MAT 347Y: Groups, rings, and fields
Homework #7.
Due on Friday, November 14 at 10:10am in class

1. Let p be a prime. Let n be a positive integer. Let E_{p^n} be the elementary abelian p-group of order p^n (in other words, E_{p^n} is the direct product of n copies of Z_p). Let $k \leq n$ be another positive integer. Find a formula for the number of subgroups of order p^k of E_{p^n}.

2. Let n be a positive integer. Prove that
$$\operatorname{Aut} D_{2n} \cong \operatorname{Hol}(Z_n).$$
Here, Hol represents the holomorph. For the definition of holomorph, see Example 5 on page 179 of the textbook.

   Note: You will need to know something else about the structure of $\operatorname{Aut} Z_n$. See the short handout I have posted next to this problem set on the website.

3. Let $p \geq 5$ be a prime. Classify groups of order $4p$ up to isomorphism.

   Note: You will need to use the Sylow theorems for this question, so you may want to wait until we see them in class, or read ahead.

4. *[Note: This problem helps decide when two semidirect products are isomorphic, and will be quite useful for your project. Depending on your order you may need to develop variations on it.]*

   Let K and H be finite groups. Let $\phi_1, \phi_2 : K \to \operatorname{Aut}(H)$ be group homomorphisms. Define $G_i := H \rtimes_{\phi_i} K$ for $i = 1, 2$.

   (a) Suppose there are automorphisms $\alpha \in \operatorname{Aut}(K)$ and $\beta \in \operatorname{Aut}(H)$ such that
   $$\phi_2 = \gamma_\beta \circ \phi_1 \circ \alpha,$$
   where γ_β is conjugation in $\operatorname{Aut}(H)$ by β. Prove that $G_1 \cong G_2$.

   (b) Prove the result from problem 6 from Section 5.5 of the textbook. (Don't pay attention to the hint given in the book. Instead, notice that it is a direct consequence of part (a).)

   (c) Suppose that H is abelian. Prove that $C_{G_1}(H) = (\ker \phi_1)H$.

   (d) Suppose that H is abelian and that $|H|$ and $|K|$ are relatively prime. Assume that $\ker \phi_1 \not\cong \ker \phi_2$. Prove that $G_1 \not\cong G_2$.