MAT 347Y: Groups, Rings, and Fields
Practice Test 1

- You are allowed to consult our textbook and your notes. You are not allowed to use the internet or any other resources.
- **Note:** The answers to most of these questions are in Chapter 6. If you want to use this as an actual practice test, do not read past Chapter 5 on your book!

For this test, WE WILL CONSIDER ONLY FINITE GROUPS.

For this test, consider that 0 is a natural number.

Definitions. Let G be any group. Recall that its centre $Z(G)$ is defined by

$$Z(G) = \{a \in G \mid ab = ba \text{ for all } b \in G\}$$

We now define inductively a subgroup $Z_i(G)$ for every natural integer i. We define

$$Z_0(G) = 1, \qquad Z_1(G) = Z(G),$$

and we say that $Z_{i+1}(G)$ is the unique subgroup of G containing $Z_i(G)$ such that

$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

The chain of subgroups

$$Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \dots$$

is called the *upper central series* of G.

A group G is called *nilpotent* if $Z_i(G) = G$ for some $i \in \mathbb{N}$. The smallest such i is called the *nilpotent class* of G.

Questions are on the back. Spend some time trying to understand the definition before you start attempting the questions. There is no particular order to attack all the questions and they are all related, so feel free to go out of order.

1. Give one example of a nilpotent group and one example of a non-nilpotent group.

2. For each of the following statements, decide whether it is true or false. If true, prove it. If false, provide a counterexample.

   (a) $Z_i(G)$ is a normal subgroup of G, for all i.

   (b) $Z_i(G)$ is a characteristic subgroup of G, for all i.

3. Which cyclic groups are nilpotent?

4. Let p be a prime. Recall that a p-group is a group with order a power of p. Which p-groups are nilpotent?

5. For which values of n is the dihedral group D_{2n} nilpotent?

6. For which values of n is the symmetric group S_n nilpotent?

7. For each of the following statements, decide whether it is true or false. If true, prove it. If false, provide a counterexample.

   (a) Let G be a group and let $H \leq G$. If G is nilpotent, then H is nilpotent.

   (b) Let G be a group and let $N \trianglelefteq G$. If G is nilpotent, then G/N is nilpotent.

   (c) Let G be a group and let $N \trianglelefteq G$. If N and G/N are both nilpotent, then G is nilpotent.

   (d) Let G and H be two groups. If G and H are both nilpotent, then their product $G \times H$ is also nilpotent.

8. For each natural number i, find a nilpotent group with nilpotence class i.