point $\gamma^*(1 - \frac{1}{l+1})$ and an arbitrary point of C^*_{l+1}. Finally, let $\gamma^*(1) = g_0$. Our definition of sets S^*_p and C^*_p implies the continuity of γ^*. Property P1 and the definition of S^*_p, C^*_p imply that the image of this path under ρ will be a rigid isotopy between $\rho(\tau_1(g_1))$ and $F(t_k)$. (Of course, only the continuity at $t = 1$ is not a priori obvious.) □

Remark 1. The statement and the proof of Theorem 3.1.1 can be generalized for the case of D-CCI's of arbitrary codimensions.

Note, that the multicodimensional version of Theorem 3.1.1 is essentially deeper than the codimension one version by the following reason: In the case of a codimension > 1 the intersecting algebraic hypersurfaces determining the D-CCI can vary discontinuously during a D-rigid isotopy.

Remark 2. We actually constructed a numeration of a set of rigidly connected components of $\Phi_{n,1,D}$ by numbers from 1 to N_o and proved that the restriction of the isotopy G on Δ_i can be lifted to the i-th rigidly connected component of $\Phi_{n,1,D}$. More precisely, for any i the map $\bar{G}_i$ constructed in the proof of Theorem 3.1.1 maps Δ_i to the i-th rigidly connected component of $\Phi_{n,1,D}$.

If two fixations belong to the same rigidly connected component of $\Phi_{n,1,d}$ then the corresponding hypersurfaces are d-rigidly isotopic. Assume that two fixations belong, correspondingly, to the i-th and the j-th rigidly connected components of $\Phi_{n,1,d}$. When are the corresponding hypersurfaces d-rigidly isotopic? The following condition is obviously sufficient:

There exist a fixation f_1 in the i-th rigidly connected component and a fixation f_2 in the j-th rigidly connected component of $\Phi_{n,1,d}$ such that the hypersurfaces $\rho(f_1)$ and $\rho(f_2)$ coincide.

The idea of the construction of the rigid isotopy between $\rho(g_1)$ and $F(t_k)$ at the end of the proof of Theorem 3.1.1 can be straightforwardly applied to demonstrate that the following weaker condition is sufficient, too:

C_{ij}: There exists a fixation f_1 in the i-th connected component of $\Phi_{n,1,d}$ such that for any $\varepsilon > 0$ there exists a fixation f_2 in the j-th connected component of $\Phi_{n,1,d}$ such that $\rho(f_2)$ ε-approximates $\rho(f_1)$.

Consider now the following graph $Gr_{n,1,d}$: The set of its vertices is in a one-to-one correspondance with the set of all rigidly connected components of $\Phi_{n,1,d}$. Two vertices are connected by an edge if and only if at least one of the conditions C_{ij} or C_{ji} holds for the corresponding rigidly connected components of $\Phi_{n,1,d}$ (numbered by i and j, respectively). The transitivity of the relation "to be d-rigidly isotopic" implies that the following condition is also sufficient to ensure that for any fixation f_1 from a rigidly connected component Φ_1 and for any fixation f_2 from a rigidly connected component Φ_2 of $\Phi_{n,1,d}$ $\rho(f_1)$ is d-rigidly isotopic to $\rho(f_2)$:

P_{Φ_1,Φ_2}: The vertices of $Gr_{n,1,d}$ corresponding to Φ_1 and to Φ_2 are in the same (connected) component of $Gr_{n,1,d}$.

Theorem 3.1.1 and Remark 2 after it imply that the condition P_{Φ_1,Φ_2} is also the necessary condition for $\rho(f_1)$ and $\rho(f_2)$ to be d-rigidly isotopic , (f_1 is any fixation from Φ_1, f_2 is any fixation from Φ_2).

Now recall that all rigidly connected components of $\Phi_{n,1,d}$ are semi-algebraic, and that there exists an algorithm which for any given n and d produces the list of presentations of all rigidly connected components of $\Phi_{n,1,d}$. Let us apply this algorithm. Now for any two fixed rigidly connected components (say, Φ_i and Φ_j) of $\Phi_{n,1,d}$ the condition C_{ij} can be written as a first-order formula in the language of the real number field theory. By virtue of the Tarski-Seidenberg theorem this fact enables us to construct effectively graph $Gr_{n,1,d}$. Then we can effectively find its components. This implies the existence of an algorithm which for any given n, d and fixations f_1, $f_2 \in \Phi_{n,1,d}$ finds the corresponding rigidly connecting components Φ_1, Φ_2 of $\Phi_{n,1,d}$ and then verifies the condition P_{Φ_1,Φ_2} for these components. As we saw above, this algorithm checks, in fact, whether or not $\rho(f_1)$ and $\rho(f_2)$ are d-rigidly isotopic. Since any fixation from $\Phi_{n,1,d}$ can be regarded, by definition, as a fixation from $\Phi_{n,1,D}$ for any $D \geqq d$, we come to the following theorem:

Theorem 3.1.2. There exists an algorithm which verifies for any given set of n, d, D and for any given pair of vectors with rational components, representing fixations of compact non-singular algebraic hypersurfaces of degree d $M_1^n, M_2^n \subset \mathbb{R}^{n+1}$, whether or not M_1^n and M_2^n are D-rigidly isotopic in $\mathbb{R}^{n+1}$.

Remark. As it was noted in Remark after the proof of Theorem 3.1.1, Theorem 3.1.1 holds for any codimension (and not only for codimension one). Therefore Theorem 3.1.2 holds for d-CCIs of any codimension. (Of course the codimension should be also included in the input data for the algorithm).

3.2. Non-recursive functions in geometry

Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a compact smooth algebraic hypersurface of degree d isotopic to the standard sphere $S^n = \{x \in \mathbb{R}^{n+1} | \ |x| = 1\}$. In the previous section we proved that there exists D (depending on Σ^n) such that Σ^n is D-rigidly isotopic to S^n. In this section we will show that it is possible to find D such that any compact smooth algebraic hypersurface of degree d isotopic to S^n is D-rigidly isotopic to S^n. Consider the minimal of such numbers D as a function of d. Denote it by $A_n(d)$. The first of the results of this section is that for any $n \geq 5$, $A_n(d)$ cannot be majorized by any recursive function of d (Theorem 3.2.1). This theorem can be generalized on regular complete intersections of compact algebraic hypersurfaces ([26]). We deduce a differential-geometric corollary of Theorem 3.2.1. For any x there exists $y \geq x$ such that any hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ isotopic to S^n and such that $\kappa_v(\Sigma^n) \leq x$ can be connected with S^n by an isotopy passing only through hypersurfaces of v-crumpledness $\leq y$. Consider the minimal y with this property as a function of x. Denote it by $C_n(x)$. Then for any $n \geq 5$, the restriction of C_n on integers cannot be majorized by any recursive function (Theorem 3.2.3). It is clear from the proofs that all results stated in this section for v-crumpledness also hold for d-crumpledness.(These results for d-crumpledness were formulated in [26]; the d-crumpledness was called just crumpledness there).

Theorem 3.2.1. (a) For any pair of positive integers d, n there exists D such that any compact non-singular algebraic hypersurfaces of degree $\leq d$, isotopic to S^n, is D-rigidly isotopic to S^n.

(b) Let $A_n(d)$ denote the minimal number $N \in \mathbb{N}$ such that any smooth algebraic hypersurface of degree $\leq d$, isotopic to S^n, is N-rigidly isotopic to S^n. For any fixed n, consider $A_n(d)$ as a function of d. If $n \geq 5$ then $A_n(d)$ cannot be majorized by any recursive function of d.

Proof.

(a) Consider the set $\Phi^*_{n,1,d}$ (defined at the beginning of Section 3.1). By the well-known property of semialgebraic sets (cf. [9] or [5]), it has only a finite number of connected components. One can choose a point f_i from any connected component of $\Phi^*_{n,1,d}$. If for some index i the hypersurface $\rho(f_i)$ is isotopic to S^n, then for some large enough D_i, $\rho(f_i)$ is D_i-rigidly isotopic to S^n (this follows from Proposition 2.3.2). Let S be the set of all indices i such that $\rho(f_i)$ is isotopic to S^n. Note that S is a finite set. Let $D = \max\{d, \max_{i\in S} D_i\}$.

