MAT 347Y: Groups, Rings, and Fields

Test 2 — February 6, 2015

You have 170 minutes. The maximum score is 40 marks.

Solve both question 1 AND question 2

1. *[10 marks]* Let R be a commutative ring with $1 \neq 0$.

   (a) Prove that the set of prime ideals of R has a minimal element with respect to inclusion.

   (b) Find one such minimal prime ideal for the case $R = \mathbb{Z}[i]$.

2. *[10 marks]* Let $\mathbb{F}_2$ be the field with 2 elements. Find all the irreducible polynomials with degree less than or equal to 4 in $\mathbb{F}_2[X]$.

Solve EITHER question 3 OR question 4, but not both

3. *[20 marks]* Define Euclidean domain and UFD. Then prove that every Euclidean domain is a UFD. You may assume all basic concepts and theorems about rings which are not specific about factorization (in other words, you may assume Chapter 7). Every other concept you use, you need to define (so your definition of UFD may take a few steps, for example.) Every other lemma you use, you need to state and proof. Use your judgement to decide how much detail to provide. Notice that, for this proof, you do not need to involve PIDs at all and you can save a lot of time.

4. *[20 marks]* Let R be an integral domain. Prove that R is a UFD if and only if $R[X]$ is a UFD. You may assume the basic theory of rings (Chapter 7) and the basic theory of UFDs (Chapter 8). Every other concept you use, you need to define. Every other lemma you use, you need to state and prove. Use your judgement to decide how much detail to provide.