MAT 347Y: Groups, rings, and fields
Homework #14
Due on Friday, March 13 at 10:10am in class

1. Let K/M and M/F be field extensions. Prove or give a counterexample:
   (a) If K/F is normal, then K/M is normal.
   (b) If K/F is normal, then M/F is normal.
   (c) If K/M and M/F are normal, then K/F is normal.

2. Prove that every field extension of degree 2 is normal.

   Warning: Do not assume that the fields do not have characteristic 2.

   Note: Are you having a deja-vu? If these first two questions make you think of similar results for groups, there is a good reason for it.

3. Let K/F be a finite field extension. Prove that the following are equivalent:
   (a) K is the splitting field of some (non-necessarily irreducible) polynomial in $F[X]$.
   (b) For any field $E \supseteq K$ and any F-homomorphism $\phi : K \to E$, $\phi(K) \subseteq K$.
   (c) For any field $E \supseteq K$ and any F-homomorphism $\phi : K \to E$, $\phi(K) = K$.
   (d) If $f(X) \in F[X]$ is an irreducible polonomial and it has a root in K, then $f(X)$ splits in K.
   (e) For every $\alpha \in K$, the minimal polynomial of α in F splits in K.
   (f) $K = F(\alpha_1, \ldots, \alpha_n)$ and the minimal polynomial of α_j in F splits in K for all $j = 1, \ldots, n$.

   Hint: You may want to use Theorems A and B.

4. Prove that the algebraic closure of a field is unique up to isomorphism.

   Hint: Let F be a field and let K_1 and K_2 be two algebraic closures. Consider the set X of triples (M_1, M_2, ϕ) where $F \subseteq M_1 \subseteq K_1$ is an intermediate extension, $F \subseteq M_2 \subseteq K_2$ is an intermediate extension, and $\phi : M_1 \to M_2$ is a field isomorphism such that $\phi(z) = z$ for all $z \in F$. Define a partial order in X by saying that $(M_1, M_2, \phi) \leq (N_1, N_2, \psi)$ iff $M_1 \subseteq N_1$, $M_2 \subseteq N_2$, and $\psi(z) = \phi(z)$ for all $z \in M_1$. Use Zorn's lemma. Beware, as there are lots of things to check.

5. The goal of this problem is to prove that a finite extension generated by separable elements is separable.

   For the first few questions, let us fix a finite, normal field extension K/F. Given intermediate extensions $F \subseteq M_1 \subseteq M_2 \subseteq K$, we define $\mathrm{Emb(M_2/M_1)}$ to be the set of M_1–homomorphisms $\varphi : M_2 \to K$.

(a) Assume $M_2 = M_1(\alpha)$. Prove that $|\operatorname{Emb}(M_2/M_1)|$ equals the number of distinct roots of $m_{\alpha,M_1}[X]$ in K. Conclude that $|\operatorname{Emb}(M_2/M_1)| \leq |M_2 : M_1|$, with equality iff α is separable over M_1.

(b) For any intermediate extensions $F \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq K$, prove that

$$|\operatorname{Emb}(M_3/M_1)| = |\operatorname{Emb}(M_3/M_2)|\,|\operatorname{Emb}(M_2/M_1)|$$

(c) Assume K/M is not separable. Prove that $|\operatorname{Emb}(K/F)| < |K : F|$.

(d) Assume K is generated over F by separable elements. Prove that $|\operatorname{Emb}(K/F)| = |K : F|$. Conclude that K/F is separable.

For the remaining questions, we remove the initial assumptions.

(e) Prove that the splitting field of a separable polynomial is a separable extension.

(f) Let K/F be a finite, separable extension. Prove that its normal closure is a finite, normal, separable extension.

(g) Let K/F be any finite extension (not necessarily normal). Assume that $K = F(\alpha_1, \ldots, \alpha_n)$ and that α_i is separable over F for all i. Prove that K/F is separable.