Any non-singular compact algebraic hypersurface of degree $\leqq d$ is d-rigidly isotopic to at least one of the hypersurfaces $\rho(f_i)$. Indeed, let p be a polynomial such that the hypersurface is its zero set and *grad* p does not vanish at any point of the hypersurface. Consider the vector f_p of coefficients of p. We know that $\rho(f_p)$ is d-rigidly isotopic to $\rho(\tau_1(f_p))$, and $\tau_1(f_p) \in \Phi^*_{n,1,d}$. There exists one of the chosen representatives f_i in the same connected component of $\Phi^*_{n,1,d}$ as $\tau_1(f_p)$. Connect $\tau_1(f_p)$ and f_i by a path in $\Phi^*_{n,1,d}$. The image of this path under ρ is a d-rigid isotopy between $\rho(\tau_1(f_p))$ and $\rho(f_i)$.

Now it follows from the transitivity of the relation "to be D-rigidly isotopic" that D is the number existence of which is stated in part (a) of Theorem 3.2.1.

(b) The proof of (b) is based on S.P. Novikov's Theorem stating that there is no algorithm checking whether or not a given smooth hypersurface $M^n \subset \mathbb{R}^{n+1}$, $(n \geq 5)$ is diffeomorphic to the n-dimensional sphere S^n ([40], ch. 10). Novikov effectively constructs a sequence of compact submanifolds $\{M_i^n\}_{i=1}^{\infty}$ of $\mathbb{R}^{n+1}$ with the following property: The set of i such that M_i^n is homeomorphic to S^n is a non-recursive set. His exposition is sketchy. In particular, he does not explain how to construct a smooth compact hypersurface in $\mathbb{R}^{n+1}$, $(n \geqq 5)$, with presribed fundamental group. (This construction is an important ingredient of the proof.) A detailed description of this well-known construction can be found in [7]. Further, a smooth *hypersurface* in $\mathbb{R}^{n+1}$, $(n \geqq 5)$, is homeomorphic to S^n if and only if it is diffeomorphic to S^n. This statement is an easy corollary of Smale's h-cobordism theorem (cf. [22], ch.9, Proposition A).

Novikov does not describe in details how the hypersurfaces M_i^n are represented by finite sets of data. The authors of [7] use Nash atlases in a similar but more general situation, when it is necessary to deal also with manifolds which are not assumed embedded as hypersurfaces. But using the standard techniques of constructive polynomial approximation of continuous functions as in section 2.3, one can easily show that a smooth hypersurface M_i^n presented by a Nash atlas can be constructively approximated by an isotopic algebraic hypersurface $A^n \subset \mathbb{R}^{n+1}$, $A^n = \{x \in \mathbb{R}^{n+1} \,\big|\, p(x) = 0\}$, *grad* $p(x) \neq 0$ for any $x \in A^n$, all coefficients of p are rational. "Constructively" means that all the coefficients of one such polynomial p can be found from a data set describing M_i^n. Hence the recognition problem, whether or not a given smooth hypersurface $M^n \subset \mathbb{R}^{n+1}$, $(n \geq 5)$ is diffeomorphic to S^n, is undecidable even for the class of non-singular algebraic hypersurfaces of $\mathbb{R}^{n+1}$. (Added in August, 1994. A detailed proof of this version of S. Novikov's theorem can be found in [30].) Furthermore, it is a well-known corollary of the h-cobordism theorem that if $n \geq 5$ then a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ is diffeomorphic to $S^n = \{x \in \mathbb{R}^{n+1} |\ |x| = 1\}$ if and only if Σ^n is isotopic to S^n (cf. [22], Ch.9, Proposition D). By the virtue of the already proven part (a) of the theorem, a non-singular algebraic hypersurface of degree d is isotopic to S^n if and only if it is $A_n(d)$-rigidly isotopic to S^n. Summarizing these observations we see that S.P. Novikov's theorem implies that the problem to determine whether or not a given non-singular algebraic hypersurface of degree d is $A_n(d)$-rigidly isotopic to S^n is algorithmically unsolvable for any $n \geq 5$.

Assume now that there exists a recursive function $\varphi(d)$ majorizing $A_n(d)$. By the definition of $A_n(d)$ a non-singular algebraic hypersurface of degree d in $\mathbb{R}^{n+1}$ is $A_n(d)$-rigidly isotopic to S^n if and only if it is isotopic to S^n and if and only if it is $\varphi(d)$-rigidly isotopic to S^n. But the (Turing) computability of φ and Theorem 3.1.2 imply the existence of an algorithm checking whether or not a given non-singular algebraic hypersurface in $\mathbb{R}^{n+1}$ of degree d is $\varphi(d)$-rigidly isotopic to S^n. We get a contradiction which proves part (b) of the theorem.□

This theorem can be generalized for the case of an arbitrary codimension. Instead of non-singular algebraic hypersurfaces one should consider regular complete intersections of k compact non-singular algebraic hypersurfaces of degree d. The exact formulation can be found in the author's papers [25], [26]. The proof is similar to the proof of Theorem 3.2.1 but one needs an effective relative version of the Weierstrass approximation theorem (an effective version of Lemma 2.2 in [3]).

Theorem 3.2.2. (a) For any n there exists a function $B_n : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$ with the following property. Any non-singular algebraic hypersurface Σ^n of degree d and of v-crumpledness $\leq x$ isotopic to S^n and such that it has a fixation in $\Phi^*_{n,1,d}$, can be connected with S^n by an isotopy passing only through hypersurfaces of v-crumpledness $\leq B_n(x, d)$.

(b) For any $n \geq 5$, any such function B_n cannot be majorized by any recursive function of $[\kappa_v(\Sigma^n)]$ and d.

Proof. (a) For any x, let $\Phi^*_{n,1,d}(x)$ denote a subset $\Phi^*_{n,1,d}$ formed by all vectors of coefficients f such that (1) $\rho(f)$ is isotopic to S^n and (2) $\kappa_d(\rho(f)) \leq x$. It is easy to show using the Tarski-Seidenberg theorem that $\Phi^*_{n,1,d}(x)$ is a semi-algebraic set. Thus, it has a finite number of connected components. For any connected component $S_i(x)$ of $\Phi^*_{n,1,d}(x)$ we choose a representative $f_{ix} \in S_i(x)$. Any non-singular algebraic hypersurface Σ^n of degree d having a fixation in $\Phi^*_{n,1,d}$ can be connected with one of the hypersurfaces $\rho(f_{ix})$ by an isotopy which passes only through hypersurfaces of d-crumpledness $\leq x$. Indeed, we can connect a fixation of Σ^n in $\Phi^*_{n,1,d}(x)$ with one of the chosen representatives f_{ix} by a continuous path. The image of this path under ρ provides the isotopy. Now we consider all pairs i, j such that $\rho(f_{ix})$ and $\rho(f_{jx})$ are isotopic. For any such pair we want to connect $\rho(f_{ix})$ and $\rho(f_{jx})$ by an isotopy F_{ij} such that $\sup_{t\in[0,1]} \kappa_d(F_{ij}(t)) < +\infty$. This is not always so since the topology of $M_n(\mathbb{R}^{n+1})$ is C^1 and, hence, κ_d is not continuous on $M_n(\mathbb{R}^{n+1})$. (However, it will be proven in Section 5 that κ_d is a lower semicontinuous functional on $M_n(\mathbb{R}^{n+1})$.) Since $\rho(f_{ix})$ and $\rho(f_{jx})$ are C^∞-smooth isotopic submanifolds of $\mathbb{R}^{n+1}$, there exists a C^∞-smooth diffeotopy $G : [0,\ 1] \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ such that $G(0, \rho(f_{ix})) = \rho(f_{ix})$ and $G(1, \rho(f_{ix})) = \rho(f_{jx})$. (Actually we need only C^2-smoothness of this diffeotopy.)

