MAT 347Y: Groups, Rings, and Fields
Test 1
October 31 2014 - Happy Halloween!

- Time allotted: 2 hours and 50 minutes.
- You are allowed to consult our textbook and your notes. You are not allowed to use the internet or any other resources.

For this test, WE WILL CONSIDER ONLY FINITE GROUPS.

Definition: Let G be a group. We say that G is *solvable* when there exist subgroups

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

for some non-negative integer m, such that:

- $G_0 = \{1\}$, $G_m = G$, and
- G_i/G_{i-1} is abelian for every $i = 1, \ldots, m$.

Notice that each G_i only has to be normal in G_{i+1}, and not necessarily in the whole G.

The notion of solvability will be very important when we study Galois Theory. It is the key to determining which polynomial equations can or cannot be solved using radicals.

Solvable groups are discussed briefly in section 3.4 of the textbook. You can read about them before you begin. The back of this page contains the test questions. I suggest you spend some moments making sure you understand the notion of solvability before you rush to solve the problems. All the questions are related, and there is no clear better order to attack them, so feel free to solve them out of order. **Do not turn this paper over until the invigilator asks you to start.**

1. Give one example of a solvable group and one example of a non-solvable group.
2. Classify all simple, solvable groups up to isomorphism.
3. Which cyclic groups are solvable?
4. For which values of n is the dihedral group D_{2n} solvable?
5. For which values of n is the symmetric group S_n solvable?
6. For each of the following statements, decide whether it is true or false. If true, prove it. If false, provide a counterexample.
   (a) Let G be a group and let $H \leq G$. If G is solvable, then H is solvable.
   (b) Let G be a group and let $N \trianglelefteq G$. If G is solvable, then G/N is solvable.
   (c) Let G be a group and let $H, K \leq G$. If H and K are both solvable, then the join $< H, K >$ is solvable.
   (d) Let G be a group and let $H, K \leq G$. Assume that $H \leq N_G(K)$. If H and K are both solvable, then HK is solvable.
   (e) *[Challenge!]* Let G be a group and let $H, K \leq G$. Assume that $HK \leq G$. If H and K are both solvable, then HK is solvable.
7. Let G be a group. Prove or disprove that the following two statements are equivalent:
   (a) G is solvable.
   (b) There exists subgroups

$$G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \ldots \trianglelefteq G_m$$

   for some non-negative integer m, such that
   - $G_0 = \{1\}$, $G_m = G$, and
   - G_i/G_{i-1} is cyclic for every $i = 1, \ldots, m$.