MAT 347Y: Groups, Rings, and Fields
Test 1 - Grading Scheme, Comments, and Solutions

Grading scheme

The test had 11 questions (counting each part of problem 6 as a separate question). Each question is worth 4 points, for a total of 44 points. However, 9 of them are bonus marks, so that achieving 35 points means a score of 100%.

General comments

For some of you, it appears that using the textbook hurt more than helped. Specifically, you attempted to use heavy theorems for questions that had easy answers. For example, some of you attempted to use Theorem 6.11, when there were directly one-line proofs. There was always an easier way to do things.

General test-taking tips

I thought this went without saying, but after grading your test, I have to comment on these things. Failing to follow this tips has probably already cost you various letter grades in many courses.

- Write your scratch work somewhere first, and then copy it on your exam booklet once you know what you are doing. Include in your exam booklet only the correct answers.
- Write neatly and organize your work. Make it easy for me to read. Make it easy for me to be satisfied.
- Indicate clearly which question you are answering. If you really must separate the answer to a question into non-consecutive pages, say so.
- **Do not bullshit.** If you do not know an answer, or if you know something is wrong, do not gloss over details hoping that I won't notice. You are more transparent than you think, and it gives a very bad impression when I realize this is what you are doing. It you write "I would need to prove X to conclude this proof, but I have not been able to do so yet" you will get partial marks. If you bullshit, you won't.
- In a question like problem 4, write your final answer at the top of the page and box it: "S_n is solvable iff $n \leq 4$." Make it easy for me to get excited about reading your answer!

Solutions

1. **Give one example of a solvable group and one example of a non-solvable group.**

   Use your work from later questions.

2. **Classify all simple, solvable groups up to isomorphism.**

   Solution: If G is simple the only possible chain is

   $$G_0 = 1 \trianglelefteq G_1 = G \tag{1}$$

   Hence, a simple group G is solvable iff it is abelian. The only simple, abelian groups are cyclic of order prime.

3. **Which cyclic groups are solvable?**

   Solution: Every abelian group G is solvable (just take the same chain as in (1)). Every cyclic group is abelian, so it is solvable.

   Note:

   - Some of your answers to this question were examples of *matar moscas a cañonazos*[1]. There is absolutely no reason why your proof should be any longer than the two sentences I wrote. If you write something longer, it gives the impression that you did not understand the definition.

4. **For which values of n is the dihedral group D_{2n} solvable?**

   Solution: All dihedral groups are solvable. Let $D_{2n} =< r, s >$ with the usual presentation. Then consider the chain:

   $$1 \trianglelefteq < r > \trianglelefteq D_{2n}$$

   We know that $< r > \trianglelefteq D_{2n}$ because it has index 2. We know $< r >$ is cyclic, and hence abelian. Finally, $D_{2n}/ < r >$ has order 2, so it is abelian.

5. **For which values of n is the symmetric group S_n solvable?**

   Solution: S_n is solvable iff $n \leq 4$.

[1]That is an old Spanish saying that translates as "killing flies with cannons".

(a) S_1 and S_2 are abelian, and hence solvable.

(b) $S_3 \cong D_6$ and hence solvable.

(c) S_4 is solvable via the chain $1 \trianglelefteq V_4 \trianglelefteq A_4 \trianglelefteq S_4$. Every quotient in this chain has order 2 or 3, and hence it is abelian.

(d) For $n \geq 5$, A_n is simple and non-abelian. By question 2, A_n is not solvable. By question 6a, S_n cannot be solvable.

Common errors:

- Here is an incomplete proof that S_n is not solvable for $n \geq 5$. We notice that the chain $1 \trianglelefteq A_n \trianglelefteq S_n$ does not work because A_n is not abelian, and since A_n is simple we cannot stick anything else between 1 and A_n. This proof is not complete because it shows this particular chain does not work, but what if there is different chain

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

with $G_{m-1} \neq A_n$? To make the proof work you have various options:

  - Notice that the only normal subgroups of S_n are 1, A_n, and S_n, so there are no other chains.
  - Use an argument similar to the Jordan-Holder theorem, to justify the uniqueness of the factors up to reordering.
  - First notice that A_n is not solvable, and then use question 6a.
  - Find (and prove) what the relation is between a decomposition series and a solving series.

6. **For each of the following statements, decide whether it is true or false. If true, prove it. If false, provide a counterexample.**

(a) **Let G be a group and let $H \leq G$. If G is solvable, then H is solvable.**
Solution: TRUE. Let

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

be a chain for G such that each G_{i+1}/G_i is abelian. I define a chain

$$H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq H_3 \trianglelefteq \ldots \trianglelefteq H_m$$

by letting $H_i := G_i \cap H$. Notice that $H_0 = 1$ and $H_m = H$. If we can prove that H_{i+1}/H_i is abelian for each i, we'll be done. To show this, consider the homomorphism

$$\varphi : H_{i+1} \to G_{i+1}/G_i$$

defined by $\varphi(x) = xG_i$. We notice that $\ker\varphi = H_i$. By the first isomorphism theorem, H_{i+1}/H_i is isomorphic to a subgroup of G_{i+1}/G_i, and hence abelian.

Common errors:

- We may not assume that $G_i \leq H \leq G_{i+1}$ for some i.
- If $G_i \leq H$ for some i, we may not simply take the chain

$$1 \trianglelefteq \ldots \trianglelefteq G_i \trianglelefteq H$$

  because in general H/G_i is not abelian.
- It is not true that $G_{i+1}/G_i \cong G_{i+1} \cap H/G_i \cap H$.