Let for any $t \in [0,\ 1]$ $F_{ij}(t) = G(t, \rho(f_{ix}))$. Now Theorem 4.19 from [10] immediately implies the finiteness of $\kappa_{ij} = \sup_{t \in [0,1]} \kappa_d(F_{ij}(t))$. Let

$$B_n^{(d)}(x,d) = \max\{x, \max_{\substack{\{i,j\} \\ \rho(fix) \text{ and} \\ \rho(fjx) \text{ are} \\ \text{isotopic}}} \kappa_{ij}\} .$$

The finiteness of $B_n^{(d)}(x,d)$ follows from the finiteness of the number of connected components of $\Phi^*_{n,1,d}(x)$. (The finiteness of the number of connected components of $\Phi^*_{n,1,d}(x)$ follows from the semialgebraicity of the function $\kappa_d \circ \rho$. Thus a similar argument for the function κ_v instead of κ_d fails since $\kappa_v \circ \rho$ is not a semialgebraic function.) Now let

$$B_n(x,d) = C_1(n)(B_n^{(d)}(C_2(n)x^n, d))^{1+\frac{1}{n}} ,$$

where $C_1(n), C_2(n)$ are constants defined in the text of Lemma 2.1.4. Lemma 2.1.4 implies that $B_n(x,d)$ is a necessary function.

(b) Suppose that $B_n(x,d)$ can be majorized by a recursive function $\alpha_n([x],d)$. Then the proof of Theorem 3.2.1(a) can be slightly modified to give an effective upper bound on $A_n(d)$. Indeed, the single non-constructive step in the proof was the usage of $\max_{i \in S} D_i$. Recall that we have chosen a point f_i from every connected component of $\Phi^*_{n,1,d}$. S denotes the set of indices i such that $\rho(f_i)$ is isotopic to S^n. By virtue of Lemma 2.3.2 $\rho(f_i)$ and S^n are D_i-rigidly isotopic for some D_i. Lemma 2.3.2 implies also that D_i can be majorized by a recursive function φ_n of d, $\kappa(p_{f_i})$ and of the maximal d-crumpledness during an isotopy between $\rho(f_i)$ and S^n. It follows from the proof of Lemma 2.3.2 that φ_n can be chosen increasing with any of its arguments. Now it is clear that the recursiveness of $\alpha_n([x],d)$ permits to find an effective (i.e. Turing computable) upper bound on $\max_{i \in S} D_i$. Indeed, we do not try to find out which indices i are in S (this is an unsolvable problem). Instead we compute an upper bound for $\varphi_n([C_2(n)\alpha_n^n([C_1(n)(2\kappa(p_{f_i}))^{1+\frac{1}{n}}], d)], d, [\kappa(p_{f_i})])$ for each i, and take the maximum over all representatives f_i. Lemmae 2.1.3, 2.1.4 and 2.3.2 imply that we obtain an effective upper bound on $A_n(d)$. But this is impossible because of Theorem 3.2.1(b). We obtained the contradiction which proves Theorem 3.2.2(b).□

Theorem 3.2.3. (a) Let n be fixed. For any $x \geqq \kappa_v(S^n)$ there exists y with the following property. Any hypersurface Σ^n of v-crumpledness $\leq x$ isotopic to S^n can be connected with S^n by an isotopy passing only through hypersurfaces of v-crumpledness $\leq y$.

(b) Consider the smallest of such y as a function of x. Denote this function by $C_n(x)$. The restriction of this function C_n on integers cannot be majorized by any recursive function.

Proof. (a) We deduce (a) from part (a) of Theorem 3.2.2. The idea is to ε-approximate Σ^n by a non-singular algebraic hypersurface Σ^n_0 such that it has a fixation in $\Phi^*_{n,1,deg\ \Sigma^n_0}$ for a sufficiently small positive ε and then construct the required isotopy combining the isotopy between Σ^n_0 and S^n, existence of which is guaranteed by Theorem 3.2.2(a) and an isotopy between Σ^n and Σ^n_0. (Note that for any d S^n has a fixation in $\Phi^*_{n,1,d}$. Namely, the vector of coefficients of the polynomial $(x_1^2 + \ldots + x_{n+1}^2)^{[d/2]} - 1$ is such fixation.) To implement this idea, we impose two constraints on ε. First, we demand that ε should be less than $r(\Sigma^n)/2$. Secondly, the value of ε should be sufficiently small to enable us to apply Lemma 2.2.2 for hypersurfaces Σ^n and Σ^n_0. The constraint on ε in the condition of Lemma 2.2.2 depends on $r(\Sigma^n_0)$. However, by the virtue of Lemma 2.3.1 Σ^n_0 can be chosen such that $\kappa_d(\Sigma^n_0)$ does not exceed $2\mu([\kappa_d(\Sigma^n)])$ independently of ε for an appropriate function μ, defined in the text of Lemma 2.3.1. Thus, one can easily see that we can define ε, for example, by the formula

$$\varepsilon = \min\{\frac{1}{4}, C\} \min\{\frac{diam(\Sigma^n)}{\kappa_d(\Sigma^n)}, \frac{diam(\Sigma^n)}{4\mu([\kappa_d(\Sigma^n)])}\},$$

where C is the constant defined in the text of Lemma 2.2.2. Now Proposition 2.3.1 and the remark following it imply that the ε-approximating algebraic hypersurface Σ^n_0 of a degree d can be chosen in such a way that

$$d \leqq f(\kappa_d(\Sigma^n)), \tag{3.2.1}$$

for some function f, and

$$\kappa_d(\Sigma^n_0) \leq 2\mu([\kappa_d(\Sigma^n)]). \tag{3.2.2}$$

Also, one can see from the construction of the approximating non-singular algebraic hypersurface in the proof of Lemma 2.3.1 that the approximating non-singular algebraic hypersurface constructed there has a fixation in $\Phi^*_{n,1,deg\ \Sigma^n_0}$. Lemma 2.2.2 guarantees that Σ^n_0 and Σ^n can be connected by an isotopy passing only through hypersurfaces of v-crumpledness $\leq \theta(\max\{\kappa_v(\Sigma^n), \kappa_v(\Sigma^n_0)\})$. Applying Lemma 2.1.4 we can deduce from the inequality (3.2.2) an upper bound for the v-crumpledness of hypersurfaces during the isotopy between Σ^n and Σ^n_0 in terms of $\kappa_d(\Sigma^n)$.

Theorem 3.2.2(a) implies that Σ^n_0 and S^n can be connected by a rigid isotopy passing only through hypersurfaces of v-crumpledness $\leq B_n(\kappa_v(\Sigma^n_0), d)$. Note that without loss of generality the function B_n can be assumed increasing with both its arguments. Hence the inequalities (3.2.1), (3.2.2) together with Lemma 2.1.4 imply the necessary estimates for the v-crumpledness of hypersurfaces during the stage of the constructed isotopy between Σ^n_0 and S^n. Thus, we proved the statement of part (a) of the theorem.

(b) Part (b) of this theorem follows immediately from Theorem 3.2.2(b) (and in fact is weaker than Theorem 3.2.2(b)). $\square$

4. Generalized knots "with thick ropes"

4.1. Existence of an infinite set of non-trivial knot "with thick rope" types in codimension one.

One of the main results of this work is the existence for any $n \geq 5$ of an infinite set of distinct non-trivial knot "with thick rope" types of embeddings $S^n \subset \mathbb{R}^{n+1}$. (The definition of knot "with thick rope" types was given in Section 1.) This shows that the theory of knot "with thick rope" types is quite different from the usual (generalized) knot theory. (Recall that for $n \geq 5$ there are no non-trivial n-dimensional knots in $\mathbb{R}^{n+1}$). More precisely, for any fixed $n \geqq 5$ Theorem 4.1 implies the existence of arbitrary large x such that a non-trivial knot "with thick rope" x-type exists. This can be regarded as an analogue of Theorem 4.3 of [26] for v-crumpledness instead of d-crumpledness (In [26] d-crumpledness was called just "crumpledness"). We prefer to formulate this theorem in the slightly more strong (but, alas, less transparent) form having in mind the applications in Section 5.2.

