**MAT 347**
**Order, dihedral groups, and presentations**
**September 12, 2014**

# Order

Let $G$ be a group. Let $a \in G$. We want to compare the powers of $a$. In other words, when do we have $a^n = a^m$?

First, we define the *order* of $a$ as the smallest positive integer $n$ such that $a^n = 1$, if there is such a thing. Otherwise we define the *order* of $a$ to be infinity. We denote the order of $a$ by $|a|$.

1. Let $G$ be a group, let $a \in G$, and let $r = |a|$. Complete the following statements and prove them:
   (a) $a^n = 1 \iff \ldots$ (something about $n$ and $r$)
   (b) $a^n = a^m \iff \ldots$ (something about $n$, $m$, and $r$)
2. Find the order of every element in $(\mathbb{Z}/18\mathbb{Z})^X$.
   *Note:* This questions is shorter than it seems.
3. Find an example of a group $G$ that contains one element of order $n$ for every positive integer $n$ and which also contains an element of order infinity.

# Dihedral groups

Consider a geometric object $C$ in $\mathbb{R}^N$. The group of symmetries of $C$ consists of all the *rigid motions* that leave $C$ invariant (invariant as a set, not pointwise). This is always a group. For example:

- the group of rigid motions of a highly asymmetric figure has only one element.
- the group of rigid motions of a line segment has two elements.
- the group of rigid motions of an equilateral triangle has six elements (can you count them?)

We define the $n$-th *dihedral group*, denoted by $D_{2n}$, as the group of symmetries of a regular $n$-gon.

4. List all the elements of $D_{12}$. Once you have your complete list, you have to prove that there are no others. Notice that saying "I cannot think of any others" is not a proof.

5. Compute the order of each element in $D_{12}$.

6. Find a pair of elements in $D_{12}$ (let's call them $\alpha$ and $\beta$) such that every element in $D_{12}$ can be written as product of powers of $\alpha$ and $\beta$. We say that *$\alpha$ and $\beta$ generate* $D_{2n}$. Actually, there are multiple such pairs. Then, list all the elements in $D_{12}$ as product of powers of $\alpha$ and $\beta$ in a canonical way. You decide what "canonical" means.

7. If we wanted to fully describe the group structure of $D_{12}$ we would need to construct its full multiplication table. This is too much. Instead, find a few simple equations involving $\alpha$ and $\beta$ that would give us enough information to complete their multiplication table. These equations contain all the information we need to fully describe the group. Part of this question consists of figuring out what the heck we mean.