MAT 347Y: Groups, Rings, and Fields
Test 2 comments and some solutions

Q1

This was the worst question on the test, despite being identical to the homework. A complete solution should have all the following steps:

- (1) We define S to be the poset of prime ideals (with respect to reverse inclusion).
- (2) S is non-empty. Notice that you cannot argue that $(0) \in S$ (because (0) may or may not be a prime ideal). Instead, argue that S has maximal ideals, which are prime.
- (4) Every chain in S has an upper (lower) bound. There are various common errors here:
  - You may not assume that your chain is countable! You may not use notation that pretends that your chain is countable.
  - You have to explain your notation.
  - To prove that an ideal $I \trianglelefteq R$ is prime, in particular you have to check that $I \neq R$.
- (1) Use Zorn's Lemma to conclude.
- (2) For part (b), the *only* correct answer is the zero ideal.

Q2

This is not as long as some of you made it to be.

- Polynomials of degree 0 are units, hence not irreducible.
- All polynomials of degree 1 are irreducible:

$$X, \qquad X+1.$$

- A polynomial of degree 2 or 3 is irreducible iff it has no roots. Notice that in $\mathbb{F}_2[X]$:
  - a polynomial has 0 as a root iff it has a non-zero constant term.
  - a polynomial has 1 as a root iff it has an odd number of non-zero terms.

  Hence we only keep polynomials with a non-zero constant term and with an odd number of non-zero terms. There are three:

$$X^2+X+1, \qquad X^3+X^2+1, \qquad X^3+X+1.$$

- A polynomial of degree 4 is reducible iff it has a root OR it is the product of two irreducibles of degree 2. There is only one irreducible of degree 2, so the only polynomial to avoid is $(X^2+X+1)^2 = (X^4+X^2+1)$. Apart from that, keep all polynomials with a non-zero constant term and with an odd number of non-zero terms. There are three more:

$$X^4+X+1, \qquad X^4+X^3+1, \qquad X^4+X^3+X^2+X+1.$$

Grading scheme:

- 1 point per correct irreducible (total of 8).
- -1 point per non-irreducible in your list.
- 2 points for a correct justification.
- If you used brute force, I accepted it as a justification only if you made no mistakes.

Q3

Grading scheme:

- 5 marks for all the definitions being correct (a lot are needed).
- 5 marks for a good structure (your plan is clear and well-written, and you have taken a good path that will produce a complete proof). This is the "big picture".
- 10 marks for the details of all the proofs.

By the way, there is no need to define GCSs or to talk about Bezout domains whatsoever.

Q4

Grading scheme:

- 5 marks for a good structure (your plan is clear and well-written, and you have taken a good path that will produce a complete proof). This is the "big picture".
- 15 marks for the details of all the proofs.
  Notice that if you do not use the fact that prime in R implies prime in $R[X]$ (which also needs to be proven) your proof cannot possibly be correct.