Theorem 4.1. Let $n \geq 5$ be fixed. There exists an infinite strictly increasing sequence of positive real numbers $\{x_i\}_{i=1}^{\infty}$ tending to infinity such that:

(1) For any i there exists a non-trivial knot "with thick rope" x_i-type K_i of embeddings $S^n \subset \mathbb{R}^{n+1}$.

(2) If a knot $k \in K_i$ then for any $j > i$, $k \notin K_j$.

Proof. It follows from Theorem 3.2.3(b) that for any x_0 there exists $x > x_0$ such that $C_n(x) > x$, where $C_n(x)$ is the function defined in the text of Theorem 3.2.3. But this means that there exists a knot k such that $\kappa_v(k) \leq x$ and k has a non-trivial knot "with thick rope" x-type. Thus, there exists a non-trivial knot "with thick rope" x-type. Denote a arbitrary non-trivial knot "with thick rope" x-type by $K(x)$.

Now choose the sequence $\{x_i\}$ by the following way. Let $x_0 = \kappa_v(S^n)$. For any i let x_i be a number greater than $\max\{2x_{i-1},\ C_n(x_{i-1})\}$ such that $C_n(x_i) > x_i$. It is easy to check that $\{x_i\}_{i=1}^{\infty}$ is the sequence existence which is stated in the theorem. One can take $K(x_i)$ for K_i. $\square$

The notion of the knot "with thick rope" can be generalized in order to permit an increase of v-crumpledness during the isotopy up to a certain specified bound. More precisely, let n be fixed. For any real x, y consider the following equivalence relation on the space of smooth hypersurfaces in $\mathbb{R}^{n+1}$ of v-crumpledness $\leqq x$, diffeomorphic to S^n. Two hypersurfaces are equivalent if and only if they can be connected by an isotopy passing only through hypersurfaces of v-crumpledness $\leqq y$. The equivalence classes will be called *knot "with thick rope" (x,y)-types.*

The following generalization of Theorem 4.1 can be proven exactly as Theorem 4.1.

Theorem 4.2. Let $f : [1, +\infty] \longrightarrow [1, +\infty]$ be an increasing function such that the function $[f([x])]$ is recursive. Let $n \geqq 5$ be fixed. There exists an increasing sequence $\{x_i\}_{i=1}^{\infty}$ converging to infinity such that:

(1) For any i there exists a non-trivial knot "with thick rope" $(x_i, f(x_i))$-type K_i of embeddings of S^n into $\mathbb{R}^{n+1}$.

(2) If a knot $k \in K_i$, then for any $j > i$ $k \notin K_j$.

Important remark. (Added in August, 1994) Recently I noticed that by a relatively minor modification of the proof of Theorem 4.1 one can prove the following stronger statement:

For any $n \geqq 5$ for all sufficiently large x there exists a non-trivial knot "with thick rope" x-type of embeddings S^n into $\mathbb{R}^{n+1}$. (In other words, for all sufficiently large x $\kappa_v^{-1}(0, x]$ is not path connected.)

Also, for any $n \geqq 5$ and for every f such that $f([x])$ is recursive there exists x_0 such that for all $x \geqq x_0$ there exists a non-trivial knot "with thick rope" $(x, f(x))$-type of embeddings of S^n into $\mathbb{R}^{n+1}$.

The details will appear elsewhere.

4.2. Knots "with thick ropes" in $\mathbb{R}^3$

We present here some observations, plausible conjectures and open questions about knots "with thick rope" x-types of embeddings S^1 into $\mathbb{R}^3$.

A. It was recently found out that the knot theory has some applications in biochemistry (since DNA and protein molecules can be knotted). Note that molecules seem to be closer to knots "with thick ropes" than to classical knots. Thus, the following observation may be of interest: Let a smooth nonselfintersecting closed space curve $\tau : S^1 \longrightarrow \mathbb{R}^3$ be given. Suppose that

$$\tau(\varphi) = (P_1(\sin\ \varphi, \cos\ \varphi), P_2(\sin\ \varphi, \cos\ \varphi), P_3(\sin\ \varphi, \cos\ \varphi))\ ,$$

where P_1, P_2, P_3 are polynomials of degree $\leq d$ with rational coefficients which are given. Assume that the projection of $\tau(S^1)$ on XY-plane has only simple crossings. Suppose further that $\tau(\varphi)$ has the trivial knot "with thick rope" x-type for some rational x which is also given.

Then there exists an algorithm which constructs an isotopy between τ and the unknot $(\cos\ \varphi, \sin\ \varphi, 0)$ and requires an amount of space which depends polynomially on d, x and the length of coefficients of the polynomials P_i. The resulting isotopy is represented by a sequence of knot diagrams $D_0, D_1, \ldots, D_m$ such that

1) D_0 is the diagram of the initial knot $\tau(\varphi)$, corresponding to its projection on the XY-plane; D_m is the trivial diagram of the unknot.

2) The diagram D_{i+1} can be derived from D_i by exactly one Reidemeister move.

The key idea of the demonstration of this fact is to show that if τ can be connected with the unknot by an isotopy passing only through knots k such that $\kappa_v(k) \leq x$, then τ can be connected with the unknot by an isotopy passing only through knots k such that $c(k) \leq poly(x, d)$. Here $c(k)$ denotes the crossing number of k and $poly$ denotes some polynomial (not depending on τ, but depending on d and x). Indeed, Jackson's theorem implies that it is possible to approximate the isotopy by an isotopy passing only through curves of the type $(P_1^*(\sin\ \varphi, \cos\ \varphi), P_2^*(\sin\ \varphi, \cos\ \varphi), P_3^*(\sin\ \varphi, \cos\ \varphi))$, where P_1^*, P_2^*, P_3^* are polynomials of degree not exceeding $poly_1(x, d)$, where $poly_1$ is a universal polynomial. Making the standard substitution $t = tg\ \frac{\varphi}{2}$, we see that such knots can be represented as $(q_1^*(t), q_2^*(t), q_3^*(t))$ where $t \in [-\infty, +\infty]$ and q_1^*, q_2^*, q_3^* are rational functions. The degrees of numerators and denumerators of polynomials q_i^* do not exceed $2poly_1(x, d)$. The Bezout theorem implies that for such knots k $c(k) \leq poly(x, d)$.

Consider now all link diagrams with $\leq poly(x, d)$ intersections as vertices of a graph. Two vertices are connected by an edge if and only if there is a Reidemeister move transforming one diagram into another. But to find a path connecting two given vertices of a graph is a PLOGSPACE problem (with respect to the number of vertices of the graph). To check whether or not a given link diagram can be transformed to another given link diagram by exactly one Reidemeister move is a PSPACE problem (with respect to the size of the diagrams). This completes the demonstration of the existence of the algorithm.

Note, however, that we do not know whether or not there exists an algorithm checking whether or not a given knot has a trivial knot "with thick rope" x-type for a given x. (As before, we consider only knots which are representable by a parametrized curve $(P_1(\sin\ \varphi, \cos\ \varphi), P_2(\sin\ \varphi, \cos\ \varphi), P_3(\sin\ \varphi, \cos\ \varphi))$ where P_1, P_2, P_3 are polynomials with rational coefficients. These coefficients are the input data.) The reason is that κ_v is not a semialgebraic function of coefficients of polynomials P_1, P_2, P_3.

B. Do there exist x and a knot $k \subset \mathbb{R}^3$ such that: (1) $\kappa_v(k) \leq x$; (2) k has a trivial knot type; (3) k has a non-trivial knot "with thick rope" x-type?

We believe that a result similar to Theorem 4.1 takes place: There is an infinite set of different knot "with thick rope" x-types – all of them having the trivial knot type.

C. Let G be the group of transformations of $M_1(\mathbb{R}^3)$ generated by all isometries and homothetic transformations of $\mathbb{R}^{n+1}$. Obviously, κ_v is constant on any orbit of G. Is it true that any locally minimal value of κ_v is always assumed on a finite set of orbits of G?

D. What is the minimal length of a rope of radius one with which it is possible to tie a non-trivial knot? In other words, what is the infimum of κ_v on the set of all space curves non-trivially knotted in the usual sense?