(b) **Let G be a group and let $N \trianglelefteq G$. If G is solvable, then G/N is solvable.**
Solution: TRUE. Let

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

be a chain for G such that each G_{i+1}/G_i is abelian. I define a chain

$$C_0 \trianglelefteq C_1 \trianglelefteq C_2 \trianglelefteq C_3 \trianglelefteq \ldots \trianglelefteq C_m$$

by letting $C_i := G_iN/N$. Notice that I can do this since N is normal, so G_iN is always a group. Moreover, $C_0 = 1$ and $C_m = G/N$. If we can prove that C_{i+1}/C_i is abelian for each i, we'll be done. We do this in two steps:

- Consider the surjective group homomorphism

$$\varphi : G_{i+1} \to G_{i+1}N/G_iN$$

  defined by $\varphi(x) = xG_iN$. Notice that $G_i \leq \ker\varphi$, so φ induces a well-defined, surjective group homomorphism

$$G_{i+1}/G_i \to G_{i+1}N/G_iN.$$

  Hence, $G_{i+1}N/G_iN$ is isomorphic to a quotient of G_{i+1}/G_i and thus abelian.
- By the Third Isomorphism Theorem, we know that $C_{i+1}/C_i \cong G_{i+1}N/G_iN$, and hence C_{i+1}/C_i is abelian.

Common errors:

- We may not assume that $G_i \leq N \leq G_{i+1}$ for some i.
- If $N \leq G_i$ for some i, we may not simply take the chain

$$1 \trianglelefteq G_i/N \trianglelefteq G_{i+1}/N \trianglelefteq \ldots$$

  because in general G_i/N is not abelian.

- We may not write G_i/N unless $N \leq G_i$.
- It is not true that $G_{i+1}/G_i \cong G_{i+1}N/G_iN$.

(c) **Let G be a group and let $H, K \leq G$. If H and K are both solvable, then the join $< H, K >$ is solvable.**

Solution: FALSE. The same counterexample to question 6(e) works.

Solution #2: (due to Saman Samikermani). Every group is the join of cyclic groups. If the claim were true, then by induction every group would be solvable.

(d) **Let G be a group and let $H, K \leq G$. Assume that $H \leq N_G(K)$. If H and K are both solvable, then HK is solvable.**

Solution: TRUE. We do this in three steps:

- We know that HK is a group, that $K \trianglelefteq HK$, and that $HK/K \cong H/H \cap K$. (See Second Isomorphism Theorem.)
- Using question 6b, since H is solvable, so is $H/H \cap K$.
- Using the theorem *proven* on page 105 of the book, since K and HK/K are solvable, so is HK.

Common errors: There are at least two bad ways to try to put together solving series $1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H_m = H$ and $1 = K_0 \trianglelefteq K_1 \trianglelefteq \ldots \trianglelefteq H = K_n = K$ into a solving series for HK:

- The series $1 = H_0K_0 \trianglelefteq H_1K_1 \trianglelefteq \ldots \trianglelefteq H_mK_m = HK$ does not work because $n \neq m$ and because $H_{i+1}K_{i+1}/HK$ is not abelian.
- The series $1 = H_0 \trianglelefteq H_1 \trianglelefteq \ldots \trianglelefteq H \trianglelefteq HK$ does not work because HK/H is not abelian.

(e) **Let G be a group and let $H, K \leq G$. Assume that $HK \leq G$. If H and K are both solvable, then HK is solvable.**

Solution: FALSE. For example, let $G = S_5$. Let H be the subgroup of G consisting of the permutations that fix 5. We know that $H \cong S_4$ so H is solvable. Let K be the subgroup generated by one 5-cycle. We know that $K \cong Z_5$ so it is solvable. Notice that $|H| = 24$ and $|K| = 5$. Hence, $|H \cap K| = 1$. Then, as a set

$$|HK| = \frac{|H| \cdot |K|}{|H \cap K|} = 120$$

Therefore $HK = S_5$. So HK is a group. But S_5 is unsolvable.

7. **Let G be a group. Prove or disprove that the following two statements are equivalent:**

   (a) **G is solvable.**

   (b) **There exists subgroups**

   $$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

   for some non-negative integer m, such that

   - $G_0 = \{1\}$**,** $G_m = G$**, and**
   - G_i/G_{i-1} **is cyclic for every** $i = 1, \ldots, m$**.**

Solution: TRUE. It is clear that (a) implies (b). To prove that (b) implies (a), we'll proceed in two steps.

- First, let's consider an abelian group A. Notice that every subgroup of an abelian group is normal. I claim that it is possible to find a chain

  $$1 = A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq A_3 \trianglelefteq \ldots \trianglelefteq A_m = A$$

  such that every A_{i+1}/A_i is cyclic. We do this by induction on $|A|$.

  - If A is cylic we are done.
  - Otherwise, pick any $1 \neq x \in A$ and take $A_1 =< x >$. Then we apply induction hypothesis to $B := A/A_1$. We get a chain

    $$1 = B_0 \trianglelefteq B_1 \trianglelefteq B_2 \trianglelefteq B_3 \trianglelefteq \ldots \trianglelefteq B_r = B$$

    such that every B_{i+1}/B_i is cyclic. By the third–fourth isomorphism theorem, we know that for each $i = 0, \ldots, r$ we can write $B_i = A_{i+1}/A_1$ for some $A_{i+1} \leq A$. Unravelling what this means we get the desired chain for A.

- Second, assume we have a chain

  $$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

  where each G_{i+1}/G_i is abelian. Apply the previous argument to obtain a chain for each G_{i+1}/G_i where each quotient is cyclic. Use the third–fourth isomorphism theorem to transform that chain into subgroups of G. Stick them all together and we get the desired result.

Note: This proof is extremely related to the the existence proof of composition series (exercise 6 on section 3.4, which was on the homework). If you had solved that exercise and understood it well, this proof is straightforward.