E. For any usual knot type $[k]$ let $\bar{\kappa}_v([k])$ denote the infimum of $\kappa_v(\gamma)$ over the space of all non-selfintersecting C^2-smooth closed curves $\gamma \subset \mathbb{R}^3$ of knot type $[k]$. The functional $\bar{\kappa}$ is a new knot type invariant. It is easy to find upper bounds for various well-known knot type invariants (say, the minimal crossing number) in terms of $\bar{\kappa}_v$. Some related knot type invariants are discussed in [13], [31].

5. Non-trivial topological hyperspheres locally minimizing crumpledness

Here we apply the results of Section 4.1 to prove the existence of an infinite set of essentially distinct local minima of κ_v on the space of $C^{1,1}$ smooth hypersurfaces in $\mathbb{R}^{n+1}$ diffeomorphic to S^n (n is any fixed number $\geqq 5$). (These minima can be viewed as "self-clenching thick hyperspheres" since there are no small normal variations of any of these topological hyperspheres Σ_i^n such that the volume of Σ_i^n remains constant and $r(\Sigma_i^n)$ increases). The sequence of the corresponding minimal values of κ_v is unbounded.

This result can be generalized for a class of crumpledness functionals containing κ_v and κ_d. The exact definition of generalized crumpledness functionals and the statement of this result together with some examples will be given in section 5.2. In section 5.1 we prove the upper semicontinuity of the functional r on the space of $C^{1,1}$-smooth compact hypersurfaces equipped with C^1-topology.

5.1. Central sets of $C^{1,1}$-smooth hypersurfaces

Let $M_n^{(1,1)}(\mathbb{R}^{n+1})$ be the space of compact $C^{1,1}$-smooth hypersurfaces equipped with the C^1-topology. This space was defined at the end of Section 1. Recall that $M_n^{(1,1)}(\mathbb{R}^{n+1})$ is $\coprod_{M^n} Emb^{(1,1)}(M^n, \mathbb{R}^{n+1})/Diff(M^n)$, where $Emb^{(1,1)}(M^n, \mathbb{R}^{n+1})$ is the space of C^1-smooth embeddings of M^n into $\mathbb{R}^{n+1}$ with $C^{1,1}$-smooth image, equipped with the C^1-topology, $Diff(M^n)$ is the group of C^1-smooth diffeomorphisms of M^n and the disjoint union is taken over the set of all compact C^1-smooth manifolds M^n admitting a C^1-embedding into $\mathbb{R}^{n+1}$ with $C^{1,1}$-smooth image. (A C^1-smooth hypersurface A^n is called $C^{1,1}$-smooth if the (outer) unit normal to A^n at a point $x \in A^n$ considered as a function of x is a Lipschitz function.) Obviously, $M_n(\mathbb{R}^{n+1}) \subset M_n^{(1,1)}(\mathbb{R}^{n+1})$. As in the case of $M_n(\mathbb{R}^{n+1})$, the topology of $M_n^{(1,1)}(\mathbb{R}^{n+1})$ can be expressed in terms of ε-approximation. (Lemma 2.2.4 for the $C^{1,1}$-smooth case was proven in Section 2.2.)

As it was already noted in Section 1 $r(M^n) > 0$ for $C^{1,1}$-smooth hypersurfaces M^n. Thus, $\kappa_d(M^n) = diam(M^n)/r(M^n)$ and $\kappa_v(M^n) = (vol(M^n))^{\frac{1}{n}}/r(M^n)$ are correctly defined functionals on $M_n^{(1,1)}(\mathbb{R}^{n+1})$. The following theorem states the lower semicontinuity of κ_d and κ_v on $M_n^{(1,1)}(\mathbb{R}^{n+1})$.

Theorem 5.1.1. The functional $r : M_n^{(1,1)}(\mathbb{R}^{n+1}) \to \mathbb{R}$ is upper semicontinuous, and functionals $\kappa_d,\ \kappa_v : M_n^{(1,1)}(\mathbb{R}^{n+1}) \to \mathbb{R}^+$ are lower semicontinuous.

Remark. The functional r is *not* continuous. This can be shown by a slight modification of the example at the end of the paper [21].

Proof of Theorem 5.1.1. Obviously, it is sufficient to prove only the upper semicontinuity of r on $M_n^{(1,1)}(\mathrm{IR}^{n+1})$. Recall that $S(\Sigma^n)$ denotes the set of points $y \in \mathrm{IR}^{n+1}$ such that there exist at least two points $x_1, x_2 \in \Sigma^n$ such that $|x_1 - y| = |x_2 - y| = dist_{\mathrm{IR}^{n+1}}(\Sigma^n, y)$. It is easy to show that $S(\Sigma^n) \subset C(\Sigma^n)$. It is known that for a C^2-smooth compact hypersurface Σ^n $C(\Sigma^n)$ coincides with the closure of $S(\Sigma^n)$ (cf. [37], [21]). It can be easily checked that the proof of the inclusion $\overline{S(\Sigma^n)} \supset C(\Sigma^n)$ (Lemma 2 in [37]) given in [37] (or in [21]) can be carried out in the $C^{1,1}$-smooth case without any modifications and, hence, dist $(\Sigma^n, \overline{S(\Sigma^n)}) = dist(\Sigma^n, \overline{C^n}) = r(\Sigma^n)$.

Let Σ^n be a $C^{1,1}$-smooth compact hypersurface. We should prove that for any ε there exists $\delta > 0$ such that if a hypersurface Σ_1^n δ-approximates Σ^n then $r(\Sigma_1^n) \leqq r(\Sigma^n) + \varepsilon$.

Let $a \in \overline{C(\Sigma^n)} = \overline{S(\Sigma^n)}$ be the closest to Σ^n point of $\overline{C(\Sigma^n)}$. Let $b \in S(\Sigma^n)$ be a point of $S(\Sigma^n)$ such that $|a - b| < \frac{\varepsilon}{2}$. Let $\sigma_1, \sigma_2 \in \Sigma^n$, $\sigma_1 \neq \sigma_2$ be any two of the closest to b points of Σ^n. Then straight lines $(\sigma_1 b)$ and $(\sigma_2 b)$ are normal to Σ^n at σ_1 and σ_2, respectively. Denote the value of the angle $\angle\sigma_1\, b\, \sigma_2$ by α and $|\sigma_1 - b| (= |\sigma_2 - b|)$ by τ. By its definition $\tau \leqq r(\Sigma^n) + \frac{\varepsilon}{2}$. Denote $\min\{\tau, \frac{\varepsilon}{2}, \frac{r(\Sigma^n)}{2}\}$ by ε_1. Consider the point b_ε situated on the ray $[\sigma_1 b)$ such that $|b_\varepsilon - \sigma_1| = \tau + \varepsilon_1$ (and, hence, $|b_\varepsilon - b| = \varepsilon_1$). Then $|\sigma_2 - b_\varepsilon| = \sqrt{\tau^2 + \varepsilon_1^2 + 2\tau\varepsilon_1 \cos\ \alpha}$. Hence

$$|\sigma_1 - b_\varepsilon| - |\sigma_2 - b_\varepsilon| = \frac{2\tau\varepsilon_1(1 - \cos\ \alpha)}{\tau + \varepsilon_1 + \sqrt{\tau^2 + \varepsilon_1^2 + 2\tau\varepsilon_1 \cos\ \alpha}} \geqq \frac{1 - \cos\ \alpha}{2}\, \varepsilon_1 \ .$$

Now let δ be a positive number. Suppose that a $C^{1,1}$-smooth hypersurface Σ_1^n δ-approximates Σ^n. Let $\sigma_{11}, \sigma_{21} \in \Sigma_1^n$ be the points situated, respectively, on the normals of Σ^n at σ_1 and σ_2, such that $|\sigma_{11} - \sigma_1| \leqq \delta$, $|\sigma_{21} - \sigma_1| \leqq \delta$. Let b_1 be the point, situated on the normal ray to Σ_1^n at σ_{11}, such that $|\sigma_{11} - b_1| = \tau + \varepsilon_1$. We suppose also that the normal goes in the same direction (inwards or outwards) with respect to Σ_1^n as the normal $[\sigma_1 b)$ with respect to Σ^n. Consider the quadrangle $b_1 b_\varepsilon \sigma_{11} \sigma_{21}$. It follows from the definition of ε-approximation and from the cosine theorem that

$$|b_1 - b_\varepsilon| \leqq |\sigma_{11} - \sigma_1| + (\tau + \varepsilon_1)\sqrt{2(1 - \cos\angle([\sigma_1 b_\varepsilon), [\sigma_{11} b_1)))} \leqq$$

$$\leqq \delta + \sqrt{2\frac{\delta}{diam(\Sigma^n)}}\ (\tau + \varepsilon_1)\ .$$

Hence

$$|b_1 - \sigma_{21}| \leqq \delta + \sqrt{\frac{2\delta}{diam(\Sigma^n)}}\ (\tau + \varepsilon_1) + |b_\varepsilon - \sigma_2| + |\sigma_2 - \sigma_{21}|\ .$$

Thus

$$|b_1 - \sigma_{21}| - |b_1 - \sigma_{11}| \leqq \delta + \sqrt{\frac{2\delta}{diam(\Sigma^n)}}\,(\tau + \varepsilon_1)$$

$$+|b_\varepsilon - \sigma_2| + \delta - |b_\varepsilon - \sigma_1| \leqq$$

$$\leqq \sqrt{2\delta}\left(\sqrt{2\delta} + \frac{2\tau}{\sqrt{diam(\Sigma^n)}}\right) - \frac{1 - cos\ \alpha}{2}\,\varepsilon_1\ .$$

Now let $\delta = \frac{(1-cos\ \alpha)^2 \varepsilon_1^2 diam(\Sigma^n)}{9(diam(\Sigma^n)+2\tau)^2}$.

By the choice of $\varepsilon_1, \delta < \frac{r(\Sigma^n)}{2}$.

It is easy to see that for this δ

$$|b_1 - \sigma_{21}| - |b_1 - \sigma_{11}| < 0$$

i.e.

$$|b_1 - \sigma_{21}| < |b_1 - \sigma_{11}|\ .$$

This means that the normal $[b_1\sigma_{11})$ to Σ_1^n is not the shortest path from b_1 to Σ_1^n. Hence $r(\Sigma_1^n) \leqq |b_1 - \sigma_{11}| = \tau + \varepsilon_1 \leqq r(\Sigma^n) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. □

5.2. Self-clenching "thick" hyperspheres

In this section we shall prove for any fixed $n \geqq 5$ the existence of an infinite set of distinct local minima of v-crumpledness in $Emb^{(1,1)}(S^n, \mathrm{I\!R}^{n+1})/Diff(S^n)$. "Distinct" means that any two of these minimizing $C^{1,1}$-smooth topological hyperspheres cannot be transformed one into the other by a composition of a homothetic transformation and an isometry. We prove an even stronger result:

Theorem 5.2.1. For any $n \geqq 5$ there exists a sequence $\{\Sigma_i^n\}_{i=1}^\infty$ of compact $C^{1,1}$-smooth hypersurfaces in $\mathrm{I\!R}^{n+1}$ diffeomorphic to the n-dimensional sphere such that:

(1) Any hypersurface Σ_i^n is a local minimum of the functional κ_v. Moreover, there is no continuous path $\gamma : [0,1] \to M_n^{(1,1)}(\mathrm{I\!R}^{n+1})$ such that (a) $\gamma(0) = \Sigma_i^n$; (b) $\kappa_v(\gamma(1)) < \kappa_v(\Sigma_i^n)$; (c) for any $t \in (0,1)$ $\kappa_v(\gamma(t)) \leqq \kappa_v(\Sigma_i^n)$.

(2) The sequence $\{\kappa_v(\Sigma_i^n)\}_{i=1}^\infty$ is a strictly increasing sequence tending to infinity.

Proof. We shall prove the theorem in three stages:

1) First we consider for every positive x the subset S_x of $Emb^{(1,1)}(S^n, \mathrm{I\!R}^{n+1})/Diff(S^n)$ composed of all hypersurfaces Σ^n of v-crumpledness $\leqq x$ contained in the (closed) ball B_2 of radius two centered at the origin, and such that $diam(\Sigma^n) = 1$. We shall show that for any $x \geqq 0$ S_x is compact. Indeed, consider for any $\Sigma^n \in S_x$ the characteristic function χ_{Σ^n} of Σ^n.

According to proven in Section 2.2 property 1) of characteristic functions of $C^{1,1}$-smooth hypersurfaces all these characteristic functions will be $C^{1,1}$-smooth.

This implies, in particular,that we can regard the set of characteristic functions of all hypersurfaces from S_x as a subset of $C^1(B_2)$. Denote this subset by S^x. Note that for any hypersurface from S_x its d-crumpledness$\leqq C_2(n)x^n$ by the virtue of Lemma 2.1.4(b). Hence for any hypersurface $\Sigma^n \in S_x$ $r(\Sigma^n) \geqq \frac{1}{C_2(n)}x^{-n}$. Property 3) of characteristic functions implies that $\|\chi_{\Sigma^n}\|_{C^1(B_2)}$ can be majorized by a function of x. Thus, S^x is a bounded subset of $C^1(B_2)$. Furthermore, property 4) of characteristic functions implies that for any $y, z \in B_2$

$$| \; grad \; \chi_{\Sigma^n}(y) - \; grad \; \chi_{\Sigma^n}(z)| \leqq (const_3 C_2^2(n))x^{2n}|y-z|. \tag{5.1}$$

Note that the right hand side of (5.1) does not depend on the function χ_{Σ^n} in the left hand side. Thus, passing to the limit we see that (5.1) holds not only for all functions from S^x but also for all functions from the closure $\bar{S}^x$ of S^x in $C^1(B_2)$. Hence by the virtue of the Ascoli-Arzela theorem the closure $\bar{S}^x$ of S^x in $C^1(B_2)$ is compact. Now we are going to deduce the compactness of S_x in $M_n^{(1,1)}(\mathbb{R}^{n+1})$ from the compactness of $\bar{S}^x$ in $C^1(B_2)$. Obviously, to achieve this goal it is sufficient to prove the following statement: Let $\{\Sigma_i^n\}$ be a sequence of hypersurfaces from S_x. If a sequence $\{\chi_{\Sigma_i^n}\}_{i=1}^{\infty} \subset S^x$ converges in $C^1(B_2)$ to a limit χ, then

(a) Zero set of χ is a non-empty compact $C^{1,1}$-smooth hypersurface Σ;

(b) $\{\Sigma_i^n\}_{i=1}^{\infty}$ converges to Σ in $M_n^{(1,1)}(\mathbb{R}^{n+1})$;

(c) $\Sigma \in S_x$.

Let us prove a). In order to show that zero set Σ of χ is non-empty consider a sequence $\{y_i\}$ such that for any i $y_i \in \Sigma_i^n (\subset B_2)$. This sequence contains a subsequence which converges in B_2. Denote its limit by y. One can easily see that $\chi(y) = 0$. The compactness of Σ follows from the obvious closeness and boundedness of Σ. To show smoothness of Σ note that property 3) of characteristic functions proved in Section 2.2 implies that for any n and any $y \in \mathbb{R}^{n+1}$

$$\chi^2_{\Sigma_i^n} + |grad \; \chi_{\Sigma_i^n}|^2 \geqq \min\{\frac{1}{4}, \frac{4}{r^2(\Sigma_i^n)}\} = \frac{1}{4}.$$

(Here we used the obvious fact that outside $N_{r(\Sigma_i^n)/4}(\Sigma_i^n)$ $|\chi_{\Sigma_i^n}| \geqq \frac{1}{2}$. The last equality follows from the fact that by the virtue of definition of S_x $diam(\Sigma^n) = 1$ and from Lemma 2.1.1) Hence $\chi^2 + |grad \; \chi|^2 \geqq \frac{1}{4}$. So $grad \; \chi$ does not vanish on Σ. By the virtue of the implicit function theorem Σ will a be C^1-smooth hypersurface. Its outer unit normal $n(y)$ coincides with $\frac{grad \; \chi(y)}{|grad \; \chi(y)|}$. As it was already mentioned $grad\chi$ is Lipschitz and $|grad \; \chi| \geqq \frac{1}{4}$ on Σ. Hence $n(y)$ is Lipschitz on Σ and Σ is $C^{1,1}$-smooth. This completes the proof of a).

To prove b) first note that for any $\epsilon > 0$ Σ_i^n belongs to the ϵ-neighborhood of Σ for all sufficiently large i. (Indeed, assume the opposite. Then for a sequence $\{i_k\} \longrightarrow \infty$ there exists $y_k \in \Sigma_{i_k}^n$ such that $dist(y_k, \Sigma) \geqq \epsilon$. Consider a converging in B_2 subsequence of $\{y_k\}$. Denote its limit by y. Then 1) $dist(y, \Sigma) \geqq \epsilon$; 2) $\chi(y) = 0$ and, thus $y \in \Sigma$. We obtained the desired contradiction.)

Let y be a point of Σ, ϵ be a positive number and $Norm_\epsilon(y)$ denote the interval of the normal line to Σ at y of length 2ϵ centered at y. Let $n(y)$ denote the outer unit normal to Σ at y. We are going to prove now that for any sufficiently small positive ϵ and for any sufficiently large i there exists a unique point $y_i \in \Sigma_i^n \bigcap Norm_\epsilon(y)$. The idea is to show that for sufficiently small ϵ and for all sufficiently large i the absolute value of the directional derivative of $\chi_{\Sigma_i^n}$ in the direction of $n(y)$ at any point of $Norm_\epsilon(y)$ is uniformly bounded from below by two. Since it is not difficult to see that for sufficiently large i $|\chi_{\Sigma_i^n}(y)| \leqq \epsilon$, this fact will imply the existence of y_i. Since we can without any loss of generality assume that δ is less than ϵ, this fact will imply that the restriction of $\chi_{\Sigma_i^n}$ on the direct line interval $Norm_\epsilon(y)$ is monotone and, as a corollary, the uniqueness of y_i. We shall demonstrate that the absolute values of directional derivatives of $\chi_{\Sigma_i^n}$ in the direction of the normal to Σ at y for sufficiently large i are uniformly bounded in a neighborhood of Σ in two steps. The first step is to note that the value of any of these directional derivatives for large i at a point z such that $dist(z, \Sigma) \leqq \epsilon$ can be made arbitrarily close to the length of gradient of $\chi_{\Sigma_i^n}$ at z by a suitable choice of $\epsilon > 0$. The reason is that the direction of $grad\chi_{\Sigma_i^n}(z)$ at points sufficiently close to Σ is close for sufficiently small ϵ and large i to the direction of $grad\chi(y)$ (i.e. to the direction of the normal to Σ at y). The second step is to show that for some $\epsilon > 0$ $|grad\chi_{\Sigma_i^n}(z)| \geqq 4$ for all z such that $dist(z, \Sigma) \leqq \epsilon$. However, we prefer to do these steps in the inverse order.

So, first note that Lemma 2.1.1 and property 3) of characteristic functions imply that if for some $z \in B_2$ $|\chi_{\Sigma_i^n}(z)| \leqq \frac{1}{2}$ (i.e. $z \in N_{r(\Sigma_i^n)/4}(\Sigma_i^n)$), then $|grad\chi_{\Sigma_i^n}(z)| = \frac{2}{r(\Sigma_i^n)} \geqq 4$. For all large enough i and any $\sigma \in \Sigma$ we can assume that $|\chi_{\Sigma_i^n}(\sigma)| \leqq \frac{1}{4}$. Since the sequence of norms $\|grad\chi_{\Sigma_i^n}\|_{C^0(B_2)}$ is uniformly bounded, then for some sufficiently small $\epsilon > 0$ and for all sufficiently large i $|\chi_{\Sigma_i^n}(z)| \leqq \frac{1}{2}$ for any z in the ϵ-neighborhood of Σ. Hence, for all sufficiently large i and any z such that $dist(z, \Sigma) \leqq \epsilon$

$$|grad\chi_{\Sigma_i^n}(z)| \geqq 4. \tag{5.2}$$

Now note that the directional derivative of $\chi_{\Sigma_i^n}(z)$ in the direction of the outer unit normal $n(y)$ to Σ at y is equal to $(grad\chi_{\Sigma_i^n}(z), n(y))$. Thus, one can write

$$(grad\chi_{\Sigma_i^n}(z), n(y)) = |grad\chi_{\Sigma_i^n}(z)| - (grad\chi_{\Sigma_i^n}(z), \frac{grad\chi_{\Sigma_i^n}(z)}{|grad\chi_{\Sigma_i^n}(z)|} - \frac{grad\chi(y)}{|grad\chi(y)|}) \geqq$$

$$\geqq |grad\chi_{\Sigma_i^n}(z)|(1 - \left|\frac{grad\chi_{\Sigma_i^n}(z)}{|grad\chi_{\Sigma_i^n}(z)|} - \frac{grad\chi(z)}{|grad\chi(z)|}\right| - \left|\frac{grad\chi(z)}{|grad\chi(z)|} - \frac{grad\chi(y)}{|grad\chi(y)|}\right|) \geqq$$

$$\geqq 4(1 - \lambda(i) - const\ x^{3n}\epsilon),$$

where $const$ is a positive constant and $\lambda(i)$ is an independent of ϵ function tending to zero, when $i \longrightarrow \infty$. The last inequality follows from the Lipschitz inequality (5.1) for $grad\chi_{\Sigma_i^n}$, from the obvious fact that $|grad\chi_{\Sigma_i^n}|$ satisfies the Lipschitz inequality with the same constant as $grad\chi_{\Sigma_i^n}$, from (5.2), and from the fact that $grad\chi_{\Sigma_i^n}(z) \longrightarrow grad\chi(z)$ when $i \longrightarrow \infty$.

Thus, when $\epsilon \leqq \frac{1}{4const\ x^{3n}}$ and i is sufficiently large to ensure that $|\lambda(i)| \leqq \frac{1}{4}$, the absolute value of directional derivative of $\chi_{\Sigma_i^n}$ in the direction of $n(y)$ is bounded from below by 2 at any point of $Norm_\epsilon(y)$. This completes the proof of existence and uniqueness of y_i.

The fact that the outer unit normal to Σ_i^n at y_i will be sufficiently close to the outer unit normal to Σ at y for sufficiently large i can be easily proven expressing these normals in terms of gradients of $\chi_{\Sigma_i^n}(y_i)$ and of $\chi(y)$ and using the fact that all maps $grad\chi_{\Sigma_i^n}$ satisfy the Lipschitz inequality with the same Lipschitz constant.

The continuity of the map $y \longrightarrow y_i$, the surjectiveness of this map and the continuity of its inverse are obvious and can be proven exactly as these properties were proven in a similar situation at the end of the proof of Lemma 2.2.4. This completes the proof of b).

In order to prove c) it is sufficiently to check that $\kappa_v(\Sigma) \leqq x$. But this fact immediately follows from already proven part b) of the statement and Theorem 5.1.1. This completes the proof of the compactness of S_x.

2) Now we consider the following equivalence relation on S_x. Two elements of S_x are equivalent iff they can be connected by a continuous path in $S_{\max\{x,\theta(x)\}}$, where θ is the function defined in the text of Lemma 2.2.2. (note that $S_x \subset S_{\max\{x,\theta(x)\}}$). The equivalence classes with respect to this relation are in the obvious one-to-one correspondence with knot "with thick ropes" $(x, \max\{x, \theta(x)\})$-types (see the definition in section 4.1). (Actually the equivalence classes with respect to this relation are subsets of knot "with thick ropes" $(x, \max\{x, \theta(x)\})$-types). Theorem 4.2 implies the existence of an increasing sequence $\{x_i\}_{i=1}^{\infty}$ and a sequence of pairwise non-intersecting equivalence classes $K_i^{(0)} \subset S_{x_i}$.

We want to prove that all sets $K_i^{(0)}$ are compact. Indeed, Theorem 5.1.1 and the $C^{1,1}$-smooth version of Lemma 2.2.2 imply that all sets $K_i^{(0)}$ are closed. Since we proved on the first stage of the proof that all sets S_x and, in particular, S_{x_i} are compact, $K_i^{(0)}$ are also compact sets.

3) Any lower semicontinuous function on a compact set has a minimum. Thus, v-crumpledness has a (global) minimum Σ_i^n on any set $K_i^{(0)}$. The construction of $K_i^{(0)}$ and Lemma 2.2.2 imply that hypersurfaces Σ_i^n are local minima of κ_v on $M_n^{(1,1)}(\mathbb{R}^{n+1})$. By the construction of the sequence $\{x_i\}$ in the proof of Theorem 4.1 (and in the similar proof of Theorem 4.2) v-crumpledness of any element of $K_{i+1}^{(0)}$ is greater than x_i. (Otherwise the element would have a trivial knot "with thick rope" x_{i+1}-type, which is impossible). This implies that the sequence of minimal values of κ_v on $K_i^{(0)}$ is strictly increasing and tends to infinity. To prove that the sequence $\{\Sigma_i^n\}$ satisfies condition (1) of Theorem 5.2.1 note that for any path γ, described in condition (1), $\gamma(1) \in K_i^{(0)}$ and $\kappa_v(\gamma(1)) < \kappa_v(\Sigma_i^n)$. Hence there is a contradiction between the assumption that

such a path exists and the definition of Σ_i^n as the global minimum of κ_v on $K_i^{(0)}$. Thus, there are no such paths γ. □

Remark. The space of all compact $C^{1,1}$-smooth hypersurfaces in $\mathbb{R}^{n+1}$ can be equipped by the topology defined using the Hausdorff distance between hypersurfaces (regarded as subsets of $\mathbb{R}^{n+1}$). This topology will not be equivalent to the topology of $M_n^{(1,1)}(\mathbb{R}^{n+1})$. However, one can easily see from the proof of Theorem 5.2.1 that any local minimum of κ_v on $M_n^{(1,1)}(\mathbb{R}^{n+1})$ will be a local minimum of κ_v on the space of hypersurfaces equipped with the Hausdorff distance topology. Indeed, assume the opposite. Then there exists a sequence $\{\Sigma_i^n\}$ converging in the Hausdorff distance topology to a local minimum of κ_v Σ such that $\kappa_v(\Sigma_i^n) < \kappa_v(\Sigma)$. Consider the sequence of characteristic functions $\{\chi_{\Sigma_i^n}\}$. Proceeding by the same way as in the proof of Theorem 5.2.1, one can show that this sequence of functions contains a converging in C^1-norm subsequence. Furthermore, one can show that the zero sets of functions of the subsequence converge to a smooth hypersurface in the topology of $M_n^{(1,1)}(\mathbb{R}^{n+1})$ exactly as in the proof of Theorem 5.2.1. The Hausdorff distance between this limit hypersurface and Σ is equal to zero. Hence this hypersurface is Σ. This yields the contradiction with the assumption that Σ is a local minimum of κ_v on $M_n^{(1,1)}(\mathbb{R}^{n+1})$.

An analysis of the proof of Theorem 5.2.1 shows that we used in the proof only the following three properties of the functional $\kappa_v : M_n^{(1,1)}(\mathbb{R}^{n+1}) \to \mathbb{R}^+$:

1) (Invariance.) κ_v is invariant under transformations of $M_n^{(1,1)}(\mathbb{R}^{n+1})$ induced by rotations, translations and homothetic transformations of $\mathbb{R}^{n+1}$.

2) (Recursive equivalence to κ_d.) There exist continuous increasing functions $\varphi_1 : \mathbb{R} \to \mathbb{R}$, $\varphi_2 : \mathbb{R} \to \mathbb{R}$ such that $[\varphi_1|_{\mathbb{N}}]$, $[\varphi_2|_{\mathbb{N}}]$ are recursive and for any $\Sigma^n \in M_n^{(1,1)}(\mathbb{R}^{n+1})$

$$\kappa_v(\Sigma^n) \leqq \varphi_1(\kappa_d(\Sigma^n)) \quad ,$$

$$\kappa_d(\Sigma^n) \leqq \varphi_2(\kappa_v(\Sigma^n)) \quad .$$

3) (Lower semicontinuity.) κ_v is a lower semicontinuous functional on $M_n^{(1,1)}(\mathbb{R}^{n+1})$.

Let us call any functional $M_n^{(1,1)}(\mathbb{R}^{n+1}) \to \mathbb{R}^+$ satisfying conditions 1) - 3) *generalized crumpledness functional.* One can prove the following generalization of Theorem 5.2.1 exactly as Theorem 5.2.1 was proved.

Theorem 5.2.2. Let κ be a generalized crumpledness functional. For any $n \geqq 5$ there exists a sequence $\{\Sigma_i^n\}_{i=1}^{\infty}$ of $C^{1,1}$-smooth hypersurfaces in $\mathbb{R}^{n+1}$, diffeomorphic to S^n, such that

1) Any hypersurface Σ_i^n is a local minimum of the funcitonal κ. Moreover, there is no continuous path $\gamma : [0,1] \to M_n^{(1,1)}(\mathbb{R}^{n+1})$ such that: (a) $\gamma(0) = \Sigma_i^n$; (b) $\kappa(\gamma(1)) < \kappa(\Sigma_i^n)$; (c) For any $t \in (0,1)$ $\kappa(\gamma(t)) \leqq \kappa(\Sigma_i^n)$.
2) The sequence $\{\kappa(\Sigma_i^n)\}_{i=1}^{\infty}$ is a strictly increasing sequence tending to infinity.

As we already saw, κ_d and κ_v are generalized crumpledness functionals. Here is another example. Let A be a positively defined $(n+1)\times(n+1)$ matrix. Let $\|x-y\|_A = (A(x-y), x-y)$ and $diam_A(\Sigma)$ denote the diameter of $\Sigma \subset \mathrm{IR}^{n+1}$ in the corresponding metric. Then κ_A defined by the formula $\kappa_A(\Sigma^n) = diam_A(\Sigma^n)/r(\Sigma^n)$ is a generalized crumpledness funcitonal. It is interesting that if $A \neq \lambda E$ then the usual unit sphere S^n is not a local minimum of κ_A.

There exists the following way to construct new generalized crumpledness functionals from the already known ones. Let $\kappa_1, \cdots, \kappa_k$ be generalized crumpledness functionals. Let $f : (\mathrm{IR}^+)^k \to \mathrm{IR}^+$ be a continuous function increasing with any of its arguments such that

(a) $[f|_{\mathrm{IN}^k}]$ is a recursive function;

(b) Suppose that we fix values of any $(k-1)$ arguments of f in an arbitrary way and consider f as the function of the remaining argument. This one variable funciton is unbounded.

It is obvious that $f(\kappa_1, \cdots, \kappa_k)$ will also be a generalized crumpledness functional. For example, $7\ln\ \kappa_v + \exp(\kappa_d)$ or $\kappa_v^{\kappa_d} \cdot \sqrt{\kappa_A}$ are generalized crumpledness functionals, and Theorem 5.2.2 implies for any fixed $n \geqq 5$ for any of these two functionals the existence of infinitely many essentially distinct local minima diffeomorphic to S^n.

Elaborating the construction above one can easily build other examples of generalized crumpledness functionals, such as $\kappa_v + \tau + diam_{\mathrm{IR}^{n+1}}/vol^{\frac{1}{n}}$ or $\tau\kappa_d + vol^{\frac{1}{n}}/diam_{\mathrm{IR}^{n+1}}$, where τ denotes the total absolute curvature functional. Again, by the virtue of Theorem 5.2.2 these functionals have an infinite set of essentially distinct local minima on the space of $C^{1,1}$-smooth topological hypersheres.

On the other hand Gromov's distortion ([14], p.113-114) is an example of a functional which is *not* a generalized crumpledness functional. One can easily see from example (2) on p.114 of [14] that there is no function φ such that $\kappa_d \leqq \varphi(distortion)$.

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